YANG-MILLS CONNECTIONS IN THE BUNDLE OF AFFINE ORTHONORMAL FRAMES

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Abstract. We get a necessary and sufficient condition for a generalized affine connection in the affine orthonormal frame bundle over a smooth manifold \((M, g)\) to be a Yang-Mills connection.

1. Introduction

Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold, \(A(M) \equiv \text{GL}(n; R) \times R^n\) the bundle of affine frames, and \(O(M, g)\) the bundle of orthonormal frames over \((M, g)\). Let \(\text{AO}(M, g)\) be a principal fiber bundle over the manifold \((M, g)\) with group \(\text{AO}(n; R) \equiv \text{O}(n; R) \times R^n\), which is a subbundle of \(A(M)\). In this paper, we call the bundle \(\text{AO}(M, g)\) the affine orthonormal frame bundle over \((M, g)\). Let \(\tilde{\gamma}: \text{AO}(M, g) \rightarrow O(M, g) = \text{AO}(M, g)/R^n\) be the natural projection, and \(\tilde{\omega}\) an arbitrarily given connection in \(\text{AO}(M, g)\). Let \(\omega\) (resp. \(\varphi\)) be the connection form (resp. 1-form) on \(O(M, g)\) such that \(\tilde{\gamma}^*\tilde{\omega} = \omega + \varphi\). Then we get the following results:

1. Assume the linear connection \(\omega\) (\(\tilde{\gamma}^*\tilde{\omega} = \omega + \varphi\)) becomes a Yang-Mills connection in \(O(M, g)\). Then we obtain a necessary and sufficient condition for the generalized affine connection form \(\tilde{\omega}\) in \(\text{AO}(M, g)\) to become a Yang-Mills connection (cf. Theorem 4.1).

2. Assume the 1-form \(\varphi\) (\(\tilde{\gamma}^*\tilde{\omega} = \omega + \varphi\)) on \(O(M, g)\) is the canonical 1-form on \(O(M, g)\), i.e., \(\varphi(X) := u^{-1}(\pi_*(X))\) for \(X \in T_u(O(M, g))\) \((u \in O(M, g))\). And, assume the linear connection \(\omega\) becomes a Yang-Mills connection. Then we obtain a necessary and sufficient condition for the affine connection form \(\tilde{\omega}\) in \(\text{AO}(M, g)\) to become a Yang-Mills connection (cf. Theorem 4.2).

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(3) As an application of (1) and (2), let $G = M$ be a compact connected semisimple Lie group, $\mathfrak{g}$ the Lie algebra of $G$, and $g$ the canonical Riemannian metric which is defined by the Killing form of the Lie algebra $\mathfrak{g}$ of $G$. Then, the linear connection form in the orthonormal frame bundle by the Levi-Civita connection for $g$ becomes a Yang-Mills connection in $O(M, g)$, but the corresponding affine connection in $AO(M, g)$ does not become a Yang-Mills connection (cf. Theorem 4.4).

2. Preliminaries

In general, when we regard $\mathbb{R}^n$ as an affine space, we denote it by $A^n$. The group $A(n; \mathbb{R})$ of all affine transformations of $A^n$ is represented by the group of all matrices of the form

$$\tilde{a} = \begin{pmatrix} a & \xi \\ 0 & 1 \end{pmatrix},$$

where $a = (a^i_j)_{i,j} \in GL(n; \mathbb{R})$ and $\xi = (\xi^i)$ ($\xi \in \mathbb{R}^n$) is a column vector. The element $\tilde{a}$ maps a point $\eta$ of $A^n$ into $a\eta + \xi$. We have the following exact sequence:

$$0 \to \mathbb{R}^n \to A(n; \mathbb{R}) \to GL(n; \mathbb{R}) \to 1.$$

Let $(M, g)$ be an $n$-dimensional Riemannian manifold, $A(M)$ ($M, A(n; R) = GL(n; R) \times R^n$) the bundle of affine frames over $M$, and $O(M, g)$ the bundle of orthonormal frames over $(M, g)$. Let $AO(M, g)$ be the principal fibre bundle over $(M, g)$ with group $AO(n; \mathbb{R}) = O(n; \mathbb{R}) \times \mathbb{R}^n$ which is a subbundle of $A(M)$. In this paper, we call the bundle $AO(M, g)$ the affine orthonormal frame bundle over $(M, g)$. Let $\tilde{\gamma} : O(M, g) \to AO(M, g)$ be the natural injection together with the group homomorphism $\gamma : O(n; \mathbb{R}) \hookrightarrow AO(n; \mathbb{R})$, and $\tilde{\omega}$ an arbitrarily given Ehresmann connection form in $AO(M, g)$, i.e.

$$\tilde{\omega}(X^*) = X \quad (X \in \mathfrak{ao}(n; R) = o(n; R) + R^n \text{ (semidirect sum)}),$$

$$R_g^*\tilde{\omega} = Ad(g^{-1})\tilde{\omega} \quad (g \in A(n; R)),$$

where $\mathfrak{ao}(n; R)$ (resp. $o(n; R)$) is the Lie algebra of $AO(n; R)$ (resp. $O(n; R)$), $X^*$ is the fundamental vector field corresponding to $X \in \mathfrak{ao}(n; R)$ which is defined on $AO(M, g)$, and $R_g^*\tilde{\omega}$ is the pull back of $\tilde{\omega}$ by the action $R_g$ on $AO(M, g)$. Let $\omega$ (resp. $\varphi$) be the Ehresmann connection form (resp. the tensorial 1-form) of type $(Ad(AO(n; R)), R^n)$ on $O(M, g)$ such that $\tilde{\gamma}^*\tilde{\omega} = \omega + \varphi$ (cf. [1]).

In this paper, we obtain the following results:
(1) Assume $\omega$ becomes a Yang-Mills connection in $O(M, g)$. Then we obtain a necessary and sufficient condition for the connection form $\tilde{\omega}$ in $AO(M, g)$ to become a Yang-Mills connection (cf. Theorem 4.1).

(2) Assume the 1-form $\varphi (\tilde{\gamma}^* \tilde{\omega} = \omega + \varphi)$ on $O(M, g)$ is the canonical 1-form on $O(M, g)$, i.e., $\varphi(X) := u^{-1}(\pi_*(X))$ for $X \in T_u(O(M, g))$ ($u \in (O(M, g))$. And, assume $\omega$ becomes a Yang-Mills connection. Then we obtain a necessary and sufficient condition for the connection form $\tilde{\omega}$ in $AO(M, g)$ to become a Yang-Mills connection (cf. Theorem 4.2).

(3) Let $G = M$ be a compact connected semisimple Lie group, $\mathfrak{g}$ the Lie algebra of $G$, and $g_o$ the canonical Riemannian metric which is defined by the Killing form of the Lie algebra $\mathfrak{g}$ of $G$. Then, the linear connection form $A$ in the orthonormal frame bundle by the Levi-Civita connection of $g_o$ becomes a Yang-Mills connection in $O(M, g)$, but the corresponding affine connection $\tilde{A}$ ($\tilde{A} = A + \theta$) in $AO(M, g)$ does not become a Yang-Mills connection, where $\theta(X) := u^{-1}(\pi_*(X))$ for $X \in T_u(O(M, g))$ ($u \in (O(M, g))$ (cf. Theorem 4.4).

Traditionally, the words ‘linear connection’ and ‘affine connection’ have been used interchangeably ([1, Theorem 3.3, p.129]). But, strictly speaking, a linear connection is a connection in $L(M)$ ($\supset O(M, g)$), and an affine connection is a connection in $A(M)$ ($\supset AO(M, g)$).

By virtue of Theorem 4.4, we show the fact that there exists a Yang-Mills linear connection in $O(M, g)$ of which the corresponding affine connection in $AO(M, g)$ does not become a Yang-Mills connection.

3. Yang-Mills connections in principal fibre bundles

Let $P(M, G)$ be a principal fibre bundle with semisimple Lie group $G$ over an $n$-dimensional closed (compact and connected) Riemannian manifold $(M, g)$, and $\mathfrak{g}$ the Lie algebra of the structure group $G$, and $\{U, V, W, \ldots\}$ an open covering of $M$ generated by local triviality of $P$. Let the mappings $\phi_{UV} : U \cap V \rightarrow G$ corresponding to the open covering $\{U, V, W, \ldots\}$ of $M$ be transition functions. If the family $A = \{A_U, A_V, A_W, \ldots\}$ of $\mathfrak{g}$-valued 1-forms which are defined on open subsets $U, V, W, \ldots$ of $M$ satisfies the following cocycle condition

\[(A_V)_x = L_{\phi_{UV}(x)}(d(\phi_{UV}))_x + Ad(\phi_{UV}(x))(A_U)_x \quad (x \in U \cap V),\]

then $A$ is said to be a connection (form) ([2, Definition 3.1.1, p.74]) in $P(M, G)$. Let $\{\sigma_U, \sigma_V, \sigma_W, \ldots\}$ be the family of local cross sections of the open neighborhoods $U, V, W, \ldots$ into $P$. Let $\mathfrak{X}_P$ be the space of all connections in $P$, and $\mathfrak{C}_P$ the space of all Ehresmann connections in $P$.
which are defined as in (2.1). Then, \( \mathfrak{A}_P \) and \( \mathfrak{C}_P \) are 1-1 correspondent as follows ([2, Theorem 3.1.4, p.76]):

if we put \( \sigma_U \ast \omega =: A_U \) for a given \( \omega \in \mathfrak{C}_P \), then the family \( A := \{ A_U, A_V, A_W, \ldots \} \) of \( g \)-valued 1-forms defined on the open neighborhoods \( U, V, W, \ldots \) satisfies the cocycle condition (3.1). On the other hand, we put \( \omega_U(Z) := A_U(Y) + X \) for \( Z = (\sigma_U \ast X) \in T_{\sigma_U}(P) \), \( Y \in T_x(M) \), \( X \in g \), and if \( Z \in T_v(P) \), then we put \( \omega_U(Z) := \text{Ad}(g^{-1}) \omega_U(R^{-1}g, Z) \). Here \( \omega_U, \omega_V, \omega_W, \ldots \) coincide on the overlapping neighborhoods of \( P \), and the family \( \omega := \{ \omega_U, \omega_V, \omega_W, \ldots \} \) satisfy the conditions as in (2.1).

The curvature form \( F(A) \) ([1, 2]) of a connection \( A \in \mathfrak{A}_P \) in the principal fibre bundle \( P(M, g) \) is given by

\[
F(A) = dA + A \wedge A. \tag{3.2}
\]

We fix an \( \text{Ad}(G) \)-invariant inner product \( < , > \) on \( g \). A Yang-Mills connection is a critical point of the Yang-Mills functional

\[
\mathcal{Y}M(A) = \frac{1}{2} \int_M \|F(A)\|^2 \, v_g \quad (A \in \mathfrak{A}_P)
\]

which is defined on the space \( \mathfrak{A}_P \), where \( v_g \) is the volume element of \((M, g)\) and \( \|F(A)\|^2 = < F(A), F(A) > \). Let \( \{ X_i \}_{i=1}^n \) be a (locally defined) orthonormal frame on \((M, g)\). Then a necessary and sufficient condition ([2, Theorem 4.1.3, p.107]) for a connection \( A \) in the principal fibre bundle \( P \) to become a Yang-Mills connection is the fact that

\[
(\delta_A F(A))(X_i) = - \sum_j \{ (\nabla_{X_j} F(A))(X_j, X_i) \}
\]

\[
+ [A(X_j), F(A)(X_j, X_i)]
\]

vanishes, where \( \delta_A \) is the formal adjoint operator of the covariant exterior differentiation \( d_A \), and \( \nabla \) is the Levi-Civita connection on \((M, g)\).

4. Yang-Mills connections in the affine orthonormal frame bundle \( AO(M, g) \) over \((M, g)\)

4.1. A generalized affine (Ehresmann) connection and an affine (Ehresmann) connection

Let \((M, g)\) be an \( n \)-dimensional compact connected smooth manifold. A (Ehresmann) connection in the bundle \( L(M) \) of all linear frames on \( M \) is called a linear connection of \( M \). A generalized affine (Ehresmann) connection is a (Ehresmann) connection in the affine frame bundle \( A(M) \)
over $M$. Moreover, a generalized affine (Ehresmann) connection $\tilde{\omega}$ in $A(M)$ is called an affine (Ehresmann) connection in $A(M)$, if the $R^n$-valued 1-form $\varphi$ ($\tilde{\gamma}^*(\tilde{\omega}) = \omega + \varphi$) on $L(M)$ is the canonical 1-form $\theta$ ($\theta(X) = u^{-1}\pi_*(X)$ ($X \in T_\omega(L(M))$), where $\gamma : GL(n; R) \hookrightarrow A(n; R)$ and $\tilde{\gamma} : L(M) \to A(M)$.

4.2. Generalized affine Yang-Mills connections in the affine orthonormal frame bundle $AO(M, g)$ over $(M, g)$

We get the following exact sequence:

$$0 \hookrightarrow R^n \overset{\alpha}{\to} AO(n; R) \overset{\beta}{\to} O(n; R) = AO(n; R)/R^n \to 1.$$ 

By the virtue of the principal fibre bundle homomorphism

$$\tilde{\beta} : AO(M, g) \to O(M, g) = AO(M, g)/R^n$$

associated with the group homomorphism

$$\beta : AO(n; R) \to O(n; R) = AO(n; R)/R^n,$$

we obtain the fact that the set of all affine (Ehresmann) connections in $AO(M, g)$ ($\subset A(M)$) and the set of all linear connections in $O(M, g)$ ($\subset L(M)$) are 1-1 correspondent (cf. [1, Theorem 3.3, p.129]).

Let $\tilde{\omega}$ be a generalized affine (Ehresmann) connection in $AO(M, g)$, and $(X_1, X_2, \ldots, X_n)$ an (locally defined) orthonormal frame on $(M, g)$. Let $\sigma_U$ be the cross section of $O(M, g)$ ($\subset AO(M, g)$) over $U$ ($\subset M$) which assigns to each $x \in U$ the linear frame $((X_1)_x, (X_2)_x, \ldots, (X_n)_x)$.

Let $\tilde{\gamma} : O(M, g) \to AO(M, g)$ be the bundle homomorphism associated with the group homomorphism $\gamma : O(n; R) \hookrightarrow AO(n; R)$. Since $\tilde{\gamma}^*\tilde{\omega} = \omega + \varphi$ on $O(M, g)$,

$$\sigma_U^*(\tilde{\gamma}^*\tilde{\omega}) = \sigma_U^*\omega + \sigma_U^*\varphi = \tilde{\omega}$$

on $U$ ($\subset M$). Here and from now on, we put

$$\sigma_U^*\tilde{\omega} =: \tilde{A}_U, \quad \sigma_U^*\omega =: A_U, \quad \sigma_U^*\varphi =: \varphi_U =: \varphi$$

on $U$ ($\subset M$). Then in (4.1), the $ao(n; R)$-valued 1-form $\tilde{A}$ and the $o(n; R)$-valued 1-form $A$ on $M$ satisfy the cocycle condition (3.1). So, $\tilde{A}$ (resp. $A$) is a connection (form) in $AO(M, g)$ (resp. $O(M, g)$). Similarly, we call $\tilde{A}$ (resp. $A$) a generalized affine connection (form) in $AO(M, g)$ (resp. the linear connection (form) in $O(M, g)$ which is related to the connection $\tilde{A}$). From the above facts, we get

$$\tilde{A}_U = A_U + \varphi_U \quad (\tilde{A} = A + \varphi, \ \text{simply}).$$
From (3.2) and (4.2), we obtain ([1, Proposition 3.4, p.130])
\[ F(\tilde{A}) = F(A) + d\varphi + A \wedge \varphi. \]

Let \((\ , \ )\) be the inner product on \(\mathfrak{o}(n; R)\) defined by \((X, Y) := -\text{trace}(XY)\) \((X, Y \in \mathfrak{o}(n; R))\), and let \(\{e_i\}_i\) be the natural basis of \(R^n\).

Now, we fix an inner product \(<\ , \ >\) on \(\mathfrak{ao}(n; R)\) such that \(\{Z_i\}_{i=1}^{(n^2-n)/2} \cup \{e_j\}_{j=1}^n\) is an orthonormal basis of \(\mathfrak{ao}(n; R)\) with respect to \(<\ , \ >\), where \(\{Z_i\}_{i=1}^{(n^2-n)/2}\) is an orthonormal basis on \((\mathfrak{o}(n; R), (\ , \ ))\). Then, the inner product \(<\ , \ >\) on \(\mathfrak{o}(n; R)\) is \(\text{Ad}(O(n; R))\)-invariant. So, \(<\ , \ >\) is \(\text{Ad}(\phi(x))\)-invariant, where \(x \in M\) and \(\phi\) are transition functions appeared by local triviality of the principal fibre bundle \(O(M, g)\).

For convenience’ sake, for a (locally defined) orthonormal frame \(\{X_i\}_{i=1}^n\) on \((M, g)\), we put
\[
(\delta_{\tilde{A}} F(\tilde{A}))(X_i) = (\delta_A F(A))(X_i) =: (\delta_AF)_i, \\
(\nabla_{X_k} d\varphi)(X_i, X_j) =: \nabla_k d\varphi_{ij}, \\
(\nabla_{X_k} A \wedge \varphi)(X_i, X_j) =: \nabla_k (A \wedge \varphi)_{ij}, \\
(F(A))(X_k, X_i) =: F_{ki}, \\
(\varphi)(X_j) =: \varphi_j, \\
A(X_k) =: A_k, \\
(A \wedge \varphi)(X_i, X_j) =: (A \wedge \varphi)_{ij}, \\
(d\varphi)(X_i, X_j) =: d\varphi_{ij}.
\]

From (3.2), (3.3), (4.3), (4.4), and the properties of the inner product \(<\ , \ >\) on \(\mathfrak{o}(n; R)\), we obtain
\[
(\delta_{\tilde{A}} F(\tilde{A}))_i = (\delta_AF)_i - \sum_k \{\nabla_k d\varphi_{ki} + \nabla_k (A \wedge \varphi)_{ki} + F_{ki} \varphi_k + [A_k, d\varphi_{ki}] + [A_k, (A \wedge \varphi)_{ki}]\}.
\]

By the help of (4.5), we get

**Theorem 4.1.** Let \(\tilde{A}\) be a generalized affine connection (form) in \(AO(M, g)\), and \(A\) a linear connection in \(O(M, g)\) such that \(\tilde{A} = A + \varphi\). Assume the linear connection \(A\) in \(O(M, g)\) is a Yang-Mills connection. Then, a necessary and sufficient condition for the connection (form) \(\tilde{A}\) to become a Yang-Mills connection is
\[
\sum_k \{\nabla_k d\varphi_{ki} + \nabla_k (A \wedge \varphi)_{ki} - F_{ki} \varphi_k + [A_k, d\varphi_{ki}] + [A_k, (A \wedge \varphi)_{ki}]\} = 0.
\]
4.3. Affine Yang-Mills connections in the affine orthonormal frame bundle $AO(M, g)$

Assume $\tilde{A}$ is an affine connection in $AO(M, g)$. Then we get

$$\tilde{A}_U = A_U + \varphi_U = A_U + \sigma_U^*\theta =: A_U + \theta_U \quad (\tilde{A} = A + \theta, \text{ briefly}),$$

where $\theta$ is the canonical 1-form on $O(M, g)$, i.e.

$$\theta(Z) := u^{-1}\pi_*(Z) \quad (Z \in T_u(O(M, g))).$$

Since $\theta_i := \theta(X_i) := (\sigma_U^*)\theta(X_i) = e_i$, we have

$$X_k(\varphi_i) = X_k(\theta_i) = 0.$$  \hspace{1cm} (4.6)

Let \{${Y^i}$\}$_i$ be the (locally defined) dual frame of the (locally defined) orthonormal frame \{${X_i}$\}$_i$ on $(M, g)$. Then we put

$$(4.7) \quad Y^i(\nabla X_iX_k) = : \Gamma_{ikt}^k, \quad [X_i, X_j] = : \sum_k C_{ij}^k X_k.$$

Since $\nabla$ is the Levi-Civita connection on $(M, g)$, we have from (4.7)

$$(4.8) \quad \Gamma_{ij}^k - \Gamma_{ji}^k = C_{ij}^k, \quad \Gamma_{ij}^k = -\Gamma_{ji}^k.$$

Then, by the help of (3.2), (4.4), (4.6) and (4.7), we get

$$\nabla_k d\theta_{ki} = -\frac{1}{2} \{X_k(\sum_l C_{kl}^i)e_l - \sum_{l,t} \{\Gamma_{kl}^t C_{ti}^l + \Gamma_{kt}^l C_{il}^t\}e_l\},$$

$$\nabla_k(A \wedge \theta)_{ki} = \frac{1}{2} \{X_k(A_k e_i - A_i e_k) + \sum_l \{\Gamma_{ki}^l (A_l e_k - A_k e_l) + \Gamma_{kk}^l (A_i e_l - A_l e_i)\}\},$$

$$(4.9) \quad F_{ki}^j = \frac{1}{2} \{X_k(A_i) - X_i(A_k) + A_k A_i - A_i A_k - \sum_l C_{ki}^l A_l\}e_k,$$

$$[A_k, d\theta_{kt}] = -\frac{1}{2} \sum_l A_k C_{kt}^l e_l,$$

$$[A_k, (A \wedge \theta)_{kt}] = \frac{1}{2} A_k (A_k e_i - A_i e_k).$$
From (4.5) and (4.9), we get

\[
(\delta_A F)_i = (\delta A F)_i - \frac{1}{2} \sum_k \left\{ X_k (A_k e_i - A_i e_k - \sum_l C_{kl}^l e_l) \\
+ A_k (A_k e_i - 2A_i e_k) + \sum_{l,t} \Gamma_{kl}^t C_{kt}^l e_l \\
+ \sum_l \Gamma_{kk}^l (A_l e_i - 2A_i e_l) + \sum_l C_{kl}^l e_l \\
+ (X_i(A_k) - X_k(A_i) + A_i A_k) e_k \\
+ 2 \sum_{l} (\Gamma_{ti}^k + \Gamma_{ik}^t + \Gamma_{kt}^i) A_k e_l \right\}.
\]

(4.10)

By virtue of (4.10), we obtain

**Theorem 4.2.** Let \( \tilde{A} \) be an affine connection (form) in \( AO(M,g) \), and \( A \) a linear connection in \( O(M,g) \) such that \( \tilde{A} = A + \theta \). Assume the linear connection \( A \) in \( O(M,g) \) is a Yang-Mills connection. Then, a necessary and sufficient condition for the connection (form) \( \tilde{A} \) to become a Yang-Mills connection is

\[
\sum_k \left\{ X_k (A_k e_i - A_i e_k - \sum_l C_{kl}^l e_l) + A_k (A_k e_i - 2A_i e_k) \\
+ \sum_{l,t} \Gamma_{kl}^t C_{kt}^l e_l + \sum_l \Gamma_{kk}^l (A_l e_i - 2A_i e_l) + \sum_l C_{kl}^l e_l \\
+ (X_i(A_k) - X_k(A_i) + A_i A_k) e_k \\
+ 2 \sum_{l} (\Gamma_{ti}^k + \Gamma_{ik}^t + \Gamma_{kt}^i) A_k e_l \right\} = 0.
\]

**4.4. The connection form in \( O(M,g) \) defined by the Levi-Civita connection for the metric \( g \) of \( (M,g) \)**

Let \( M = G \) be an \( n \)-dimensional closed (connected and compact) semisimple Lie group. Let \( g \) be the canonical bi-invariant Riemannian metric induced by the Killing form of the Lie algebra \( \mathfrak{g} \) of the group \( M \). Let \( \{ X_i \}_i \) be an orthonormal basis on \( (\mathfrak{g}, (\ , \ ) := g_e) \), where \( e \) is the identity element of the group \( M \). Let \( \nabla \) be the Levi-Civita connection on \( (M = G, g) \), and \( \{ Y^k \}_k \) be the dual frame of the orthonormal frame \( \{ X_i \}_i \) on \( (M, g) \). Now, we put \( Y^k(\nabla X_j X_i) =: \Gamma_{ji}^k, \sum_k \Gamma_{kj} Y^k =: a^i_j \), and \( (a^i_j)_{i,j} =: A \). Then, \( A \) is a linear connection (form) in \( O(M,g) \). Let \( \tilde{A} \) be an affine connection in \( AO(M,g) \) such that \( \tilde{A} = A + \theta \) on \( O(M,g) \), where \( \theta \) is the canonical form on \( O(M,g) \).
From these facts, we get

\begin{equation}
A_k = A(X_k) = (\Gamma_{kij})_{i,j}
\end{equation}

which belongs to \(\mathfrak{o}(n; R)\). We fix an inner product \((\cdot, \cdot)\) on \(\mathfrak{o}(n; R)\) such that \((X, Y) := g_{ij}(X, Y) (X, Y \in \mathfrak{o}(n; R))\). Then the inner product \((\cdot, \cdot)\) is \(\text{Ad}(O(n; R))-\)invariant. So, we obtain from (4.8) and (4.11)

\begin{equation}
\Gamma_{kij} = -\Gamma_{kji}, \quad 2\Gamma_{kij} = C_{kij}, \quad \Gamma_{kij} = \Gamma_{jik} = \Gamma_{ikj}.
\end{equation}

Since \(C_{kij}\) are constants, we have

\begin{equation}
X_k(C_{ij}^k) = 2X_k(\Gamma_{ij}^k) = 0.
\end{equation}

From (3.2), (4.11) and (4.12), we get

\begin{equation}
(F(A))(X_k, X_i) := F_{ki} = \frac{1}{2} \sum_l (\Gamma_{kl}^i \Gamma_{lij} - \Gamma_{il}^i \Gamma_{kji} - 2\Gamma_{ki}^i \Gamma_{ijl})_{s,j}
\end{equation}

which belongs to \(\mathfrak{o}(n; R)\). From (4.4), (4.11), (4.12) and (4.14), we get

\begin{equation}
\sum_k [A_k, F_{ki}]_s^j = \sum_{k,l,t} (3\Gamma_{kj}^t \Gamma_{kil}^j - \Gamma_{ilt}^j \Gamma_{kj}^l - 2\Gamma_{kt}^j \Gamma_{ijl}^s + \frac{1}{2} \Gamma_{kst}^j \Gamma_{ktl}^i \Gamma_{lij})
\end{equation}

which is the \((s, j)\)th component of \(\sum_k [A_k, F_{ki}] (\in \mathfrak{o}(n; R))\). From (4.12) and (4.14), we obtain

\begin{equation}
(\sum_k \nabla_k F_{ki})_j^s = \sum_{k,l,t} (\Gamma_{kl}^i \Gamma_{it}^s \Gamma_{kj}^l + \Gamma_{ki}^i \Gamma_{kjt}^l \Gamma_{lj}^s)
\end{equation}

which is the \((s, j)\)th component of \(\sum_k \nabla_k F_{ki} (\in \mathfrak{o}(n; R))\). We get from (3.3), (4.4), (4.12), (4.15) and (4.16),

\begin{equation}
(\delta_A F)_i = -\sum_{k,l,t} (4\Gamma_{kj}^t \Gamma_{kl}^i \Gamma_{it}^s + \Gamma_{klt}^j \Gamma_{kl}^i \Gamma_{lj}^s)
\end{equation}

\begin{equation}
+ \frac{1}{2} \Gamma_{ks}^t \Gamma_{kt}^j \Gamma_{ls}^i + \frac{1}{2} \Gamma_{kj}^t \Gamma_{kl}^j \Gamma_{ls}^i).
\end{equation}

Since \([[X_j, X_s], X_k]] + [[X_s, X_k], X_j]] + [[X_k, X_j], X_s] = 0\), we have

\begin{equation}
\sum_l C_{js}^l C_{ik}^l = \sum_l (C_{ks}^l C_{lj}^i + C_{jk}^l C_{is}^j).
\end{equation}

From (4.12) and (4.18), we get

\begin{equation}
\sum_{k,l,t} \Gamma_{kij}^l \Gamma_{is}^t = 2 \sum_{k,l,t} \Gamma_{kji}^l \Gamma_{tk}^i.
\end{equation}
Similarly, we have from (4.12) and (4.19)
\[
\frac{1}{2} \sum_{k,l,t} \Gamma_{klt}^i \Gamma_{kl}^j = \sum_{k,l,t} \Gamma_{jk}^i \Gamma_{kt}^l \Gamma_{lt}^s,
\]
(4.20)
\[
\frac{1}{2} \sum_{k,l,t} \Gamma_{kls}^i \Gamma_{kjt}^s = \sum_{k,l,t} \Gamma_{jk}^i \Gamma_{kt}^l \Gamma_{lt}^s.
\]
By virtue of (4.12), (4.17), (4.19) and (4.20), we get
\[
(\delta_A F)_i = 0.
\]
Thus we obtain the following

**Lemma 4.3.** Let $M$ be a closed semisimple Lie group, and $g$ the canonical bi-invariant Riemannian metric which is induced from the Killing form of the Lie algebra $\mathfrak{g}$ of the Lie group $M$. Let $A$ be the connection form in $O(M, g)$ which is defined by the Levi-Civita connection on $(M, g)$. Then, $A$ becomes a Yang-Mills connection.

Moreover from (4.10), (4.11), (4.13) and (4.21), we obtain
\[
(\delta_{\tilde{A}} \tilde{F})_i = -\frac{1}{2} \sum_k \{A_k(A_k e_i - 2A_i e_k) + \sum_{l,t} \Gamma_{kl}^i C_{kt}^l e_l \\
+ \sum_l \Gamma_{kl}^i (A_l e_t - A_t e_l + \sum_t C_{tl}^i e_l) \\
+ A_i A_k e_k + 2 \sum_{l,t} (\Gamma_{kl}^k + \Gamma_{kl}^l + \Gamma_{kl}^t) A_k e_l \}.
\]
(4.22)
We get from (4.11), (4.12) and (4.22), we get
\[
(\delta_{\tilde{A}} \tilde{F})_i = -\frac{1}{2} \sum_k \{ \sum_i A_k (\Gamma_{ki}^l e_t - 2\Gamma_{ik}^l e_t) \\
+ 2 \sum_{l,t} (\Gamma_{kl}^l + 3\Gamma_{ki}^k) \Gamma_{kt}^l e_l \}.
\]
(4.23)
By the help of (4.11), (4.12) and (4.23), we have
\[
(\delta_{\tilde{A}} \tilde{F})_i = -\frac{1}{2} \sum_{k,l,t} \Gamma_{kl}^i \Gamma_{kl}^l e_t = -\frac{1}{8} \sum_{k,l,t} C_{kl}^i C_{kl}^l e_l.
\]
(4.24)
So, from (4.24) we get the following:

*If the affine connection $\tilde{A}$ ($\tilde{A} = A + \theta$) in $AO(M, g)$ becomes a Yang-Mills connection, then the Lie algebra $\mathfrak{g}$ of $M$ ($= G$) is abelian.*

This contradicts the fact that $\mathfrak{g}$ is semisimple.
Combining this result with Lemma 4.3, we obtain

**Theorem 4.4.** Let $M$ be a closed semisimple Lie group, and $g$ the canonical bi-invariant Riemannian metric which is induced from the Killing form of the Lie algebra $g$ of the group $M$. Let $A$ be the connection form in $O(M, g)$ which is defined by the Levi-Civita connection on $(M, g)$, and $\tilde{A}$ the affine connection in $AO(M, g)$ such that $\tilde{A} = A + \theta$. Then, $A$ becomes a Yang-Mills connection in $O(M, g)$, but $\tilde{A}$ does not become a Yang-Mills connection in $AO(M, g)$.

**References**


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