REEB FLOW INVARIANT UNIT TANGENT SPHERE BUNDLES

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Abstract. For unit tangent sphere bundles $T^1M$ with the standard contact metric structure $(\eta, \tilde{g}, \phi, \xi)$, we have two fundamental operators that is, $h = \frac{1}{2} L_\xi \phi$ and $\ell = \tilde{R}(\cdot, \xi)\xi$, where $L_\xi$ denotes Lie differentiation for the Reeb vector field $\xi$ and $\tilde{R}$ denotes the Riemannian curvature tensor of $T^1M$. In this paper, we study the Reeb flow invariancy of the corresponding $(0,2)$-tensor fields $H$ and $L$ of $h$ and $\ell$, respectively.

1. Introduction

For a given contact metric structure $(\eta, \tilde{g}, \phi, \xi)$, a symmetry type occurs when the geodesic flow generated by $\xi$, which is called the Reeb flow, leaves some structure tensors invariant. This is always the case for $\xi$ and $\eta$ since $L_\xi \xi = 0$ and $L_\xi \eta = 0$. The metric $\tilde{g}$ is left invariant by the Reeb flow (or equivalently, the flow consists of local isometries or $\xi$ is a Killing vector field) if and only if $\phi$ is preserved under the Reeb flow. Apart from the defining structure tensors $\eta, \tilde{g}, \phi$ and $\xi$, two other operators play a fundamental role in contact metric geometry, namely, the structural operator $h = \frac{1}{2} L_\xi \phi$ and the characteristic Jacobi operator $\ell = \tilde{R}(\cdot, \xi)\xi$, where $L_\xi$ denotes Lie differentiation in the characteristic direction $\xi$.

An important topic in the study of the contact metric structure $(\eta, \tilde{g}, \phi, \xi)$ on unit tangent sphere bundles $T^1M$ is to determine those Riemannian manifolds $(M, g)$ for which the corresponding contact metric structure enjoys such a symmetry along the Reeb flow. In fact, Y. Tashiro [(11)] proved that $\xi$ is a Killing vector on the unit tangent
sphere bundle \( T_1M \) if and only if \((M,g)\) has constant curvature \(c = 1\). E. Boeckx and the present authors ([6]) proved that \( T_1M \) satisfies \( \mathcal{L}_\xi \ell = 0 \) if and only if \((M,g)\) is of constant curvature \(c = 1\) and \( T_1M \) satisfies \( \mathcal{L}_\xi \epsilon = 0 \) if and only if \((M,g)\) is of constant curvature \(c = 0\) or \(c = 1\).

In the present paper, we define the \((0,2)\)-tensor fields \( L \) and \( H \) by

\[ L(\mathring{X}, \mathring{Y}) = g(\ell \mathring{X}, \mathring{Y}) \] \[ H(\mathring{X}, \mathring{Y}) = g(\mathring{h} \mathring{X}, \mathring{Y}) \]

for any vector fields \( \mathring{X} \) and \( \mathring{Y} \) on \( \mathring{M} \) and we investigate when the \((0,2)\)-tensor fields \( L \) and \( H \) on \( T_1M \) are preserved by the geodesic flow. Namely, we prove:

**Theorem 1.** Let \( T_1M \) be the unit tangent sphere bundle with the standard contact metric structure \((\eta, \mathring{g}, \phi, \xi)\). Then \( T_1M \) satisfies \( \mathcal{L}_\xi L = 0 \) if and only if \((M,g)\) is of constant curvature \(c = -4\), \(c = 0\) or \(c = 1\).

**Theorem 2.** Let \( T_1M \) be the unit tangent sphere bundle with the standard contact metric structure \((\eta, \mathring{g}, \phi, \xi)\). Then \( T_1M \) satisfies \( \mathcal{L}_\xi H = 0 \) if and only if \((M,g)\) is of constant curvature \(c = -1\) or \(c = 1\).

From the results in [6] and Theorems 1 and 2, we find an evident distinction of the Reeb flow invariancy between \( h, \ell \) and the corresponding \((0,2)\)-tensor fields \( H, L \), respectively. That is, \( T_1H(-1) \) satisfies \( \mathcal{L}_\xi H = 0 \), but \( \mathcal{L}_\xi h \neq 0 \). And \( T_1H(-4) \) satisfies \( \mathcal{L}_\xi L = 0 \), but \( \mathcal{L}_\xi \ell \neq 0 \).

2. The standard contact metric structure on a unit tangent sphere bundle

We start by reviewing some fundamental facts on contact metric manifolds. We refer to [1] for more details. All manifolds are assumed to be connected and of class \( C^\infty \). A \((2n - 1)\)-dimensional manifold \( \mathring{M} \) is said to be a contact manifold if it admits a global 1-form \( \eta \) such that \( \eta \wedge (d\eta)^{n-1} \neq 0 \) everywhere on \( \mathring{M} \), where the exponent denotes the \((n-1)\)-th exterior power of the exterior derivative \( d\eta \) of \( \eta \). We call such \( \eta \) a contact form of \( \mathring{M} \). It is well known that given a contact form \( \eta \), there exists a unique vector field \( \xi \), which is called the characteristic vector field or the Reeb vector field, satisfying \( \eta(\xi) = 1 \) and \( d\eta(\xi, \mathring{X}) = 0 \) for any vector field \( \mathring{X} \) on \( \mathring{M} \). A Riemannian metric \( \mathring{g} \) on \( \mathring{M} \) is an associated metric to a contact form \( \eta \) if there exists a \((1,1)\)-tensor field \( \phi \) satisfying

\[ \eta(\mathring{X}) = \mathring{g}(\mathring{X}, \xi), \quad \eta(\phi \mathring{X}, \phi \mathring{Y}) = \mathring{g}(\mathring{X}, \phi \mathring{Y}), \quad \phi^2 \mathring{X} = -\mathring{X} + \eta(\mathring{X})\xi, \]

where \( \mathring{X} \) and \( \mathring{Y} \) are vector fields on \( \mathring{M} \). From (2.1) it follows that

\[ \phi \xi = 0, \quad \eta \circ \phi = 0, \quad \mathring{g}(\phi \mathring{X}, \phi \mathring{Y}) = \mathring{g}(\mathring{X}, \mathring{Y}) - \eta(\mathring{X})\eta(\mathring{Y}). \]
A Riemannian manifold $\bar{M}$ equipped with structure tensors $(\eta, \bar{g}, \phi, \xi)$ satisfying (2.1) is said to be a contact metric manifold. Given a contact metric manifold $\bar{M}$, we define the structural operator $h$ by $h = \frac{1}{2} L_\xi \phi$, where $L$ denotes Lie differentiation. Then we may observe that $h$ is self-adjoint and satisfies

\begin{align}
(2.2) & \quad h \xi = 0 \quad \text{and} \quad h \phi = -\phi h, \\
(2.3) & \quad \bar{\nabla}_X \xi = -\phi \bar{X} - \phi h \bar{X},
\end{align}

where $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$. From (2.2) and (2.3) we see that each trajectory of $\xi$ is a geodesic. We denote by $\bar{R}$ the Riemannian curvature tensor defined by

\[\bar{R}(\bar{X}, \bar{Y}) \bar{Z} = \bar{\nabla}_\bar{X}(\bar{\nabla}_\bar{Y} \bar{Z}) - \bar{\nabla}_\bar{Y}(\bar{\nabla}_\bar{X} \bar{Z}) - \bar{\nabla}_{[\bar{X}, \bar{Y}]} \bar{Z}\]

for all vector fields $\bar{X}, \bar{Y}, \bar{Z}$ on $\bar{M}$. Along a trajectory of $\xi$, the Jacobi operator $\ell = \bar{R}(\cdot, \xi) \xi$ is a symmetric (1,1)-tensor field. We call it the characteristic Jacobi operator. We have

\begin{align}
(2.4) & \quad \ell = \phi \ell \phi - 2(h^2 + \phi^2), \\
(2.5) & \quad \bar{\nabla}_\xi h = \phi - \phi \ell - \phi h^2.
\end{align}

A contact metric manifold for which $\xi$ is Killing is called a $K$-contact manifold. It is easy to see that a contact metric manifold is $K$-contact if and only if $h = 0$ or, equivalently, $\ell = I - \eta \otimes \xi$.

The basic facts and fundamental formulae about tangent bundles are well-known (cf. [7], [9], [13]). We briefly review some notations and definitions. Let $(M, g)$ be an $n$-dimensional Riemannian manifold and $\nabla$ the associated Levi-Civita connection. $R$ denotes its Riemannian curvature tensor. The tangent bundle over $(M, g)$ is denoted by $TM$ and consists of pairs $(p, u)$, where $p$ is a point in $M$ and $u$ a tangent vector to $M$ at $p$. The mapping $\pi : TM \to M$, $\pi(p, u) = p$, is the natural projection from $TM$ onto $M$. For a vector field $X$ on $M$, its vertical lift $X^v$ on $TM$ is the vector field defined by $X^v \omega = \omega(X) \circ \pi$, where $\omega$ is a 1-form on $M$. For the Levi-Civita connection $\nabla$ on $M$, the horizontal lift $X^h$ of $X$ is defined by $X^h \omega = \nabla_X \omega$. The tangent bundle $TM$ can be endowed in a natural way with a Riemannian metric $\bar{g}$, the so-called Sasaki metric, depending only on the Riemannian metric $g$ on $M$. It is determined by

\[\bar{g}(X^h, Y^h) = \bar{g}(X^v, Y^v) = g(X, Y) \circ \pi, \quad \bar{g}(X^h, Y^v) = 0\]
for all vector fields $X$ and $Y$ on $M$. Also, $TM$ admits an almost complex structure tensor $J$ defined by $JX^h = X^v$ and $JX^v = -X^h$. Then $\tilde{g}$ is a Hermitian metric for the almost complex structure $J$.

The unit tangent sphere bundle $\tilde{\pi} : T_1 M \to M$ is a hypersurface of $TM$ given by $g_p(u, u) = 1$. Note that $\tilde{\pi} = \pi \circ i$, where $i$ is the immersion of $T_1 M$ into $TM$. A unit normal vector field $N = u^v$ to $T_1 M$ is given by the vertical lift of $u$ for $(p, u)$. The horizontal lift of a vector is tangent to $T_1 M$, but the vertical lift of a vector is not tangent to $T_1 M$ in general. So, we define the tangential lift of $X$ to $(p, u) \in T_1 M$ by

$$X^t_{(p, u)} = (X - g(X, u)u)^v.$$

Clearly, the tangent space $T_{(p, u)} T_1 M$ is spanned by vectors of the form $X^h$ and $X^t$, where $X \in T_pM$.

We now define the standard contact metric structure of the unit tangent sphere bundle $T_1 M$ over a Riemannian manifold $(M, g)$. The metric $g'$ on $T_1 M$ is induced from the Sasaki metric $\tilde{g}$ on $TM$. Using the almost complex structure $J$ on $TM$, we define a unit vector field $\xi'$, a 1-form $\eta'$ and a $(1,1)$-tensor field $\phi'$ on $T_1 M$ by

$$\xi' = -JN, \quad \phi' = J - \eta' \otimes N.$$

Since $g'(\tilde{X}, \phi' \tilde{Y}) = 2d\eta'(\tilde{X}, \tilde{Y})$, $(\eta', g', \phi', \xi')$ is not a contact metric structure. Rectifying this by

$$\xi = 2\xi', \quad \eta = \frac{1}{2} \eta', \quad \phi = \phi', \quad \tilde{g} = \frac{1}{4} g',$$

we get the standard contact metric structure $(\eta, \tilde{g}, \phi, \xi)$. Here the tensor $\phi$ is explicitly given by

$$\phi X^t = -X^h + \frac{1}{2} g(X, u) \xi, \quad \phi X^h = X^t,$$

where $X$ and $Y$ are vector fields on $M$.

From now on, we consider $T_1 M = (T_1 M; \eta, \tilde{g}, \phi, \xi)$ with the standard contact metric structure. Then the Levi-Civita connection $\tilde{\nabla}$ of $T_1 M$ is described by

$$\tilde{\nabla}_X Y^t = -g(Y, u) X^t,$$

$$\tilde{\nabla}_X Y^h = \frac{1}{2} (R(u, X) Y)^h,$$

$$\tilde{\nabla}_X Y^t = (\nabla_X Y)^t + \frac{1}{2} (R(u, Y) X)^h,$$

$$\tilde{\nabla}_X Y^h = (\nabla_X Y)^h - \frac{1}{2} (R(X, Y) u)^t$$

(2.7)
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for all vector fields $X$ and $Y$ on $M$ (cf. [2], [3]).

For the Riemannian curvature tensor $\bar{R}$, we give only the two expressions we need for the characteristic Jacobi operator $\ell$:

$$\bar{R}(X^t, Y^h)Z^h = -\frac{1}{2} \{ R(Y, Z)(X - g(X, u)u) \}^t$$

$$+ \frac{1}{4} \{ R(Y, R(u, X)Z)u \}^t$$

$$- \frac{1}{2} \{ (\nabla_Y R)(u, X)Z \}^h,$$

(2.8)

$$\bar{R}(X^h, Y^h)Z^h = (R(X, Y)Z)^h + \frac{1}{2} \{ R(u, R(X, Y)u)Z \}^h$$

$$- \frac{1}{4} \{ R(u, R(Y, Z)u)X - R(u, R(X, Z)u)Y \}^h$$

$$+ \frac{1}{2} \{ (\nabla_Z R)(X, Y)u \}^t$$

for all vector fields $X$, $Y$ and $Z$ on $M$. From $\xi = 2u^h$ and (2.7), it follows

(2.9) $$\nabla_X^t \xi = -2\phi X^t - (R_u X)^t, \quad \nabla_X^h \xi = -(R_u X)^t$$

where $R_u = R(\cdot, u)u$ is the Jacobi operator associated with the unit vector $u$. From (2.3) and (2.9), it follows that

(2.10) $$hX^t = X^t - (R_u X)^t,$$

$$hX^h = -X^h + \frac{1}{2} g(X, u)\xi + (R_u X)^h.$$ 

Using the formulae (2.8), we get

(2.11) $$\ell X^t = (R_u^2 X)^t + 2(R_u^1 X)^h,$$

$$\ell X^h = 4(R_u X)^h - 3(R_u^2 X)^h + 2(R_u^1 X)^t$$

where $R_u^1 = (\nabla_u R)(\cdot, u)u$ and $R_u^2 = R(R(\cdot, u)u, u)u$. By using (2.5), (2.6) and (2.8) we obtain

(2.12) $$\nabla^t \xi \ell X^t = -2(R_u X)^h + 2(R_u^2 X)^h - 2(R_u^1 X)^t,$$

$$\nabla^h \xi \ell X^h = -2(R_u X)^t + 2(R_u^2 X)^t + 2(R_u^1 X)^h.$$ 

Finally, from (2.7) and (2.11) we compute

(2.13) $$\nabla^t \ell X^t = 4(R_u^2 R_u + R_u R_u^1)X + 4(R_u^1 X + R_u^2 X - R_u^1 X)^h,$$

$$\nabla^h \ell X^h = 8(R_u^1 R_u X - R_u^1 R_u^1 R_u X)^h + 4(R_u^2 X + R_u^2 X - R_u^2 X)^t.$$ 

The above formulae (2.9) ~ (2.13) are also found in [4], [5], [6].
We define the \((0,2)\)-tensor fields \(H\) and \(L\) by
\[
H(\bar{X}, \bar{Y}) = g(h\bar{X}, \bar{Y})
\]
and
\[
L(\bar{X}, \bar{Y}) = g(\ell\bar{X}, \bar{Y})
\]
for any vector fields \(\bar{X}\) and \(\bar{Y}\) on \(\tilde{M}\), respectively.

3. Proofs of Theorems

Proof of Theorem 1.

At first, from the definition of Lie differentiation and (2.3), we have
\[
(\mathcal{L}_\xi L)(\bar{X}, \bar{Y}) = \xi L(\bar{X}, \bar{Y}) - L(\mathcal{L}_\xi \bar{X}, \bar{Y}) - L(\bar{X}, \mathcal{L}_\xi \bar{Y})
\]
\[
= g((\tilde{\nabla}_\xi \ell)\bar{X}, \bar{Y}) + g(\ell(\nabla_{\bar{X}}\xi), \bar{Y}) + g(\ell\bar{X}, \nabla_{\bar{Y}}\xi)
\]
\[
= g((\tilde{\nabla}_\xi \ell)\bar{X}, \bar{Y}) + g(-\ell\phi\bar{X} - \phi h\bar{X}, \bar{Y}) + g(\ell\bar{X}, -\phi\bar{Y} - \phi h\bar{Y}).
\]
(3.1)

From (3.1), we see that the condition \(\mathcal{L}_\xi L = 0\) is equivalent to
\[
\tilde{\nabla}_\xi \ell = \ell \phi - \phi \ell + \phi h - h \phi \ell.
\]
(3.2)

Now we suppose that \(T_1 M\) satisfies \(\mathcal{L}_\xi L = 0\). Then from (3.2), by a straightforward calculation, we have two equations:
\[
0 = (4R'_uX + R'_uR_uX + R_uR'_uX)^t
\]
\[
+ (2R''_uX - 3R^2_uX - R^3_uX + 4R_uX)^h,
\]
(3.3)
\[
0 = (4R'_uX - 5R'_uR_uX - 5R_uR'_uX)^h
\]
\[
+ (2R''_uX - 3R^2_uX - R^3_uX + 4R_uX)^t.
\]
(3.4)

These equations are equivalent to the conditions:
\[
4R'_uX + R'_uR_uX + R_uR'_uX = 0,
\]
(3.5)
\[
4R'_uX - 5R'_uR_uX - 5R_uR'_uX = 0,
\]
(3.6)
\[
2R''_uX - 3R^2_uX - R^3_uX + 4R_uX = 0
\]
(3.7)

for all vector fields \(X\) on \(M\). From (3.5) and (3.6), we obtain \(R'_uX = 0\). This implies that \((M, g)\) is a locally symmetric space ([8], [12]). Further, we see from (3.7) that the eigenvalues of \(R_u\) are constant and equal to 0 or 1 or \(-4\), i.e., \((M, g)\) is a globally Osserman space (i.e., the eigenvalues of \(R_u\) do not depend on the point \(p\) and not on the choice of unit vector \(u\) at \(p\)). However, a locally symmetric globally Osserman space is locally flat or locally isometric to a rank one symmetric space ([10]). Therefore, we conclude that \(M\) is a space of constant curvature \(c = 0\) or \(c = 1\) or \(-4\). Conversely, if \((M, g)\) is of constant curvature \(c\), then we can
calculate the following explicit expressions for $h$, $\ell$, $\bar{\nabla}_h h$ and $\bar{\nabla}_\ell \ell$ from (2.10) $\sim$ (2.13):

$$hX^t = (1 - c)X^t, \quad hX^h = (c - 1)(X^h - \frac{1}{2} g(X, u)\xi),$$

$$\ell X^t = c^2 X^t, \quad \ell X^h = (4c^2 - 3c^3)(X^h - \frac{1}{2} g(X, u)\xi),$$

$$\bar{\nabla}_h X^t = 2(c^2 - c)(X^h - \frac{1}{2} g(X, u)\xi), \quad (\bar{\nabla}_h h)X^t = 2(c^2 - c)X^t,$$

$$\bar{\nabla}_h h = 2(c^2 - c^3)X^h - \frac{1}{2} g(X, u)\xi), \quad (\bar{\nabla}_\ell X^t = 4(c^2 - c^3)(X^h - \frac{1}{2} g(X, u)\xi),\quad (\bar{\nabla}_\ell h)X^h = 4(c^2 - c^3)X^t,$$

for vector fields $X$ on $M$. From (3.8), we easily check that $T_1 M$ satisfies (3.2) when $c = -4$, $c = 0$ or $c = 1$. □

**Proof of Theorem 2.**

From the definition of Lie differentiation and (2.3), we have

$$(\mathcal{L}_h H)(\bar{X}, \bar{Y}) = \xi H(\bar{X}, \bar{Y}) - H(\mathcal{L}_\phi \bar{X}, \bar{Y}) + H(\bar{X}, \mathcal{L}_h \bar{Y})$$

$$= g((\bar{\nabla}_h h)\bar{X}, \bar{Y}) + g(h(\bar{\nabla}_h \xi), \bar{Y}) + g(h\bar{X}, \bar{\nabla}_h \xi)$$

$$= g((\bar{\nabla}_\phi h)\bar{X}, \bar{Y}) + g(-\phi h\bar{X} - h\phi h\bar{X}, \bar{Y}) + g(h\bar{X}, -\phi \bar{Y} - \phi h\bar{Y})$$

$$= g((\bar{\nabla}_h h)\bar{X}, \bar{Y}) + 2g(\phi h\bar{X}, \bar{Y}).$$

From (3.9), we see that the condition $\mathcal{L}_h H = 0$ is equivalent to (3.10)

$$\bar{\nabla}_h = 2h\phi.$$

We suppose that $T_1 M$ satisfies $\mathcal{L}_h H = 0$. Then from (3.10), by a straightforward calculation, we have two equations:

$$R^2_u X - X)^h = 0,$$

$$R^2_u X - X)^t + (R^2_u X)^h = 0$$

for any vector field $X$ perpendicular to $u$ on $M$. From (3.11) and (3.12), we obtain $R^2_u X - X = 0$ and $R^2_u X = 0$. Using the similar arguments as in the proof of Theorem 1, we can conclude that the base manifold $(M, g)$ must be locally symmetric and of constant curvature 1 or $-1$. Conversely, when $(M, g)$ has constant curvature $c = -1$ or $c = 1$, we show that (3.10) holds. □

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