ON RIGHT $f$-DERIVATIONS OF INCLINE ALGEBRAS

SANG MOON LEE AND KYUNG HO KIM

Abstract. In this paper, we introduce the concept of a right $f$-derivation in incline algebras and give some properties of incline algebras. Also, the concept of $d$-ideal is introduced in an incline algebra with respect to right $f$-derivations.

1. Introduction

Z. Q. Cao, K. H. Kim and F. W. Roush [4] introduced the notion of incline algebras in their book and later it was developed by some authors [1, 2, 3, 5]. Ahn et al [1] introduced the notion of quotient incline and obtained the structure of incline algebras. N. O. Alshehri [3] introduced the notion of derivation in incline algebra. Incline algebra is a generalization of both Boolean and fuzzy algebra and it is a special type of semiring. It has both a semiring structure and a poset structure. It can also be used to represent automata and other mathematical systems, to study inequalities for non-negative matrices of polynomials. In this paper, we introduce the concept of a right $f$-derivation in incline algebras and give some properties of incline algebras. Also, the concept of $d$-ideal is introduced in an incline algebra with respect to right $f$-derivations.

2. Preliminaries

An incline algebra is a set $K$ with two binary operations denoted by “$+$” and “$*$” satisfying the following axioms:

(K1) $x + y = y + x$,
\[(K2) \ x + (y + z) = (x + y) + z, \]
\[(K3) \ x \ast (y \ast z) = (x \ast y) \ast z, \]
\[(K4) \ x \ast (y + z) = (x \ast y) + (x \ast z), \]
\[(K5) \ (y + z) \ast x = (y \ast x) + (z \ast x), \]
\[(K6) \ x + x = x, \]
\[(K7) \ x + (x \ast y) = x, \]
\[(K8) \ y + (x \ast y) = y, \]
for all \(x, y, z \in K\).

For convenience, we pronounce “+” (resp. “\ast”) as addition (resp. multiplication). Every distributive lattice is an incline algebra. An incline algebra is a distributive lattice if and only if \(x \ast x = x\) for all \(x \in K\). Note that \(x \leq y \iff x + y = y\) for all \(x, y \in K\). It is easy to see that “\(\leq\)” is a partial order on \(K\) and that for any \(x, y \in K\), the element \(x + y\) is the least upper bound of \(\{x, y\}\). We say that \(\leq\) is induced by operation +.

In an incline algebra \(K\), the following properties hold.
\[(K9) \ x \ast y \leq x \text{ and } y \ast x \leq x \text{ for all } x, y \in K, \]
\[(K10) \ y \leq z \text{ implies } x \ast y \leq x \ast z \text{ and } y \ast x \leq z \ast x \text{ for all } x, y, z \in K, \]
\[(K11) \text{ If } x \leq y \text{ and } a \leq b, \text{ then } x + a \leq y + b, \text{ and } x \ast a \leq y \ast b \text{ for all } x, y, a, b \in K.\]

Furthermore, an incline algebra \(K\) is said to be commutative if \(x \ast y = y \ast x\) for all \(x, y \in K\).

A subincline of an incline algebra \(K\) is a non-empty subset \(M\) of \(K\) which is closed under the addition and multiplication. A subincline \(M\) is called an ideal if \(x \in M\) and \(y \leq x\) then \(y \in M\). An element “0” in an incline algebra \(K\) is a zero element if \(x + 0 = x = 0 + x\) and \(x \ast 0 = 0 = 0 \ast x\) for any \(x \in K\). An non-zero element “1” is called a multiplicative identity if \(x \ast 1 = 1 \ast x = x\) for any \(x \in K\). A non-zero element \(a \in K\) is said to be a left (resp. right) zero divisor if there exists a non-zero \(b \in K\) such that \(a \ast b = 0\) (resp. \(b \ast a = 0\)) A zero divisor is an element of \(K\) which is both a left zero divisor and a right zero divisor. An incline algebra \(K\) with multiplicative identity 1 and zero element 0 is called an integral incline if it has no zero divisors. By a homomorphism of inclines, we mean a mapping \(f\) from an incline algebra \(K\) into an incline algebra \(L\) such that \(f(x + y) = f(x) + f(y)\) and \(f(x \ast y) = f(x) \ast f(y)\) for all \(x, y \in K\).
On right $f$-derivations of incline algebras 887

**Definition 2.1.** Let $K$ be an incline algebra. By a right derivation of $K$, we mean a self map $d$ of $K$ satisfying the identities

$$d(x + y) = d(x) + d(y) \text{ and } d(x \ast y) = (d(x) \ast y) + (d(y) \ast x)$$

for all $x, y \in K$.

### 3. Right $f$-derivations of incline algebras

In what follows, let $K$ denote an incline algebra with a zero-element unless otherwise specified.

**Definition 3.1.** Let $K$ be an incline algebra and let $f$ be an endomorphism on $K$. By a right $f$-derivation of $K$, we mean a self map $d$ of $K$ satisfying the identities

$$d(x + y) = d(x) + d(y) \text{ and } d(x \ast y) = (d(x) \ast f(y)) + (d(y) \ast f(x))$$

for all $x, y \in K$.

**Example 3.2.** Let $K = \{0, a, b, 1\}$ be a set in which “$+$” and “$\ast$” is defined by

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Then it is easy to check that $(K, +, \ast)$ is an incline algebra. Define a map $d : K \to K$ by

$$d(x) = \begin{cases} a & \text{if } x = a, b, 1 \\ 0 & \text{if } x = 0 \end{cases}$$

and define an endomorphism $f : K \to K$ by

$$f(x) = \begin{cases} b & \text{if } x = a, b \\ 1 & \text{if } x = 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Then we can see that $d$ is a right $f$-derivation of the incline algebra $K$.

**Example 3.3.** Let $K = \{a, b, c, d\}$ be a set in which “$+$” and “$\ast$” is defined by

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Then it is easy to check that $(K, +, \cdot)$ is an incline algebra. Define a map $d : K \to K$ by
\[
d(x) = \begin{cases} 
a & \text{if } x = a, b, c \\
d & \text{if } x = d
\end{cases}
\]
and define an endomorphism $f : K \to K$ by
\[
f(x) = \begin{cases} 
a & \text{if } x = a, c \\
d & \text{if } x = b, d
\end{cases}
\]
It is easily checked that $d$ is a right $f$-derivation of an incline algebra $K$.

**Proposition 3.4.** Let $K$ be a commutative incline algebra and let $f$ be an endomorphism on $K$. Then for a fixed $a \in K$, the mapping $d_a : K \to K$ given by $d_a(x) = f(x) \cdot a$, for every $x \in K$, is a right $f$-derivation of $K$.

**Proof.** Let $K$ be a commutative incline algebra. Then for a fixed $a \in K$, we have
\[
d_a(x \cdot y) = f(x \cdot y) \cdot a = (f(x \cdot y) \cdot a) + (f(x \cdot y) \cdot a)
\]
\[
= ((f(x) \cdot f(y)) \cdot a) + ((f(x) \cdot f(y)) \cdot a)
\]
\[
= ((f(x) \cdot a) \cdot f(y)) + ((f(y) \cdot a) \cdot f(x))
\]
\[
= d_a(x) \cdot f(y) + d_a(y) \cdot f(x)
\]
for all $x, y \in K$. This completes the proof. \qed

**Proposition 3.5.** Let $K$ be a commutative incline algebra and $f$ be an endomorphism on $K$. Then $d_{a+b} = d_a + d_b$ for all $a, b \in K$.

**Proof.** Let $K$ be a commutative incline algebra and $a, b \in K$. Then for all $c \in K$, we have
\[
d_{a+b}(c) = f(c) \cdot (a + b) = (f(c) \cdot a) + (f(c) \cdot b)
\]
\[
= d_a(c) + d_b(c) = (d_a + d_b)(c).
\]

**Proposition 3.6.** Let $d$ be a right $f$-derivation of an incline algebra $K$. If $f(0) = 0$, then we have $d(0) = 0$. 

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Proof. Let \( d \) be a right \( f \)-derivation of an incline algebra. Then we have
\[
\begin{align*}
d(0) &= d(0 \ast 0) = d(0) \ast f(0) + d(0) \ast f(0) \\
&= d(0) \ast 0 + d(0) \ast 0 = 0 + 0 \\
&= 0.
\end{align*}
\]

Proposition 3.7. Let \( d \) be a right \( f \)-derivation of an incline algebra \( K \). If \( K \) is a distributive lattice, then \( d(x) \leq f(x) \) for all \( x \in K \).

Proof. Let \( d \) be a right \( f \)-derivation of \( K \) and let \( K \) be a distributive lattice. Then
\[
d(x) = d(x \ast x) = d(x) \ast f(x) + d(x) \ast f(x) \\
&= d(x) \ast f(x) \leq f(x)
\]
from (K9) for all \( x \in K \).

Proposition 3.8. Let \( K \) be an incline algebra and let \( d \) be a right \( f \)-derivation of \( K \). Then we have \( d(x \ast y) \leq d(x) \) for all \( x, y \in K \).

Proof. Let \( x, y \in K \). By using (K9), we get \( d(x) \ast f(y) \leq d(x) \) and \( d(y) \ast f(x) \leq d(y) \). Thus we get
\[
d(x \ast y) = (d(x) \ast f(y)) + (d(y) \ast f(x)) \leq d(x) + d(y) = d(x + y).
\]

Proposition 3.9. Let \( K \) be an incline algebra and a distributive lattice. Define \( d^2(x) = d(d(x)) \) for all \( x \in K \). If \( d^2 = d \) and \( f \circ d = f \), then \( d(x \ast d(x)) = d(x) \) for all \( x \in K \).

Proof. Let \( K \) be an incline algebra and \( x \in K \). Then
\[
d(x \ast d(x)) = (d(x) \ast f(d(x))) + (d^2(x) \ast f(x)) \\
&= (d(x) \ast d(x)) + (d^2(x) \ast f(x)) \\
&= d(x) + (d(x) \ast f(x)) \\
&= d(x).
\]

Proposition 3.10. Let \( K \) be an incline algebra and let \( d \) be a right \( f \)-derivation of \( K \). Then for all \( x, y \in K \), \( d(x \ast y) \leq d(x) \) and \( d(x \ast y) \leq d(y) \).

Proof. Let \( x, y \in K \). Then by using (K7), we obtain
\[
d(x) = d(x + x \ast y) = d(x) + d(x \ast y).
\]
Hence we get \( d(x \ast y) \leq d(x) \). Also, \( d(y) = d(y + x \ast y)) = d(y) + d(x \ast y) \), and so \( d(x \ast y) \leq d(y) \).

Definition 3.11. Let \( K \) be an incline algebra. A mapping \( f \) is \textit{isotone} if \( x \leq y \) implies \( f(x) \leq f(y) \) for all \( x, y \in K \).
Proposition 3.12. Let $d$ be a right $f$-derivation of an incline algebra $K$. Then $d$ is isotone.

Proof. Let $x, y \in K$ be such that $x \leq y$. Then $x + y = y$. Hence we have $d(y) = d(x + y) = d(x) + d(y)$, which implies $d(x) \leq d(y)$. This completes the proof. \qed

Proposition 3.13. A sum of two $f$-right derivations of an incline algebra $K$ is again a right $f$-derivation of $K$.

Proof. Let $d_1$ and $d_2$ be two right $f$-derivations of $K$ respectively. Then we have for all $a, b \in K$,
\[
(d_1 + d_2)(a * b) = d_1(a * b) + d_2(a * b)
\]
\[
= d_1(a) * f(b) + d_1(b) * f(a) + d_2(a) * f(b) + d_2(b) * f(a)
\]
\[
= d_1(a) * f(b) + d_2(a) * f(b) + d_1(b) * f(a) + d_2(b) * f(a)
\]
\[
= d_1(a) * f(b) + (d_1 + d_2)(b) * f(a).
\]
Clearly, $(d_1 + d_2)(a + b) = (d_1 + d_2)(a) + (d_1 + d_2)(b)$ for all $a, b \in K$. This completes the proof. \qed

Theorem 3.14. Let $K$ be a commutative incline algebra and let $d_1, d_2$ be right $f$-derivations of $K$, respectively. Define $(d_1 \circ d_2)(x) = d_1(d_2(x))$ for all $x \in K$. If $d_1d_2 = 0$, $d_1 \circ f = f \circ d_1$ and $d_2 \circ f = f \circ d_2$, then $d_2d_1$ is a right $f$-derivation of $K$.

Proof. Let $K$ be a commutative incline algebra and $x, y \in K$. Then we have
\[
0 = d_1d_2(x * y) = d_1(d_2(x) * f(y) + d_2(y) * f(x))
\]
\[
= d_1d_2(x) * f^2(y) + d_1(f(y)) * f(d_2(x)) + d_1d_2(y) * f^2(x)
\]
\[
+ d_1(f(x)) * f(d_2(y))
\]
\[
= d_1(f(y)) * f(d_2(x)) + d_1(f(x)) * f(d_2(y))
\]
\[
= f(d_1(y)) * d_2(f(y)) + f(d_1(x)) * d_2(f(y)).
\]
Then
\[
d_2d_1(x * y) = d_2(d_1(x) * f(y) + d_1(y) * f(x))
\]
\[
= d_2d_1(x) * f^2(y) + d_2(f(y)) * f(d_1(x))
\]
\[
+ d_2d_1(y) * f^2(x) + d_2(f(x)) * f(d_1(y))
\]
\[
= d_2d_1(x) * f(y) + d_2d_1(y) * f(x).
\]
Finally, for all $x, y \in K$, we get
\[
d_2d_1(x + y) = d_2(d_1(x) + d_1(y)) = d_2d_1(x) + d_2d_1(y).
\]
This implies that $d_2 d_1$ is a right derivation of a commutative incline algebra $K$. □

**Definition 3.15.** Let $K$ be an incline algebra and let $d$ be a non-trivial right $f$-derivation of $K$. An ideal $I$ of $K$ is called a $d$-ideal if $d(I) = I$.

Since $d(0) = 0$, it can be easily observed that the zero ideal $\{0\}$ is a $d$-ideal of $K$. If $d$ is onto, then $d(K) = K$, which implies $K$ is a $d$-ideal of $K$.

**Example 3.16.** In Example 3.2, let $I = \{0, a\}$. Then $I$ is an ideal of $K$. It can be verified that $d(I) = I$. Therefore, $I$ is an $d$-ideal of $K$.

**Lemma 3.17.** Let $d$ be a right $f$-derivation of $K$ and let $I, J$ be any two $d$-ideals of $K$. Then we have $I \subseteq J$ implies $d(I) \subseteq d(J)$.

**Proof.** Let $I \subseteq J$ and $x \in d(I)$. Then we have $x = d(y)$ for some $y \in I \subseteq J$. Hence we get $x = d(y) \in d(J)$. Therefore, $d(I) \subseteq d(J)$. □

**Proposition 3.18.** Let $K$ be an incline algebra. Then, a sum of any two $d$-ideals is also a $d$-ideal of $K$.

**Proof.** Let $I$ and $J$ be $d$-ideals of $K$. Then $I + J = d(I) + d(J) = d(I + J)$. Hence $I + J$ is a $d$-ideal of $K$. □

Let $d$ be a right $f$-derivation of $K$. Define a set $Kerd$ by

$$Kerd := \{x \in K \mid d(x) = 0\}$$

for all $x \in K$.

**Proposition 3.19.** Let $d$ be a right $f$-derivation of an incline algebra $K$. Then $Kerd$ is a subincline of $K$.

**Proof.** Let $x, y \in Kerd$. Then $d(x) = 0, d(y) = 0$ and

\[
d(x \ast y) = (d(x) \ast f(y)) + (d(y) \ast f(x)) = (0 \ast f(y)) + (0 \ast f(x)) = 0 + 0 = 0,
\]

and

\[
d(x + y) = d(x) + d(y) = 0 + 0 = 0.
\]

Therefore, $x \ast y, x + y \in Kerd$. This completes the proof. □
Proposition 3.20. Let $d$ be a right $f$-derivation of an integral incline algebra $K$. If $f$ is an one to one function, then $\text{Kerd}$ is an ideal of $K$.

Proof. By Proposition 3.19, $\text{Kerd}$ is a subincline of $K$. Now let $x \in K$ and $y \in \text{Kerd}$ such that $x \leq y$. Then $d(y) = 0$ and

$$0 = d(y) = d(y + x \ast y) = d(y) + d(x \ast y) = 0 + d(x \ast y),$$

which implies $d(x \ast y) = 0$. Hence we have

$$0 = d(x \ast y) = (d(x) \ast f(y)) + (d(y) \ast f(x)) = d(x) \ast f(y).$$

Since $K$ has no zero divisors, either $d(x) = 0$ or $f(y) = 0$. If $d(x) = 0$, then $x \in \text{Kerd}$. If $f(y) = 0$, then $y = 0$ and so $x \leq y = 0$, i.e., $x = 0$, which implies $x \in \text{Kerd}$. \hfill $\Box$

Let $d$ be a right $f$-derivation of $K$. Define a set $\text{Fix}_d(K)$ by

$$\text{Fix}_d(K) := \{x \in K \mid d(x) = f(x)\}$$

for all $x \in K$.

Proposition 3.21. Let $K$ be a commutative incline algebra and let $d$ be a right $f$-derivation. Then $\text{Fix}_d(K)$ is a subincline of $K$.

Proof. Let $x, y \in \text{Fix}_d(K)$. Then we have $d(x) = f(x)$ and $d(y) = f(y)$, and so

$$d(x \ast y) = d(x) \ast f(y) + d(y) \ast f(x) = f(x) \ast f(y) + f(y) \ast f(x)$$

$$= f(x) \ast f(y) + f(x) \ast f(y) = f(x) \ast f(y) = f(x + y).$$

Now

$$d(x + y) = d(x) + d(y) = f(x) + f(y) = f(x + y),$$

which implies $x + y, x \ast y \in \text{Fix}_d(K)$. This completes the proof. \hfill $\Box$

Definition 3.22. Let $K$ be an incline algebra. An element $a \in K$ is said to be additively left cancellative if for all $a, b \in K$, $a + b = a + c \Rightarrow b = c$. An element $a \in K$ is said to be additively right cancellative if for all $a, b \in K$, $b + a = c + a \Rightarrow b = c$. It is said to be additively cancellative if it is both left and right cancellative. If every element of $K$ is additively left cancellative, it is said to be additively left cancellative. If every element of $K$ is additively right cancellative, it is said to be additively right cancellative.

Definition 3.23. A subincline $I$ of an incline algebra $K$ is called a $k$-ideal if $x + y \in I$ and $y \in I$, then $x \in I$.

Example 3.24. In Example 3.2, $I = \{0, a, b\}$ is an $k$-ideal of $K$. 
Theorem 3.25. Let $K$ be a commutative incline algebra and additively right cancellative. If $d$ is a right $f$-derivation of $K$, then $Fix_d(K)$ is a $k$-ideal of $K$.

Proof. By Proposition 3.21, $Fix_d(K)$ is a subincline of $K$. Let $x + y, y \in Fix_d(K)$. Then $d(y) = f(y)$ and $f(x + y) = d(x + y)$. Hence $f(x) + f(y) = d(x + y) = d(x) + d(y) = d(x) + f(y)$, which implies $x \in Fix_d(K)$. Hence $Fix_d(K)$ is a $k$-ideal of $K$.

Proposition 3.26. Let $K$ be an incline algebra and let $d$ be a right $f$-derivation of $K$. Then $Kerd$ is a $k$-ideal of $K$.

Proof. From Proposition 3.19, $Kerd$ is a subincline of $K$. Let $x + y \in K$ and $y \in Kerd$. Then we have $d(x + y) = 0$ and $d(y) = 0$, and so
\[0 = d(x + y) = d(x) + d(y) = d(x) + 0 = d(x).
\]
This implies $x \in Kerd$. 

References


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