ON $W_4$-BIRECURRENT MANIFOLDS

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Abstract. In the present paper, we introduce a type of Riemannian manifolds (namely, $W_4$-birecurrent manifold) and study the several properties of such a manifold on which some geometric conditions are imposed.

1. Introduction

In [7], Lichnerowicz introduced a recurrent manifold of second order (called birecurrent manifold) and obtained some interesting results. Later the weak notion of this manifold was studied by many authors including Thompson [9], Chaki and Roy Chowdhury [2], Ewert-Krzemieniewski [3,4], Garai [5]. In [8], Pokhariyal defined some curvature tensors with the help of Weyl’s projective curvature tensor and studied their physical and geometrical properties. One of the curvature tensors mentioned in [8] was the $W_4$-curvature tensor defined by

$$W_4(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{(n-1)}[g(X,Z)r(Y,T) - g(X,Y)r(Z,T)],$$

or in local coordinates,

$$W_{4ijkl} = R_{ijkl} + \frac{1}{(n-1)}[g_{ik}r_{jl} - g_{ij}r_{kl}],$$

where $R$ and $r$ are the Riemannian curvature tensor and the Ricci tensor, respectively. The $W_4$-curvature tensor has no symmetry but it satisfies the cyclic property


and by way of contraction, it reduces to the Ricci tensor. Hence a $W_4$-flat manifold (i.e., a Riemannian manifold with $W_4 = 0$) is Ricci-flat.

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which yields from (1.1) that the manifold is flat. Moreover it was shown that the vanishing of the divergence of $W_4$-curvature tensor in an electromagnetic field implies a purely electric field [6,8,10]. A Riemannian manifold $(M^n,g)$ is said to be a birecurrent manifold if the manifold satisfies the following condition
\[(\nabla_U \nabla_V R)(X,Y,Z,T) = A(U,V)R(X,Y,Z,T)\]
or in local coordinates,
\[(1.2)\quad R_{ijkl;qp} = A_{pq}R_{ijkl},\]
where $A$ is the associated tensor of type $(0,2)$ which is nonzero.

This paper is concerned with the manifolds in which the $W_4$-curvature tensor is second order recurrent. More precisely, we deals with a Riemannian manifold $(M^n,g)$ called $W_4$-birecurrent manifold such that the $W_4$-curvature tensor $W_4$ of $(M^n,g)$ satisfies the following relation
or in local coordinates,
\[(1.3)\quad W_{4ijkl;qp} = A_{pq}W_{4ijkl},\]
where $A$ is the associated tensor of type $(0,2)$ which is nonzero.

Here the semicolon denotes covariant derivative with respect to the metric.

2. Some properties of $W_4$-birecurrent manifold

We first prove the following.

**Theorem 2.1.** Every $W_4$-birecurrent manifold is a birecurrent manifold with the same associated tensor.

**Proof.** Contracting (1.3) over $i$ and $l$, we obtain
\[(2.4)\quad r_{jk;qp} = A_{pq}r_{jk},\]
From (1.1), (1.3) and (2.4), it follows that
\[(2.5)\quad R_{ijkl;qp} = A_{pq}R_{ijkl},\]
which completes the proof. □

As a consequence we have

**Theorem 2.2.** Let $(M^n,g)$ be a $W_4$-birecurrent manifold. Then either the manifold is flat or the associated tensor $A$ in (1.2) is symmetric.
Proof. Taking account of (2.5), we have
\begin{equation}
(R_{ijkl}R^{ijkl})_{;qp} = 2R_{ijkl;qp}R^{ijkl} + 2R_{ijkl;q}R^{ijkl;p}.
\end{equation}
Therefore from (2.5) and (2.6) we obtain
\[ 0 = 2(A_{pq} - A_{qp})R_{ijkl}R^{ijkl}, \]
which yields either
\[ A_{pq} = A_{qp} \]
or
\[ R_{ijkl} = 0. \]
This completes the proof.

Concerning the scalar curvature of \( W_4 \)-birecurrent manifold, we can state the followings.

**Theorem 2.3.** Let \((M^n, g)\) be a \( W_4 \)-birecurrent manifold. Then the scalar curvature \( s \) of \((M^n, g)\) cannot be a nonzero constant.

**Proof.** Contracting (1.3) over \( i \) and \( l \), and then contracting the relation obtained thus over \( j \) and \( k \), we have
\begin{equation}
s_{;qp} = A_{pq}s.
\end{equation}
If \( s = \text{constant}(\neq 0) \), then we have from (2.7) \( A_{pq} = 0 \), which is inadmissible because of the defining condition for the \( W_4 \)-birecurrent manifold. Therefore the scalar curvature \( s \) of \((M^n, g)\) cannot be a nonzero constant.

**Theorem 2.4.** Let \((M^n, g)\) be a \( W_4 \)-birecurrent manifold. If the scalar curvature \( s \) of \((M^n, g)\) is nonzero, then the relation \( A_{pq} = \frac{1}{s}s_{;qp} \) holds.

**Proof.** By virtue of (2.7) and \( s \neq 0 \) (and so nonconstant by Theorem 2.3), we have \( A_{pq} = \frac{1}{s}s_{;qp} \), which completes the proof.

Let \((M^n, g)\) be a Riemannian manifold satisfying the condition
\begin{equation}
R_{ijkl;q} = 0.
\end{equation}
Then the manifold is said to be locally symmetric.

Now we can state the following.

**Theorem 2.5.** Let \((M^n, g)\) be a \( W_4 \)-birecurrent manifold. If \((M^n, g)\) is locally symmetric, then the manifold is flat.
Proof. Taking account of Theorem 2.1 and (2.8), we have
\[ 0 = A_{pq}R_{ijkl}, \]
which yields \( R_{ijkl} = 0 \) because the associated tensor \( A \) is nonzero. This completes the proof. \( \Box \)

3. \( W_4 \)-birecurrent Einstein manifolds

Let \( (M^n, g) \) be a Riemannian manifold satisfying the relation
\[ r_{jk} = \frac{s}{n} g_{jk}. \] (3.9)
Then we call the manifold an Einstein manifold. It is well known that the scalar curvature \( s \) of an Einstein manifold is constant \([1]\).

A Riemannian manifold is said to be a \( W_4 \)-birecurrent Einstein manifold if the manifold is simultaneously an Einstein manifold and a \( W_4 \)-birecurrent manifold.

Concerning a \( W_4 \)-birecurrent Einstein manifold, we have

**Theorem 3.1.** Let \( (M^n, g) \) be a \( W_4 \)-birecurrent Einstein manifold. Then the Ricci tensor \( r \) of \( (M^n, g) \) is zero.

*Proof.* Taking account of Theorem 2.3 and \( s = \text{constant} \), we have from (3.9) \( r_{jk} = 0 \), which completes the proof. \( \Box \)

**Theorem 3.2.** Let \( (M^n, g) \) be a \( W_4 \)-birecurrent Einstein manifold. Then either the manifold is flat or \( A^q_i A_{pq} = 0 \).

*Proof.* By virtue of Theorem 2.1 and the second Bianchi identity, we get
\[ 0 = R_{ijklq} + R_{ijlqk} + R_{ijqklp} \]
(3.10)
\[ = A_{pq}R_{ijkl} + A_{pl}R_{ijqk} + A_{pk}R_{ijlq}. \]
Contracting (3.10) over \( i \) and \( l \) and using the fact of the vanishing Ricci tensor by Theorem 3.1, we obtain
\[ A^q_i R_{ijqk} = 0 \]
or equivalently
\[ A^q_i R_{qkji} = 0 \] (3.11)
Multiplying \( A^q_i \) to (3.10), we get
\[ 0 = A^q_i A_{pq}R_{ijkl} + A^q_i A_{pl}R_{ijqk} + A^q_i A_{pk}R_{ijlq}. \] (3.12)
From (3.11) and (3.12), it follows that
\[ A^q_i A_{pq} R_{ijkl} = 0, \]
which yields either \( A^q_i A_{pq} = 0 \) or \( R_{ijkl} = 0 \). This completes the proof. \( \square \)

4. Vector fields and \( W_4 \)-birecurrent manifold

A vector field \( V \) on a Riemannian manifold \((M^n, g)\) is said to be concurrent if it satisfies the relation
\[
V^i_{;j} = \rho \delta^i_j
\]
where \( \rho = \text{constant} \neq 0 \) and \( \delta^i_j \) denotes the Kronecker delta.

Now we can state the following.

**Theorem 4.1.** Let \((M^n, g)\) be a \( W_4 \)-birecurrent manifold admitting a concurrent vector field \( V \). If the scalar curvature \( s \) of \((M^n, g)\) is nonzero, then we have \( A_{ij} V^i V^j = 6 \rho^2 \).

**Proof.** By virtue of (4.13) and the Ricci identity, we have
\[
0 = V^i_{;jk} - V^i_{;kj} = V^i R^i_{hjk}.
\]
Contracting (4.14) over \( i \) and \( k \), we obtain
\[
V^h r_{hj} = 0.
\]
Differentiating (4.15) covariantly, we obtain
\[
V^h r_{hj} + V^h r_{hj;l} = 0.
\]
From (4.13) and (4.16), it follows that
\[
\rho \delta^h_l r_{hj} + V^h r_{hj;l} = 0.
\]
Multiplying \( g^{lj} \) to (4.17) and contracting with respect to the indices, we get
\[
\rho s + V^h r_{hj;l} g^{lj} = 0.
\]
Taking account of (4.18) and the second Bianchi identity, we have
\[
\rho s + \frac{1}{2} V^h s_{;h} = 0.
\]
Differentiating (4.19) covariantly, we get from (4.13)
\[
2 \rho s_{;m} + V^h s_{;hm} + \rho \delta^h_m s_{;h} = 0,
\]
which reduces to
\[
3 \rho s_{;m} + V^h s_{;hm} = 0.
\]
Multiplying $V^m$ to (4.20), we have 

$$3\rho s_m V^m + V^h V^m s_{hm} = 0,$$

which yields from (2.7) and (4.19)

$$-6\rho^2 s + V^h V^m A_{mh} s = 0$$

or equivalently

$$(4.21) \quad (A_{mh} V^h V^m - 6\rho^2) s = 0.$$ 

Hence from (4.21) and $s \neq 0$, we have

$$(4.22) \quad A_{mh} V^m V^h = 6\rho^2,$$

which completes the proof. \(\square\)

A vector field $V$ on a Riemannian manifold $(M^n, g)$ is called parallel if it satisfies

$$V^i_{ij} = 0.$$ 

Consequently we have

**Corollary 4.2.** Let $(M^n, g)$ be a $W_4$-birecurrent manifold admitting a parallel vector field $V$. If the scalar curvature $s$ of $(M^n, g)$ is nonzero, then we have $A_{ij} V^i V^j = 0$.

**Proof.** By virtue of (4.13), (4.22) and (4.23), we have $A_{ij} V^i V^j = 0$, which completes the proof. \(\square\)

On the other hand, we can state the following.

**Theorem 4.3.** Let $(M^n, g)$ be a $W_4$-birecurrent manifold admitting a parallel vector field $V$. If the scalar curvature $s$ of $(M^n, g)$ is nonzero, then we have $A_{pq} V^q = 0$.

**Proof.** By virtue of the Ricci identity, we have

$$0 = V^h_{ijk} - V^h_{kji} = V^i R^h_{ijk}.$$ 

Contracting (4.24) over $h$ and $k$, we obtain

$$V^i r_{ij} = 0.$$ 

On the other hand, differentiating (4.24) covariantly, we have from $V^i_{;i} = 0$

$$V^i R^h_{ijk;l} = 0.$$ 

Contracting (4.26) over $h$ and $k$, we have

$$V^i r_{ijl} = 0.$$
Taking account of (4.26) and the second Bianchi identity, we have
\begin{equation}
V^i R^h_{lkji; i} = 0.
\end{equation}
(4.28)
Contracting (4.28) over \( h \) and \( j \), and then contracting the relation obtained thus over \( l \) and \( k \), we get
\begin{equation}
V^i r_{lk; i} = 0
\end{equation}
(4.29)
and
\begin{equation}
V^i s_{; i} = 0.
\end{equation}
(4.30)
Taking account of \( V^i_{; m} = 0 \), (4.28) and (4.29), we have
\begin{equation}
V^i R^h_{lk; ijm} = 0
\end{equation}
(4.31)
and
\begin{equation}
V^i r_{lk; ijm} = 0.
\end{equation}
(4.32)
Multiplying \( V^q \) to (1.3), we have from (1.1), (4.31) and (4.32)
\begin{equation}
0 = V^q W_{4ijkl; qp} = V^q A_{pq} W_{4ijkl}.
\end{equation}
(4.33)
Contracting (4.33) over \( i \) and \( l \), and then contracting the relation obtained thus over \( j \) and \( k \), we obtain
\begin{equation}
0 = V^q A_{pq} r_{jk}
\end{equation}
(4.34)
and
\begin{equation}
0 = V^q A_{pq} s.
\end{equation}
(4.35)
By virtue of \( s \neq 0 \), we have from (4.35) \( A_{pq} V^q = 0 \), which completes the proof.

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References


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