PROPERTIES OF HYPERHOLOMORPHIC FUNCTIONS
ON DUAL SEDENION NUMBERS

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Abstract. The aim of this paper is to define hyperholomorphic functions with dual sedenion variables on \( S \times S \), where \( S \cong \mathbb{C}^8 \). By the condition of harmonicity, we research properties of hyperholomorphic functions of dual sedenion variables in Clifford analysis.

1. Introduction

The dual numbers extended the real numbers by adjoining one new element \( \varepsilon \) with the property \( \varepsilon^2 = 0 \). Every dual number has the form \( z = x + \varepsilon y \) with \( x \) and \( y \) uniquely determined real numbers. The collection of dual numbers forms a particular two-dimensional commutative unital associative algebra over the real numbers. Hörmander [1] and Krantz [6] introduced the concepts of several complex variables and theories to complex analysis in several variables. K. Nôno [8, 9] researched the properties of hyperholomorphic functions of quaternion variable and octonion variable in the 1980s. We [2, 7, 10] found the theorem about hyperholomorphic functions of dual quaternion in \( \mathbb{C}^2 \times \mathbb{C}^2 \) and investigated the extensions of quaternions. Also, we [3, 4, 5] researched properties of the regularity functions on a variety of quaternionic variables and the corresponding Cauchy-Riemann system on each several types.

2. Preliminaries

The multiplication of these unit sedenions follows:
\[
\begin{array}{cccccccccccccccc}
\times & e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} & e_{12} & e_{13} & e_{14} & e_{15} \\
e_0 & e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} & e_{12} & e_{13} & e_{14} & e_{15} \\
e_1 & e_1 & e_{-1} & e_{-2} & e_5 & -e_4 & e_7 & -e_6 & e_9 & -e_8 & e_{11} & -e_{10} & e_{13} & -e_{12} & e_{15} & -e_{14} \\
e_2 & e_2 & e_{-3} & e_{-1} & e_6 & -e_7 & -e_4 & e_5 & e_{10} & -e_{11} & -e_8 & e_9 & e_{14} & -e_{15} & -e_{12} & e_{13} \\
e_3 & e_3 & e_2 & e_{-1} & e_7 & e_6 & -e_5 & e_{-4} & e_1 & e_11 & e_{10} & -e_9 & -e_{18} & e_{15} & e_{14} & -e_{13} \\
e_4 & e_4 & -e_5 & -e_6 & -e_7 & -1 & e_1 & e_2 & e_3 & e_{12} & e_{13} & e_{14} & e_{15} & -e_8 & -e_9 & -e_{10} \\
e_5 & e_5 & e_4 & e_7 & -e_6 & -e_1 & -1 & e_3 & e_{-2} & e_13 & -e_{12} & e_{15} & -e_{14} & e_9 & -e_8 & e_{11} \\
e_6 & e_6 & -e_7 & e_4 & e_5 & -e_2 & -e_3 & -1 & e_1 & e_{14} & -e_{15} & -e_{12} & e_{13} & e_{10} & -e_{11} & -e_8 \\
e_7 & e_7 & e_6 & -e_5 & e_4 & -e_3 & e_2 & e_{-1} & -1 & e_15 & e_{14} & -e_{13} & -e_{12} & e_{11} & e_{10} & -e_9 \\
e_8 & e_8 & -e_9 & -e_{10} & -e_{11} & -e_{12} & -e_{13} & -e_{14} & -e_{15} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
e_9 & e_9 & e_8 & e_11 & -e_{10} & -e_{13} & e_{12} & e_{15} & e_{14} & e_{-1} & -1 & e_3 & e_{-2} & e_{-5} & e_4 & e_7 \\
e_{10} & e_{10} & -e_{11} & e_8 & e_9 & -e_{14} & -e_{15} & e_{12} & e_{13} & -e_{2} & -e_{3} & -1 & e_1 & e_{-6} & e_{-7} & e_4 & e_5 \\
e_{11} & e_{11} & e_{10} & -e_{9} & e_8 & -e_{15} & e_{14} & -e_{13} & e_{12} & -e_{3} & e_2 & -e_{1} & -1 & e_7 & e_6 & -e_5 & e_4 \\
e_{12} & e_{12} & -e_{13} & -e_{14} & -e_{15} & e_8 & -e_9 & -e_{10} & -e_{11} & -e_{4} & e_5 & e_6 & e_7 & -1 & e_1 & e_2 & e_3 \\
e_{13} & e_{13} & e_{12} & e_{15} & -e_{14} & e_9 & e_8 & e_{11} & -e_{10} & -e_{5} & -e_4 & e_7 & -e_6 & e_{-1} & -1 & e_3 & e_2 \\
e_{14} & e_{14} & -e_{15} & e_{12} & e_{13} & e_{10} & -e_{11} & e_8 & e_9 & -e_{6} & -e_7 & -e_4 & e_5 & -e_2 & -e_{3} & -1 & e_1 \\
e_{15} & e_{15} & e_{14} & -e_{13} & e_{12} & e_{11} & e_{10} & -e_9 & e_8 & -e_7 & e_6 & -e_5 & -e_4 & e_{-3} & e_{-2} & -e_1 & -1 \\
\end{array}
\]
The algebra of dual numbers is a ring that is a local ring since the principal ideal generated by $\varepsilon$ is its only maximal ideal. Dual numbers form the coefficients of dual quaternions. Using matrices, dual numbers can be represented as

$$
\varepsilon = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a + b\varepsilon = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.
$$

The sum and product of dual numbers are then calculated with ordinary matrix addition and matrix multiplication; both operations are commutative and associative within the algebra of dual numbers.

For the field $S$ of sedenions, we give hyperholomorphic functions and a harmonic function with dual sedenion variables on $S \times S$ and research properties of functions of dual sedenion variables.

The element $e_0$ is the identity of $S$ and $e_1$ identifies the imaginary unit $\sqrt{-1}$ in the $\mathbb{C}$-field of complex numbers. A sedenion $z$ is given by

$$
(2.1) \quad z = x_0 + \sum_{i=0}^{15} e_i x_i, \quad (x_i \in \mathbb{R}, \ i = 0, 1, \ldots, 15)
$$

is an sixteen dimensional non-commutative and non-associative $\mathbb{R}$-field generated by sixteen base elements $e_l (l = 0, 1, \ldots, 15)$ with the following non-commutative multiplication rules:

$$
e_l^2 = -1, \quad e_i e_j = -e_j e_i, \quad e_i e_j e_k = e_i (e_j e_k) \ (i \neq j \neq k, i \neq 0, j \neq 0, k \neq 0).
$$

An sedenion $z$ given by (2.1) is regarded as $z = z_1 + z_2 e_2 + z_3 e_4 + z_4 e_6 + z_5 e_8 + z_6 e_{10} + z_7 e_{12} + z_8 e_{14} \in S$, where $z_1 := x_0 + e_1 x_1, z_2 := x_2 + e_1 x_3, z_3 := x_4 + e_1 x_5, z_4 := x_6 + e_1 x_7, z_5 := x_8 + e_1 x_9, z_6 := x_{10} + e_1 x_{11}, z_7 := x_{12} + e_1 x_{13}$ and $z_8 := x_{14} + e_1 x_{15}$ are complex numbers in $S$. Thus, we identify $S$ with $\mathbb{C}^8$.

We write the sedenion $z = \sum_{l=0}^{15} e_l x_l$ and the sedenion conjugate $z^* = x_0 - \sum_{l=1}^{15} e_l x_l$. Also, the absolute value $|z|$ of $z$ and an inverse $z^{-1}$ of $z$ in $S$ are defined by

$$
|z| = \sqrt{\sum_{l=1}^{8} |z_l|^2}, \quad z^{-1} = \frac{z^*}{|z|^2} \ (z \neq 0).
$$

Thus, the sedenion $z \in S$ have the following forms:

$$
z = \sum_{l=0}^{15} e_l x_l = z_1 + z_2 e_2 + z_3 e_4 + z_4 e_6 + z_5 e_8 + z_6 e_{10} + z_7 e_{12} + z_8 e_{14} = Z_1 + Z_2 e_4 + Z_3 e_8 + Z_4 e_{12} = P_1 + P_2 e_8 \in S
$$
and

\[ z^* = x_0 - \sum_{l=1}^{15} e_l x_l \]
\[ = x_1 - z_2 e_2 - z_3 e_4 - z_4 e_6 - z_5 e_8 - z_6 e_{10} - z_7 e_{12} - z_8 e_{14} \]
\[ = Z_1 - Z_2 e_4 - Z_3 e_8 - Z_4 e_{12} = P_1^* - P_2 e_8 \in S, \]

where \( Z_1 = z_1 + z_2 e_2, Z_2 = z_3 + z_4 e_2, Z_3 = z_5 + z_6 e_2 \) and \( Z_4 = z_7 + z_8 e_2 \) are quaternion numbers in \( \mathbb{C}^2 \), \( P_1 = Z_1 + Z_2 e_4 \) and \( P_2 = Z_3 + Z_4 e_4 \) are octonion numbers in \( \mathbb{C}^4 \).

We use the following differential operators:

\[
\begin{align*}
\frac{\partial}{\partial Z_1} &:= \frac{\partial}{\partial z_1} - e_4 \frac{\partial}{\partial z_2}, & \frac{\partial}{\partial Z_1^*} &= \frac{\partial}{\partial z_1} + e_2 \frac{\partial}{\partial z_2}, \\
\frac{\partial}{\partial Z_2} &:= \frac{\partial}{\partial z_3} - e_4 \frac{\partial}{\partial z_4}, & \frac{\partial}{\partial Z_2^*} &= \frac{\partial}{\partial z_3} + e_2 \frac{\partial}{\partial z_4}, \\
\frac{\partial}{\partial Z_3} &:= \frac{\partial}{\partial z_5} - e_4 \frac{\partial}{\partial z_6}, & \frac{\partial}{\partial Z_3^*} &= \frac{\partial}{\partial z_5} + e_2 \frac{\partial}{\partial z_6}, \\
\frac{\partial}{\partial Z_4} &:= \frac{\partial}{\partial z_7} - e_4 \frac{\partial}{\partial z_8}, & \frac{\partial}{\partial Z_4^*} &= \frac{\partial}{\partial z_7} + e_2 \frac{\partial}{\partial z_8},
\end{align*}
\]

where \( \partial/\partial z_l, \partial/\partial \bar{z}_l \) (\( l = 1, 2, \ldots, 8 \)) are usual differential operators used in complex analysis. And, we use the following differential operators:

\[
\begin{align*}
\frac{\partial}{\partial P_1} &:= \frac{\partial}{\partial Z_1} - e_4 \frac{\partial}{\partial Z_2}, & \frac{\partial}{\partial P_1^*} &= \frac{\partial}{\partial Z_1^*} + e_4 \frac{\partial}{\partial Z_2^*}, \\
\frac{\partial}{\partial P_2} &:= \frac{\partial}{\partial Z_3} - e_4 \frac{\partial}{\partial Z_4}, & \frac{\partial}{\partial P_2^*} &= \frac{\partial}{\partial Z_3^*} + e_4 \frac{\partial}{\partial Z_4^*}.
\end{align*}
\]

Also, we use the following sedenion differential operators:

\[
D := \frac{\partial}{\partial P_1} - e_8 \frac{\partial}{\partial P_2}, \quad D^* = \frac{\partial}{\partial P_1^*} + e_8 \frac{\partial}{\partial P_2^*}.
\]

The operator

\[ DD^* = \sum_{l=1}^{8} \frac{\partial^2}{\partial z_l \partial \bar{z}_l} = \frac{1}{4} \sum_{l=0}^{15} \frac{\partial^2}{\partial x_l^2} \]

is the usual complex Laplacian \( \Delta \).
3. Regular function with sedenion variables

Let \( \Omega = \Omega_1 \times \Omega_2 \) be an open set in \( \mathcal{S} \times \mathcal{S} \). The function \( f(z) \) is defined by the following form in \( \Omega \) with valued in \( \mathcal{S} \):

\[
f : \Omega \to \mathcal{S}, \quad (z, w) \in \Omega \mapsto f(z, w) = \phi_1(z, w) + \phi_2(z, w)e_8.
\]

In detail,

\[
f(z, w) = \sum_{l=0}^{15} u_le_l = \sum_{m=1}^{8} f_me_{2m} = \sum_{n=0}^{3} F_ne_{4n} = \phi_1 + \phi_2e_8 \in \mathcal{S},
\]

where \( u_l \) \((l = 0, 1, \ldots, 15)\) are real-valued functions, \( f_m = u_{2m-2} + u_{2m-1}e_1 \) \((m = 1, 2, \ldots, 8)\) are complex-valued functions, \( F_n = f_{2n-1} + f_{2n}e_2 \) \((n = 1, 2, 3, 4)\) are quaternion-valued functions, \( \phi_1 = F_1 + F_2e_4 \) and \( \phi_2 = F_3 + F_4e_4 \) are octonion-valued functions in \( \mathcal{S} \).

We use the following differential operators:

\[
\frac{\partial}{\partial Z_1} := \frac{\partial}{\partial z_1} - e_2\frac{\partial}{\partial z_2}, \quad \frac{\partial}{\partial Z_1^*} = \frac{\partial}{\partial z_1} + e_2\frac{\partial}{\partial z_2},
\]
\[
\frac{\partial}{\partial Z_2} := \frac{\partial}{\partial z_3} - e_2\frac{\partial}{\partial z_4}, \quad \frac{\partial}{\partial Z_2^*} = \frac{\partial}{\partial z_3} + e_2\frac{\partial}{\partial z_4},
\]
\[
\frac{\partial}{\partial Z_3} := \frac{\partial}{\partial z_5} - e_2\frac{\partial}{\partial z_6}, \quad \frac{\partial}{\partial Z_3^*} = \frac{\partial}{\partial z_5} + e_2\frac{\partial}{\partial z_6},
\]
\[
\frac{\partial}{\partial Z_4} := \frac{\partial}{\partial z_7} - e_2\frac{\partial}{\partial z_8}, \quad \frac{\partial}{\partial Z_4^*} = \frac{\partial}{\partial z_7} + e_2\frac{\partial}{\partial z_8},
\]
\[
\frac{\partial}{\partial W_1} := \frac{\partial}{\partial w_1} - e_2\frac{\partial}{\partial w_2}, \quad \frac{\partial}{\partial W_1^*} = \frac{\partial}{\partial w_1} + e_2\frac{\partial}{\partial w_2},
\]
\[
\frac{\partial}{\partial W_2} := \frac{\partial}{\partial w_3} - e_2\frac{\partial}{\partial w_4}, \quad \frac{\partial}{\partial W_2^*} = \frac{\partial}{\partial w_3} + e_2\frac{\partial}{\partial w_4},
\]
\[
\frac{\partial}{\partial W_3} := \frac{\partial}{\partial w_5} - e_2\frac{\partial}{\partial w_6}, \quad \frac{\partial}{\partial W_3^*} = \frac{\partial}{\partial w_5} + e_2\frac{\partial}{\partial w_6},
\]
\[
\frac{\partial}{\partial W_4} := \frac{\partial}{\partial w_7} - e_2\frac{\partial}{\partial w_8}, \quad \frac{\partial}{\partial W_4^*} = \frac{\partial}{\partial w_7} + e_2\frac{\partial}{\partial w_8},
\]

where \( \partial/\partial z_l, \partial/\partial z_l^* \) \((l = 1, 2, \ldots, 8)\) are usual differential operators used in complex analysis. And, we use the following differential operators:
\[ \frac{\partial}{\partial P_1} := \frac{\partial}{\partial Z_1} - e_4 \frac{\partial}{\partial Z_2}, \quad \frac{\partial}{\partial P_1^*} = \frac{\partial}{\partial Z_1^*} + e_4 \frac{\partial}{\partial Z_2^*}, \]
\[ \frac{\partial}{\partial P_2} := \frac{\partial}{\partial Z_3} - e_4 \frac{\partial}{\partial Z_4}, \quad \frac{\partial}{\partial P_2^*} = \frac{\partial}{\partial Z_3^*} + e_4 \frac{\partial}{\partial Z_4^*}, \]
\[ \frac{\partial}{\partial Q_1} := \frac{\partial}{\partial W_1} - e_4 \frac{\partial}{\partial W_2}, \quad \frac{\partial}{\partial Q_1^*} = \frac{\partial}{\partial W_1^*} + e_4 \frac{\partial}{\partial W_2^*}, \]
\[ \frac{\partial}{\partial Q_2} := \frac{\partial}{\partial W_3} - e_4 \frac{\partial}{\partial W_4}, \quad \frac{\partial}{\partial Q_2^*} = \frac{\partial}{\partial W_3^*} + e_4 \frac{\partial}{\partial W_4^*}. \]

Also, we use the following sedenion differential operators:
\[ D := \frac{\partial}{\partial P_1} - e_8 \frac{\partial}{\partial P_2}, \quad D^* = \frac{\partial}{\partial P_1^*} + e_8 \frac{\partial}{\partial P_2^*}, \]
\[ E := \frac{\partial}{\partial Q_1} - e_8 \frac{\partial}{\partial Q_2}, \quad E^* = \frac{\partial}{\partial Q_1^*} + e_8 \frac{\partial}{\partial Q_2^*}. \]
\[ H := D + \varepsilon E, \quad H^* := D^* + \varepsilon E^*. \]

**Definition 3.1.** Let \( \Omega \) be an open set in \( S \times S \). A function \( f(z, w) = \phi_1(z, w) + \phi_2(z, w)e_8 \) is said to be bihyperholomorphic in \( \Omega \) if the following two conditions are satisfied:

1. \( \phi_r(z, w) \) \((r = 1, 2)\) are continuously differential functions in \( \Omega \).
2. The function \( f \) satisfies

\[ D^* f = 0, \quad f E^* = 0 \]

in \( \Omega \).

**Theorem 3.1.** Let \( G \) and \( T \) be functions with values in sedenions of class \( C^\infty \) on \( \Omega \) in \( S \times S \). Then the Cousin 1 problem according to sedenion-valued functions on \( \Omega \) has a solution.

**Proof.** Consider the inhomogeneous Cauchy-Riemann system of partial differential equations
\[ D^* f = G, \quad f E^* = T. \]

If the system (3.1) has a solution \( f \) with values in sedenions, then
\[ GE^* = D^* f E^* = D^* T. \]

For the system (3.2), the solvability condition
\[ GE^* = D^* T \]
Properties of hyperholomorphic functions on dual sedenion numbers

is necessary for the existence of a solution $f$ to the system (3.1). Let $\Omega$ be a product domain $\Omega_1 \times \Omega_2$ of simply connected domains in $S$ and $S$, respectively, and let

$$Rf := (D^* f, f E^*).$$

Then $Rf = (G, T) = 0$ satisfies (3.3). For $Rf : \Omega \to S$ of class $C^2$, if

$$A(G, T) = GE^* - D^* T,$$

then the Cousin 1 problem according to sedenion-valued functions on $\Omega$ has a solution.

Let $\Omega = \Omega_1 \times \Omega_2$ be an open set in $S \times S$. The function $F(\zeta)$ is defined by the following form in $\Omega$ with valued in $S \times S$:

$$F : \Omega \to S \times S,$$

$$\zeta = z + \varepsilon w \in \Omega \mapsto F(\zeta) = F(z + \varepsilon w) = \psi_1(z, w) + \varepsilon \psi_2(z, w) = (\phi_1(z, w) + \phi_2(z, w)e_8) + \varepsilon(\varphi_1(z, w) + \varphi_2(z, w)e_8),$$

where

$$\psi_2(z, w) = \sum_{l=0}^{15} v_l e_l = \sum_{m=1}^{8} g_m e_{2m-2} = \sum_{n=1}^{4} G_n e_{4n-4} = \varphi_1 + \varphi_2 e_8 \in S$$

with $v_l$ ($l = 0, 1, \cdots, 15$) are real-valued functions, $g_m = v_{2m-2} + v_{2m-1} e_1$ ($m = 1, 2, \cdots, 8$) are complex-valued functions, $G_n = g_{2n-1} + g_{2n} e_2$ ($n = 1, 2, 3, 4$) are quaternion-valued functions, $\varphi_1 = G_1 + G_2 e_4$ and $\varphi_2 = G_3 + G_4 e_4$ are octonion-valued functions in $S$.

**Definition 3.2.** Let $\Omega$ be an open set in $S \times S$. A function $F(\zeta)$ is said to be $L(R)$-hyperholomorphic in $\Omega$ if the following two conditions are satisfied:

1. $f_k(z), g_k(w)$ ($k = 1, \cdots, 8$) are continuously differential functions in $\Omega$.
2. $H^* F = 0$ ($FH^* = 0$) in $\Omega$.

When we deal with a $L$-hyperholomorphic function $F(\zeta)$ in $\Omega \subset S \times S$, for simplicity, we often say that $F(\zeta)$ is a hyperholomorphic function in $\Omega \subset S \times S$. 
The Equation (3.4) is applied to $F(\zeta)$ as follows:

$$H^*F = (D^* + \varepsilon E^*)(\psi_1(z, w) + \varepsilon \psi_2(z, w))$$
$$= (D^* \psi_1(z, w) + \varepsilon (E^* \psi_1(z, w) + D^* \psi_2(z, w))$$
$$= (D^* (F_1(z) + F_2(z)e_4 + F_3(z)e_8 + F_4(z)e_{12})$$
$$+ \varepsilon (E^* (F_1(z) + F_2(z)e_4 + F_3(z)e_8 + F_4(z)e_{12})$$
$$+ D^* (G_1(w) + G_2(w)e_4 + G_3(w)e_8 + G_4(w)e_{12})))$$
$$= \left( \frac{\partial f_1}{\partial P_1} - \frac{\partial f_5}{\partial Q_1} \right) + \left( \frac{\partial f_2}{\partial P_1} + \frac{\partial f_6}{\partial Q_1} \right)e_2 + \left( \frac{\partial f_3}{\partial P_1} + \frac{\partial f_7}{\partial Q_1} \right)e_4 + \left( \frac{\partial f_9}{\partial P_1} + \frac{\partial f_{10}}{\partial Q_1} \right)e_{10} + \left( \frac{\partial f_1}{\partial P_1} + \frac{\partial f_5}{\partial Q_1} \right)e_{12} + \left( \frac{\partial f_2}{\partial P_1} + \frac{\partial f_6}{\partial Q_1} \right)e_{14}.$$

If the following equations

$$\frac{\partial f_1}{\partial P_1} = \frac{\partial f_5}{\partial P_1}, \quad \frac{\partial f_2}{\partial P_1} = \frac{\partial f_6}{\partial P_1}, \quad \frac{\partial f_3}{\partial P_1} = \frac{\partial f_7}{\partial P_1}, \quad \frac{\partial f_9}{\partial P_1} = \frac{\partial f_{10}}{\partial P_1}, \quad \frac{\partial f_4}{\partial P_1} = \frac{\partial f_8}{\partial P_1},$$
$$\frac{\partial f_5}{\partial P_1} = \frac{\partial f_1}{\partial P_1}, \quad \frac{\partial f_6}{\partial P_1} = \frac{\partial f_2}{\partial P_1}, \quad \frac{\partial f_7}{\partial P_1} = \frac{\partial f_3}{\partial P_1}, \quad \frac{\partial f_9}{\partial P_1} = \frac{\partial f_{10}}{\partial P_1}, \quad \frac{\partial f_8}{\partial P_1} = \frac{\partial f_4}{\partial P_1},$$
$$\frac{\partial g_1}{\partial P_1} = \frac{\partial g_5}{\partial P_1}, \quad \frac{\partial g_2}{\partial P_1} = \frac{\partial g_6}{\partial P_1}, \quad \frac{\partial g_3}{\partial P_1} = \frac{\partial g_4}{\partial P_1}, \quad \frac{\partial g_5}{\partial P_1} = \frac{\partial g_8}{\partial P_1}, \quad \frac{\partial g_9}{\partial P_1} = \frac{\partial g_{10}}{\partial P_1},$$
$$\frac{\partial g_5}{\partial Q_1} = \frac{\partial g_9}{\partial Q_1}, \quad \frac{\partial g_6}{\partial Q_1} = \frac{\partial g_{10}}{\partial Q_1}, \quad \frac{\partial g_7}{\partial Q_1} = \frac{\partial g_8}{\partial Q_1}, \quad \frac{\partial g_7}{\partial Q_1} = \frac{\partial g_8}{\partial Q_1}, \quad \frac{\partial g_7}{\partial Q_1} = \frac{\partial g_8}{\partial Q_1},$$

are satisfied, the function $F(\zeta)$ is a hyperholomorphic function on $\Omega$. These are the corresponding s-Cauchy-Riemann equations on $S \times S$. 
Definition 3.3. Let $\Omega$ be an open set in $\mathbb{S} \times \mathbb{S}$ and let

\[ F = (f_1, f_2, \cdots, f_8, g_1, g_2, \cdots, g_8) : \Omega \to \mathbb{S} \times \mathbb{S}. \]

This mapping is said to be harmonic if all its components $f_j$ and $g_j$ ($j = 1, 2, \cdots, 8$) of $F$ are harmonic on $\Omega$. The system (3.5) is called a generalized Cauchy-Riemann system if every solution $F(\zeta)$ has only harmonic components $f_j$ and $g_j$ ($j = 1, 2, \cdots, 8$).

Remark 3.1. We redefine the Equations (3.5) as follows:

\[
\frac{\partial f_1}{\partial \zeta_1} = \frac{\partial f_5}{\partial \zeta_5}, \quad \frac{\partial f_1}{\partial \zeta_2} = \frac{\partial f_5}{\partial \zeta_6}, \quad \frac{\partial f_1}{\partial \zeta_3} = \frac{\partial f_5}{\partial \zeta_7}, \quad \frac{\partial f_1}{\partial \zeta_4} = \frac{\partial f_5}{\partial \zeta_8}, \quad \frac{\partial f_2}{\partial \zeta_1} = \frac{\partial f_6}{\partial \zeta_5}, \quad \frac{\partial f_2}{\partial \zeta_2} = \frac{\partial f_6}{\partial \zeta_6}, \quad \frac{\partial f_2}{\partial \zeta_3} = \frac{\partial f_6}{\partial \zeta_7}, \quad \frac{\partial f_2}{\partial \zeta_4} = \frac{\partial f_6}{\partial \zeta_8},
\]

\[
\frac{\partial f_3}{\partial \zeta_1} = \frac{\partial f_7}{\partial \zeta_5}, \quad \frac{\partial f_3}{\partial \zeta_2} = \frac{\partial f_7}{\partial \zeta_6}, \quad \frac{\partial f_3}{\partial \zeta_3} = \frac{\partial f_7}{\partial \zeta_7}, \quad \frac{\partial f_3}{\partial \zeta_4} = \frac{\partial f_7}{\partial \zeta_8}, \quad \frac{\partial f_4}{\partial \zeta_1} = \frac{\partial f_8}{\partial \zeta_5}, \quad \frac{\partial f_4}{\partial \zeta_2} = \frac{\partial f_8}{\partial \zeta_6}, \quad \frac{\partial f_4}{\partial \zeta_3} = \frac{\partial f_8}{\partial \zeta_7}, \quad \frac{\partial f_4}{\partial \zeta_4} = \frac{\partial f_8}{\partial \zeta_8}.
\]

(3.6)

We call that the Equations (3.6) are the condition of harmonicity of the hyperholomorphic function $F(\zeta)$ on $\Omega$ in $\mathbb{S} \times \mathbb{S}$.

Remark 3.2. We redefine the Equations (3.5) as follows:

\[
\frac{\partial f_1}{\partial \eta_1} = \frac{\partial f_5}{\partial \eta_5}, \quad \frac{\partial f_1}{\partial \eta_2} = -\frac{\partial f_5}{\partial \eta_4}, \quad \frac{\partial f_1}{\partial \eta_3} = -\frac{\partial f_5}{\partial \eta_3}, \quad \frac{\partial f_1}{\partial \eta_4} = \frac{\partial f_5}{\partial \eta_2}, \quad \frac{\partial f_2}{\partial \eta_1} = \frac{\partial f_6}{\partial \eta_5}, \quad \frac{\partial f_2}{\partial \eta_2} = -\frac{\partial f_6}{\partial \eta_4}, \quad \frac{\partial f_2}{\partial \eta_3} = -\frac{\partial f_6}{\partial \eta_3}, \quad \frac{\partial f_2}{\partial \eta_4} = \frac{\partial f_6}{\partial \eta_2},
\]

\[
\frac{\partial g_1}{\partial \eta_1} + \frac{\partial f_1}{\partial \eta_1} = \frac{\partial g_3}{\partial \eta_3} + \frac{\partial f_3}{\partial \eta_3}, \quad \frac{\partial g_1}{\partial \eta_2} + \frac{\partial f_1}{\partial \eta_2} = -\frac{\partial g_3}{\partial \eta_4} - \frac{\partial f_3}{\partial \eta_4}.
\]
Now, we need the following:

\[
\frac{\partial g_1}{\partial z_1} = \frac{\partial f_3}{\partial w_1}, \quad \frac{\partial g_1}{\partial z_2} = \frac{\partial f_3}{\partial w_2}, \quad \frac{\partial g_1}{\partial z_3} = \frac{\partial f_3}{\partial w_3}, \quad \frac{\partial g_2}{\partial z_1} = \frac{\partial f_2}{\partial w_1}, \quad \frac{\partial g_2}{\partial z_2} = \frac{\partial f_2}{\partial w_2}, \quad \frac{\partial g_2}{\partial z_3} = \frac{\partial f_2}{\partial w_3}, \quad \frac{\partial g_3}{\partial z_1} = \frac{\partial f_1}{\partial w_1}, \quad \frac{\partial g_3}{\partial z_2} = \frac{\partial f_1}{\partial w_2}, \quad \frac{\partial g_3}{\partial z_3} = \frac{\partial f_1}{\partial w_3}.
\]

(3.7)

We call that the Equations (3.7) and (3.7) are the condition of harmonicity.

**Theorem 3.2.** If the hyperholomorphic function

\[
F(\zeta) = f_1(z) + f_2(z)e_2 + f_3(z)e_4 + f_4(z)e_6 + f_5(z)e_8 + f_6(z)e_{10}
+ f_7(z)e_{12} + f_8(z)e_{14} + \varepsilon(g_1(w) + g_2(w)e_2 + g_3(w)e_4
+ g_4(w)e_6 + g_5(w)e_8 + g_6(w)e_{10} + g_7(w)e_{12} + g_8(w)e_{14})
\]
satisfies the condition of harmonicity (3.6) and (3.7) in an open set \( \Omega \) in \( S \times S \), then the functions \( f_k(z) \), \( g_k(w) \) (\( k = 1, 2, \cdots, 8 \)) are harmonic in \( \Omega \).

**Proof.** We have

\[
HH^* f_1 = (D + \varepsilon E)(D^* + \varepsilon E^*) f_1 = (DD^* + \varepsilon(DE^* + D^*E)) f_1
\]

\[
= \frac{\partial^2 f_1}{\partial z_1 \partial z_1} + \frac{\partial^2 f_1}{\partial z_2 \partial z_2} + \frac{\partial^2 f_1}{\partial z_3 \partial z_3} + \frac{\partial^2 f_1}{\partial z_4 \partial z_4} + \frac{\partial^2 f_1}{\partial z_5 \partial z_5} + \frac{\partial^2 f_1}{\partial z_6 \partial z_6}
+ \varepsilon \left( \frac{\partial^2 f_1}{\partial z_1 \partial w_1} + \frac{\partial^2 f_1}{\partial z_2 \partial w_2} + \frac{\partial^2 f_1}{\partial z_3 \partial w_3} + \frac{\partial^2 f_1}{\partial z_4 \partial w_4} + \frac{\partial^2 f_1}{\partial z_5 \partial w_5} + \frac{\partial^2 f_1}{\partial z_6 \partial w_6}
+ \frac{\partial^2 f_1}{\partial z_7 \partial w_7} + \frac{\partial^2 f_1}{\partial z_8 \partial w_8} + \frac{\partial^2 f_1}{\partial z_9 \partial w_9} + \frac{\partial^2 f_1}{\partial z_10 \partial w_{10}} + \frac{\partial^2 f_1}{\partial z_11 \partial w_{11}} + \frac{\partial^2 f_1}{\partial z_12 \partial w_{12}} \right)
\]
Let $H. S. Jung, S. J. Ha, K. H. Lee, S. M. Lim and K. H. Shon,$
$J. E. Kim, S. J. Lim and K. H. Shon,$
$J. E. Kim, S. J. Lim and K. H. Shon,$
$w$ $J. E. Kim, S. J. Lim and K. H. Shon,$
$z$
$L. Hörmander,$

If the system (3.5) is a generalized Cauchy-Riemann system on $\Omega$. 

$\Omega$

\[ \frac{\partial}{\partial z_1} \left( \frac{\partial f_5}{\partial z_5} \right) + \frac{\partial}{\partial z_2} \left( \frac{\partial f_5}{\partial z_6} \right) + \frac{\partial}{\partial z_3} \left( \frac{\partial f_5}{\partial z_7} \right) + \frac{\partial}{\partial z_4} \left( \frac{\partial f_5}{\partial z_8} \right) 
+ \frac{\partial}{\partial z_5} \left( - \frac{\partial f_5}{\partial z_1} - \frac{\partial f_5}{\partial z_2} - \frac{\partial f_5}{\partial z_3} - \frac{\partial f_5}{\partial z_4} + \frac{\partial f_5}{\partial w_1} + \frac{\partial f_5}{\partial w_2} + \frac{\partial f_5}{\partial w_3} + \frac{\partial f_5}{\partial w_4} + \frac{\partial f_5}{\partial w_5} + \frac{\partial f_5}{\partial w_6} + \frac{\partial f_5}{\partial w_7} + \frac{\partial f_5}{\partial w_8} \right) \]

$= 0,$

and the functions $f_k(z), g_k(w) (k = 1, 2, \ldots, 8)$ are proved by the similar method as in the proof of the case of $f_1$. \qed

**Proposition 3.1.** Let $\Omega$ be a domain in $S \times S$ and $F(\zeta)$ be a $L$-hyperholomorphic functions in $\Omega$. If $HH^*F(\zeta) = 0$, then the system (3.5) is a generalized Cauchy-Riemann system in $\Omega$.

**Proof.** If $HH^*F = 0$, then $\sum_{j=1}^{8} \left( \frac{\partial^2 f_j}{\partial z_j \partial z_j} + \frac{\partial^2 f_j}{\partial w_j \partial w_j} \right) = 0$. Therefore, the components $f_j, g_j (j = 1, 2, \ldots, 8)$ are harmonic in $\Omega$. That is, the system (3.5) is a generalized Cauchy-Riemann system on $\Omega$. \qed

**References**


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