SUM AND PRODUCT THEOREMS OF RELATIVE TYPE AND RELATIVE WEAK TYPE OF ENTIRE FUNCTIONS

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Abstract. Orders and types of entire functions have been actively investigated by many authors. In this paper, we aim at investigating some basic properties in connection with sum and product of relative type and relative weak type of entire functions.

1. Introduction, Definitions and Notations.

Let $f$ be an entire function defined in the complex plane $\mathbb{C}$. The function $M_f(r)$ on $|z| = r$ is defined as follows:

$$M_f(r) := \max_{|z|=r} |f(z)|,$$

which is known as maximum modulus function corresponding to $f$.

It is noted that, if $f$ is non-constant, then $M_f(r)$ is strictly increasing and continuous, and its inverse $M_f^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and satisfies $\lim_{s \to \infty} M_f^{-1}(s) = \infty$.

On the other hand, the Nevanlinna’s characteristic function of $f$ denoted by $T_f(r)$ is defined as

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta,$$

where $\log^+ x = \max \{\log x, 0\}$ for all $x > 0$.

We begin by recalling the following definitions.

Received November 26, 2014. Accepted December 16, 2014.
2010 Mathematics Subject Classification: 30D20, 30D30, 30D35.
Key words and phrases: Entire functions; Relative order (relative lower order); Relative type (relative lower type); Relative weak type; Regular relative growth.
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Definition 1. The order $\rho_f$ and lower order $\lambda_f$ of an entire function $f$ are defined as

$$\rho_f := \limsup_{r \to \infty} \frac{\log^{[2]} M_f(r)}{\log r} \quad \text{and} \quad \lambda_f := \liminf_{r \to \infty} \frac{\log^{[2]} M_f(r)}{\log r}.$$  

An entire function whose order and lower order are the same is said to be of regular growth. Entire functions which are not of regular growth are said to be of irregular growth.

Definition 2. The type $\sigma_f$ and lower type $\sigma_f$ of an entire function $f$ are defined as

$$\sigma_f := \limsup_{r \to \infty} \frac{\log M_f(r)}{r^{\rho_f}} \quad \text{and} \quad \sigma_f := \liminf_{r \to \infty} \frac{\log M_f(r)}{r^{\rho_f}} \quad (0 < \rho_f < \infty).$$

Datta and Jha [3] introduced to define weak type of an entire function of finite positive lower order in the following way:

Definition 3. The weak type $\tau_f$ and the growth indicator $\tau_f$ of an entire function $f$ of finite positive lower order $\lambda_f$ are defined by

$$\tau_f := \limsup_{r \to \infty} \frac{\log M_f(r)}{r^{\lambda_f}} \quad \text{and} \quad \tau_f := \liminf_{r \to \infty} \frac{\log M_f(r)}{r^{\lambda_f}} \quad (0 < \lambda_f < \infty).$$

For any two given entire functions $f$ and $g$, the ratio $\frac{M_f(r)}{M_g(r)}$ as $r \to \infty$ is called the growth of $f$ with respect to $g$ in terms of their maximum moduli. From Definition 1, it is seen that the order of an entire function $f$ which is generally used for computational purpose is defined in terms of the growth of $f$ with respect to the exponential function as follows:

$$\rho_f := \limsup_{r \to \infty} \frac{\log \log M_f(r)}{\log \log M_{\exp(z)}(r)} \quad = \limsup_{r \to \infty} \frac{\log \log M_f(r)}{\log r}.$$  

Bernal [1, 2] introduced to define relative order of an entire function $g$ with respect to an entire function $f$ denoted by $\rho_f(g)$ to avoid comparing growth with just the exponential function $\exp(z)$ as follows:

$$\rho_f(g) := \inf \{\mu > 0 : M_g(r) < M_f(r^\mu) \quad \text{for all} \ r > r_0(\mu) > 0\}$$

$$= \limsup_{r \to \infty} \frac{\log M_f^{-1} M_g(r)}{\log r}.$$  

It is easy to see that the above definition coincides with the classical one if $f(z) = \exp(z)$ (cf. [16]).
Similarly one can define the relative lower order of $g$ with respect to $f$, denoted by $\lambda_f(g)$, as follows:

$$\lambda_f(g) := \lim_{r \to \infty} \inf \frac{\log M_f^{-1}M_g(r)}{\log r}.$$ 

An entire function $g$ is said to be of regular relative growth with respect to $f$ if its relative order with respect to $f$ coincides with its relative lower order with respect to the same function $f$.

To compare the relative growth of two entire functions having same nonzero finite relative order with respect to another entire function, Roy [15] recently introduced the notion of relative type of two entire functions in the following manner.

**Definition 4.** Let $f$ and $g$ be any two entire functions such that $0 < \rho_g(f) < \infty$. Then the relative type $\sigma_g(f)$ of $f$ with respect to $g$ is defined as follows:

$$\sigma_g(f) := \inf \left\{ k > 0 : M_f(r) < M_g \left( kr^{\rho_g(f)} \right) \text{ for all sufficiently large values of } r \right\}$$

$$= \lim_{r \to \infty} \sup \frac{M_f^{-1}M_f(r)}{r^{\rho_g(f)}}.$$ 

Likewise one can define the relative lower type of an entire function $f$ with respect to another entire function $g$ denoted by $\sigma_g(f)$ as follows:

$$\sigma_g(f) := \lim_{r \to \infty} \inf \frac{M_f^{-1}M_f(r)}{r^{\rho_g(f)}} \quad (0 < \rho_g(f) < \infty).$$

Similarly one can define the relative lower order of $g$ with respect to $f$, denoted by $\lambda_f(g)$, as follows:

$$\lambda_f(g) := \lim_{r \to \infty} \inf \frac{\log M_f^{-1}M_g(r)}{\log r}.$$ 

An entire function $g$ is said to be of regular relative growth with respect to $f$ if its relative order with respect to $f$ coincides with its relative lower order with respect to the same function $f$.

To compare the relative growth of two entire functions having same nonzero finite relative order with respect to another entire function, Roy [15] recently introduced the notion of relative type of two entire functions in the following manner.

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$$\sigma_g(f) := \inf \left\{ k > 0 : M_f(r) < M_g \left( kr^{\rho_g(f)} \right) \text{ for all sufficiently large values of } r \right\}$$

$$= \lim_{r \to \infty} \sup \frac{M_f^{-1}M_f(r)}{r^{\rho_g(f)}}.$$ 

Likewise one can define the relative lower type of an entire function $f$ with respect to another entire function $g$ denoted by $\sigma_g(f)$ as follows:

$$\sigma_g(f) := \lim_{r \to \infty} \inf \frac{M_f^{-1}M_f(r)}{r^{\rho_g(f)}} \quad (0 < \rho_g(f) < \infty).$$

Analogously to determine the relative growth of two entire functions having same nonzero finite relative lower order with respect to another entire function, Datta and Biswas [5] introduced to define relative weak type of an entire function $f$ with respect to another entire function $g$ of finite positive relative lower order $\lambda_g(f)$ in the following way.

**Definition 5.** The relative weak type $\tau_g(f)$ of an entire function $f$ with respect to another entire function $g$ having finite positive relative lower order $\lambda_g(f)$ is defined as follows:

$$\tau_g(f) := \lim_{r \to \infty} \inf \frac{M_f^{-1}M_f(r)}{r^{\lambda_g(f)}}.$$ 

Also one may define the growth indicator $\tau_g(f)$ of an entire function $f$ with respect to an entire function $g$ in the following way:

$$\tau_g(f) := \lim_{r \to \infty} \sup \frac{M_f^{-1}M_f(r)}{r^{\lambda_g(f)}} \quad (0 < \lambda_g(f) < \infty).$$
Choosing \( g(z) = \exp(z) \), one may easily verify that Definition 4 and Definition 5 coincide with the classical definitions of type (lower type) and weak type, respectively.

In this connection, the following definition is introduced (see [2]).

**Definition 6.** A non-constant entire function \( f \) is said to have Property (A) if for any \( \sigma > 1 \) and for all large \( r \), \( |M_f(r)|^2 \leq M_f(r^\sigma) \) holds.

For examples of functions with or without the Property (A), one may refer to [2].

Here, in this paper, we aim at investigating some basic properties of relative type and relative weak type of entire functions under somewhat different conditions. Throughout this paper, for entire functions \( f_i \) and \( g_k \) \((i, k = 1, 2)\), we assume that \( \sigma_{f_i}(g_k), \tau_{f_i}(g_k) \) and \( \sigma_{f_i} \) are all nonzero finite.

It is also remarked in passing that the standard definitions and notations in the theory of entire functions, for which one may refer to [17], are not given here.

2. Some Known and New Results

Determination of the order and type of entire functions are very important to study the basic growth properties in the value distribution theory. In this regard, during the past decades, many researchers have made close investigations on this research subject to yield many results, for example, some of which are recalled here.

**Theorem A** ([9]). Let \( f \) and \( g \) be any two entire functions of order \( \rho_f \) and \( \rho_g \) respectively. Then

\[
\rho_{f+g} = \rho_g \quad \text{when} \quad \rho_f < \rho_g \quad \text{and} \quad \rho_{f+g} \leq \rho_g \quad \text{when} \quad \rho_f \leq \rho_g.
\]

**Theorem B** ([12]). Let \( f \) and \( g \) be any two entire functions with order \( \rho_f, \rho_g \), and type \( \sigma_f, \sigma_g \), respectively. Then

\[
\rho_{f+g} \leq \max \{\rho_f, \rho_g\}, \quad \rho_{f+g} \leq \max \{\rho_f, \rho_g\}
\]

and

\[
\sigma_{f+g} \leq \max \{\sigma_f, \sigma_g\}, \quad \sigma_{f+g} \leq \sigma_f + \sigma_g.
\]
Relative Type and Relative Weak Type of Entire Functions

Detailed investigations on the properties of relative order of entire functions have been made in [2], [8], [10] and [11]. In this connection we state the following two theorems.

**Theorem C** ([2]). Let \( f_1, g_1 \) and \( g_2 \) be any three entire functions. If \( \rho_{f_1}(g_i) = \max \{ \rho_{f_1}(g_k) \mid k, i = 1, 2 \} \), then
\[
\rho_{f_1}(g_1 \pm g_2) \leq \rho_{f_1}(g_i) \quad (i = 1, 2),
\]
whose equality holds when \( \rho_{f_1}(g_1) \neq \rho_{f_1}(g_2) \).

**Theorem D** ([2, 14]). Let \( f_1, g_1 \) and \( g_2 \) be any three entire functions. If \( \rho_{f_1}(g_i) = \max \{ \rho_{f_1}(g_k) \mid k, i = 1, 2 \} \), then
\[
\rho_{f_1}(g_1 \cdot g_2) \leq \rho_{f_1}(g_i) \quad (i = 1, 2),
\]
whose equality holds when \( \rho_{f_1}(g_1) \neq \rho_{f_1}(g_2) \).

Similar results hold for the quotient \( \frac{g_1}{g_2} \) provided \( \frac{g_1}{g_2} \) is entire.

Datta et al. [4] proved the following two theorems for relative lower order.

**Theorem E** ([4]). Let \( f_1, f_2 \) and \( g_1 \) be any three entire functions. If \( \lambda_{f_1}(g_1) = \min \{ \lambda_{f_k}(g_1) \mid k, i = 1, 2 \} \), then
\[
\lambda_{f_1 \pm f_2}(g_1) \geq \lambda_{f_i}(g_1) \quad (i = 1, 2),
\]
whose equality holds when \( \lambda_{f_1}(g_1) \neq \lambda_{f_2}(g_1) \).

**Theorem F** ([4]). Let \( f_1, f_2 \) and \( g_1 \) be any three entire functions. If \( \lambda_{f_1}(g_1) = \min \{ \lambda_{f_k}(g_1) \mid k, i = 1, 2 \} \) and \( g_1 \) has the Property (A), then
\[
\lambda_{f_1 \cdot f_2}(g_1) \geq \lambda_{f_i}(g_1) \quad (i = 1, 2),
\]
whose equality holds when \( \lambda_{f_1}(g_1) \neq \lambda_{f_2}(g_1) \).

Similar results hold for the quotient \( \frac{f_1}{f_2} \) provided \( f_2 \) is entire.

Extending the results, Datta et al. [6] established the following theorems under somewhat different conditions.

**Theorem G** ([6]). Let \( f_1, f_2, g_1 \) and \( g_2 \) be any four entire functions.

(i) If \( \rho_{f_1}(g_1) = \min \{ \rho_{f_k}(g_1) \mid k, i = 1, 2 \} \) and \( g_1 \) is of regular relative growth with respect to at least any one of \( f_1 \) or \( f_2 \), then
\[
\rho_{f_1 \pm f_2}(g_1) \geq \rho_{f_i}(g_1) \quad (i = 1, 2),
\]
whose equality holds when \( \rho_{f_1}(g_1) \neq \rho_{f_2}(g_1) \).
(ii) If \( \lambda_{f_1}(g_i) = \max \{ \lambda_{f_1}(g_k) \mid k = 1, 2 \} \) and at least \( g_1 \) or \( g_2 \) is of regular relative growth with respect to \( f_1 \), then

\[
\lambda_{f_1}(g_1 \pm g_2) \leq \lambda_{f_1}(g_i) \quad (i = 1, 2),
\]

whose equality holds when \( \lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2) \).

**Theorem H** ([6]). Let \( f_1, f_2, g_1 \) and \( g_2 \) be any four entire functions.

(i) If \( \rho_{f_k}(g_1) = \min \{ \rho_{f_k}(g_1) \mid k = 1, 2 \} \), \( g_1 \) has the Property (A) and is of regular relative growth with respect to at least any one of \( f_1 \) or \( f_2 \), then

\[
\rho_{f_1, f_2}(g_1) \geq \rho_{f_1}(g_1) \quad (i = 1, 2),
\]

whose equality holds when \( \rho_{f_1}(g_1) \neq \rho_{f_2}(g_1) \).

Similar results hold for the quotient \( \frac{f_1}{f_2} \) provided \( \frac{f_1}{f_2} \) is entire.

(ii) If \( \lambda_{f_1}(g_i) = \max \{ \lambda_{f_1}(g_k) \mid k = 1, 2 \} \), \( f_1 \) has the Property (A) and at least \( g_1 \) or \( g_2 \) is of regular relative growth with respect to \( f_1 \), then

\[
\lambda_{f_1}(g_1 \cdot g_2) \leq \lambda_{f_1}(g_i) \quad (i = 1, 2),
\]

whose equality holds when \( \lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2) \).

Similar results hold for the quotient \( \frac{g_1}{g_2} \) provided \( \frac{g_1}{g_2} \) is entire.

**Theorem I** ([6]). Let \( f_1, f_2, g_1 \) and \( g_2 \) be any four entire functions.

(i) If \( g_1 \neq \rho_{f_2}(g_1) \), \( \rho_{f_1}(g_2) \neq \rho_{f_2}(g_2) \) and \( g_1 \) and \( g_2 \) are both of regular relative growth with respect to at least any one of \( f_1 \) or \( f_2 \), then

\[
\rho_{f_1 \pm f_2}(g_1 \pm g_2) \leq \max \{ \min \{ \rho_{f_1}(g_1), \rho_{f_2}(g_1) \}, \min \{ \rho_{f_1}(g_2), \rho_{f_2}(g_2) \} \},
\]

whose equality holds when

\[
\min \{ \rho_{f_1}(g_1), \rho_{f_2}(g_1) \} \neq \min \{ \rho_{f_1}(g_2), \rho_{f_2}(g_2) \}.
\]

(ii) If \( \lambda_{f_1}(g_1) \neq \lambda_{f_2}(g_1) \), \( \lambda_{f_1}(g_2) \neq \lambda_{f_2}(g_2) \) and at least \( g_1 \) or \( g_2 \) is of regular relative growth with respect to \( f_1 \) and \( f_2 \), respectively, then

\[
\lambda_{f_1 \pm f_2}(g_1 \pm g_2) \geq \min \{ \max \{ \lambda_{f_1}(g_1), \lambda_{f_2}(g_1) \}, \max \{ \lambda_{f_1}(g_2), \lambda_{f_2}(g_2) \} \},
\]

whose equality holds when

\[
\max \{ \lambda_{f_1}(g_1), \lambda_{f_2}(g_1) \} \neq \max \{ \lambda_{f_1}(g_2), \lambda_{f_2}(g_2) \}.
\]

**Theorem J** ([6]). Let \( f_1, f_2, g_1 \) and \( g_2 \) be any four entire functions.
If (a) $\rho_{f_1}(g_1) \neq \rho_{f_2}(g_1)$, (b) $\rho_{f_1}(g_2) \neq \rho_{f_2}(g_2)$ (c) $f_1 \cdot f_2$, $g_1$ and $g_2$ have the Property (A) and (d) $g_1$ and $g_2$ are both of regular relative growth with respect to at least any one of $f_1$ or $f_2$, then

\[
\rho_{f_1,f_2}(g_1 \cdot g_2) = \max \{ \rho_{f_1}(g_1), \rho_{f_2}(g_1) \} \geq \min \{ \rho_{f_1}(g_2), \rho_{f_2}(g_2) \}.
\]

and

\[
\rho_{f_1,f_2}(g_1/g_2) = \max \{ \rho_{f_1}(g_1), \rho_{f_2}(g_1) \} \geq \min \{ \rho_{f_1}(g_2), \rho_{f_2}(g_2) \}.
\]

In the cases of relative type and relative weak type, it therefore seems natural to make parallel investigations of their basic properties. In this connection, Roy [15] proved only the following theorem.

**Theorem K** ([15]). Let $f_1$, $g_1$ and $g_2$ be any three entire functions. If (i) $\rho_{f_1}(g_i) = \max \{ \rho_{f_1}(g_k) \mid k, i = 1, 2 \}$ and (ii) $\rho_{f_1}(g_1) \neq \rho_{f_1}(g_2)$, then

\[
\sigma_{f_1}(g_1 \pm g_2) = \sigma_{f_1}(g_i).
\]

Here, under somewhat different conditions, we present the following theorems related to relative type (relative lower type ) and relative weak type that extend the previous results in some sense.

**Theorem 1.** Let $f_1$, $f_2$, $g_1$ and $g_2$ be any four entire functions such that $\rho_{f_k}(g_k)$ ($k = 1, 2$) are non-zero finite.
If \((A)\) \(\rho_{f_1}(g_k) = \max\{\rho_{f_1}(g_k) \mid k = 1, 2\}\) and \((B)\) \(\rho_{f_1}(g_1) \neq \rho_{f_1}(g_2)\), then
\[\sigma_{f_1}(g_1 \pm g_2) = \sigma_{f_1}(g_i) \quad (i = 1, 2)\.

If \((A)\) \(\rho_{f_1}(g_k) = \min\{\rho_{f_1}(g_k) \mid k = 1, 2\}\), \((B)\) \(\rho_{f_1}(g_1) \neq \rho_{f_2}(g_1)\) and \((C)\) \(g_1\) is of regular relative growth with respect to at least any one of \(f_1\) or \(f_2\), then
\[
(i) \quad \sigma_{f_1 \pm f_2}(g_1) = \sigma_{f_1}(g_1) \quad (i = 1, 2)
\]
and
\[
(ii) \quad \sigma_{f_1 \pm f_2}(g_1) = \sigma_{f_1}(g_1) \quad (i = 1, 2).
\]

Assume the functions \(f_1, f_2, g_1\) and \(g_2\) satisfy the following conditions:
\[
(A) \quad \rho_{f_1}(g_k) = \max\{\min\{\rho_{f_1}(g_1) \mid k = 1, 2\}, \min\{\rho_{f_2}(g_1), \rho_{f_2}(g_2)\}\};
\]
\[
(B) \quad \rho_{f_1}(g_1) \neq \rho_{f_2}(g_1);
\]
\[
(C) \quad \rho_{f_1}(g_2) \neq \rho_{f_2}(g_2);
\]
\[
(D) \quad \min\{\rho_{f_1}(g_1), \rho_{f_2}(g_1)\} \neq \min\{\rho_{f_1}(g_2), \rho_{f_2}(g_2)\};
\]
\[
(E) \quad g_1\) and \(g_2\) are both of regular relative growth with respect to at least any one of \(f_1\) or \(f_2\).
\]
Then we have
\[
(i) \quad (i) \quad \sigma_{f_1 \pm f_2}(g_1 \pm g_2) = \sigma_{f_1}(g_k) \quad (i, k = 1, 2)
\]
and
\[
(ii) \quad \sigma_{f_1 \pm f_2}(g_1 \pm g_2) = \sigma_{f_1}(g_k) \quad (i, k = 1, 2).
\]

**Theorem 2.** Let \(f_1, f_2, g_1\) and \(g_2\) be any four entire functions such that \(\lambda_{f_k}(g_k)\) \((k = 1, 2)\) are non-zero finite.

**I** The following conditions are assumed to be satisfied:
\[
(A) \quad \lambda_{f_1}(g_k) = \max\{\lambda_{f_1}(g_k) \mid k = 1, 2\};
\]
\[
(B) \quad \lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2);
\]
\[
(C) \quad \lambda_{f_2}(g_1) \neq \lambda_{f_2}(g_2);
\]
\[
(D) \quad \min\{\lambda_{f_1}(g_1), \lambda_{f_2}(g_1)\} \neq \min\{\lambda_{f_1}(g_2), \lambda_{f_2}(g_2)\};
\]
\[
(E) \quad g_1\) and \(g_2\) are both of regular relative growth with respect to at least any one of \(f_1\) or \(f_2\).
\]
Then we have
\[
(i) \quad \tau_{f_1}(g_1 \pm g_2) = \tau_{f_1}(g_1) \quad (i = 1, 2).
\]
and
\[
(ii) \quad \tau_{f_1}(g_1 \pm g_2) = \tau_{f_1}(g_1) \quad (i = 1, 2).
\]

**II** The following two conditions are assumed to be satisfied:
\[
(A) \quad \lambda_{f_1}(g_1) = \min\{\lambda_{f_k}(g_1) \mid k = 1, 2\}
\]
and
\[
(B) \quad \lambda_{f_1}(g_1) \neq \lambda_{f_2}(g_1).
\]
Then we have
\[
(i) \quad \tau_{f_1 \pm f_2}(g_1) = \tau_{f_1}(g_1) \quad (i = 1, 2)
\]
and
\[(\text{ii}) \quad \tau_{f_1 \pm f_2} (g_1) = \tau_{f_i} (g_1) \quad (i = 1, 2).\]

(III) The following conditions are assumed to be satisfied:

(A) $\lambda_{f_i} (g_k) = \min \{ \max \{ \lambda_{f_1} (g_1), \lambda_{f_2} (g_1) \}, \max \{ \lambda_{f_1} (g_2), \lambda_{f_2} (g_2) \} \}$;
(B) $\lambda_{f_i} (g_1) \neq \lambda_{f_2} (g_1)$;
(C) $\lambda_{f_1} (g_2) \neq \lambda_{f_2} (g_2)$;
(D) $\max \{ \lambda_{f_1} (g_1), \lambda_{f_2} (g_1) \} \neq \max \{ \lambda_{f_1} (g_2), \lambda_{f_2} (g_2) \}$;
(E) At least $g_1$ or $g_2$ is of regular relative growth with respect to $f_1$ and $f_2$, respectively.

Then we have

(i) $\tau_{f_1 \pm f_2} (g_1 \pm g_2) = \tau_{f_i} (g_k) \quad (i, k = 1, 2)$
and

(ii) $\tau_{f_1 \pm f_2} (g_1 \pm g_2) = \tau_{f_i} (g_k) \quad (i, k = 1, 2)$.

**Theorem 3.** Let $f_1$, $f_2$, $g_1$ and $g_2$ be any four entire functions such that $\rho_{f_k} (g_k) \quad (k = 1, 2)$ are non-zero finite.

(I) The following conditions are assumed to be satisfied:

(A) $\rho_{f_1} (g_k) = \max \{ \rho_{f_1} (g_k) \mid k, i = 1, 2 \}$;
(B) $\rho_{f_1} (g_1) \neq \rho_{f_2} (g_2)$;
(C) $f_1$ has the Property (A).

Then we have

(i) $\sigma_{f_1} (g_1 \cdot g_2) \leq \sigma_{f_1} (g_i) \quad (i = 1, 2)$, whose equality holds only when $2^{\rho_{f_1} (g_i)} \leq 1$.
(ii) $\overline{\sigma}_{f_1} (g_1 \cdot g_2) \leq \overline{\sigma}_{f_1} (g_i) \quad (i = 1, 2)$, whose equality holds only when $2^{\rho_{f_1} (g_i)} \leq 1$.

(II) The following conditions are assumed to be satisfied:

(A) $\rho_{f_1} (g_1) = \min \{ \rho_{f_k} (g_1) \mid k = 1, 2 \}$;
(B) $\rho_{f_1} (g_1) \neq \rho_{f_2} (g_1)$;
(C) $g_1$ has the Property (A) and also $g_1$ is of regular relative growth with respect to at least any one of $f_1$ or $f_2$.

Then we have

(i) $\sigma_{f_1} (f_2 (g_1) \geq \sigma_{f_1} (g_1) \quad (i = 1, 2)$, whose equality holds only when $2^{\rho_{f_1} (g_i)} \geq 1$.
and

(ii) $\overline{\sigma}_{f_1} (f_2 (g_1) \geq \overline{\sigma}_{f_1} (g_1) \quad (i = 1, 2)$, whose equality holds only when $2^{\rho_{f_1} (g_i)} \geq 1$.

(III) The following conditions are assumed to be satisfied:

(A) $\rho_{f_1} (g_k) = \max \{ \min \{ \rho_{f_1} (g_1), \rho_{f_2} (g_1) \}, \min \{ \rho_{f_1} (g_2), \rho_{f_2} (g_2) \} \}$;
(B) $\rho_{f_1} (g_1) \neq \rho_{f_2} (g_1)$;
Theorem 4. Let $f_1$, $f_2$, $g_1$ and $g_2$ be any four entire functions such that $\rho_{f_k}(g_k)$ $(k = 1, 2)$ are non-zero finite.

(I) The following conditions are assumed to be satisfied:

(A) $\lambda(f_1, g_1) = \max \{\lambda(f_k, g_k) \mid k, i = 1, 2\}$;

(B) $\lambda(f_1, g_1) \neq \lambda(f_2, g_1)$;

(C) $f_1$ has the Property (A) and at least $g_1$ or $g_2$ is of regular relative growth with respect to $f_1$.

Then we have

(i) $\tau_{f_1, f_2}(g_1) \leq \tau_{f_1, f_2}(g_i) \ (i = 1, 2)$, whose equality holds only when $2^{\lambda_{f_1}(g_i)} \leq 1$.

(ii) $\bar{\tau}_{f_1, f_2}(g_1) \leq \bar{\tau}_{f_1, f_2}(g_i) \ (i = 1, 2)$, whose equality holds only when $2^{\lambda_{f_1}(g_i)} \leq 1$.

(II) The following conditions are assumed to be satisfied:

(A) $\lambda(f_1, g_1) = \min \{\lambda(f_k, g_1) \mid k = 1, 2\}$;

(B) $\lambda(f_1, g_1) \neq \lambda(f_2, g_1)$;

(C) $g_1$ has the Property (A).

Then we have

(i) $\tau_{f_1, f_2}(g_1) \geq \tau_{f_1, f_2}(g_i) \ (i = 1, 2)$, whose equality holds only when $2^{\lambda_{f_1}(g_i)} \geq 1$.

(ii) $\bar{\tau}_{f_1, f_2}(g_1) \geq \bar{\tau}_{f_1, f_2}(g_i) \ (i = 1, 2)$, whose equality holds only when $2^{\lambda_{f_1}(g_i)} \geq 1$.

(III) The following conditions are assumed to be satisfied:

(A) $\lambda(f_1, g_1) = \min \{\max\{\lambda(f_1, g_1), \lambda(f_2, g_1)\}, \max\{\lambda(f_1, g_2), \lambda(f_2, g_2)\}\}$;

(B) $\lambda(f_1, g_1) \neq \lambda(f_2, g_1)$;
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(C) $\lambda_{f_1}(g_2) \neq \lambda_{f_2}(g_2)$;
(D) $\max\{\lambda_{f_1}(g_1), \lambda_{f_2}(g_1)\} \neq \max\{\lambda_{f_1}(g_2), \lambda_{f_2}(g_2)\}$;
(E) $g_1 \cdot g_2, f_1$ and $f_2$ have the Property (A);
(F) At least $g_1$ or $g_2$ is of regular relative growth with respect to $f_1$ and $f_2$, respectively;
(G) $2^{\lambda_{f_1} f_2(g_k)} \leq 1$ and $2^{\lambda_{f_2} f_2(g_k)} \geq 1$.

Then we have

(i) $\tau_{f_1, f_2}(g_1 \cdot g_2) = \tau_{f_1}(g_k)$ ($i, k = 1, 2$)
and
(ii) $\tau_{f_1, f_2}(g_1 \cdot g_2) = \tau_{f_2}(g_k)$ ($i, k = 1, 2$).

Similar results for the above three cases hold for the quotient $\frac{f_1}{f_2}$ provided $\frac{f_1}{f_2}$ is entire.

Here we reconsider the equalities in Theorem C to Theorem H under somewhat different conditions and give our assertions as in following four theorems.

**Theorem 5.** Let $f_1, f_2, g_1$ and $g_2$ be any four entire functions.

(I) If either $\sigma_{f_1}(g_1) \neq \sigma_{f_1}(g_2)$ or $\sigma_{f_1}(g_1) \neq \sigma_{f_1}(g_2)$ holds, then

$$\rho_{f_1}(g_1 \pm g_2) = \rho_{f_1}(g_1) = \rho_{f_1}(g_2).$$

(II) The following two conditions are assumed to be satisfied:
(A) Either $\sigma_{f_1}(g_1) \neq \sigma_{f_2}(g_1)$ or $\sigma_{f_1}(g_1) \neq \sigma_{f_2}(g_1)$ holds;
(B) $g_1$ is of regular relative growth with respect to at least any one of $f_1$ or $f_2$.

Then we have

$$\rho_{f_1 \pm f_2}(g_1) = \rho_{f_1}(g_1) = \rho_{f_2}(g_1).$$

**Theorem 6.** Let $f_1, f_2, g_1$ and $g_2$ be any four entire functions.

(I) The following conditions are assumed to be satisfied:
(A) Either $\tau_{f_1}(g_1) \neq \tau_{f_1}(g_2)$ or $\tau_{f_1}(g_1) \neq \tau_{f_1}(g_2)$ holds;
(B) At least $g_1$ or $g_2$ is of regular relative growth with respect to $f_1$.

Then we have

$$\lambda_{f_1}(g_1 \pm g_2) = \lambda_{f_1}(g_1) = \lambda_{f_1}(g_2).$$

(II) If either $\tau_{f_1}(g_1) \neq \tau_{f_2}(g_1)$ or $\tau_{f_1}(g_1) \neq \tau_{f_2}(g_1)$ holds, then

$$\lambda_{f_1 \pm f_2}(g_1) = \lambda_{f_1}(g_1) = \lambda_{f_2}(g_1).$$
Theorem 7. Let \( f_1, f_2, g_1 \) and \( g_2 \) be any four entire functions.

(I) The following conditions are assumed to be satisfied:
(A) Either \( \sigma_{f_1} (g_1) \neq \sigma_{f_1} (g_2) \) or \( \sigma_{f_1} (g_1) \neq \sigma_{f_1} (g_2) \) holds;
(B) \( f_1 \) has the Property (A);
(C) \( 2^{\rho_{f_1} (g_1)} \geq 1 \).

Then we have
\[
\rho_{f_1} (g_1 \cdot g_2) = \rho_{f_1} (g_1) = \rho_{f_1} (g_2).
\]

(II) The following conditions are assumed to be satisfied:
(A) Either \( \sigma_{f_1} (g_1) \neq \sigma_{f_2} (g_1) \) or \( \sigma_{f_1} (g_1) \neq \sigma_{f_2} (g_1) \) holds;
(B) \( g_1 \) has the Property (A) and is of regular relative growth with respect to at least any one of \( f_1 \) or \( f_2 \);
(C) \( 2^{\rho_{f_1} (g_1)} \geq 1 \).

Then we have
\[
\rho_{f_1} \cdot f_2 (g_1) = \rho_{f_1} (g_1) = \rho_{f_2} (g_1).
\]

Similar results for the above two cases hold for the quotient \( \frac{f_1}{f_2} \) provided \( \frac{f_1}{f_2} \) is entire.

Theorem 8. Let \( f_1, f_2, g_1 \) and \( g_2 \) be any four entire functions.

(I) The following conditions are assumed to be satisfied:
(A) Either \( \tau_{f_1} (g_1) \neq \tau_{f_1} (g_2) \) or \( \tau_{f_1} (g_1) \neq \tau_{f_1} (g_2) \) holds;
(B) \( f_1 \) has the Property (A) and at least \( g_1 \) or \( g_2 \) is of regular relative growth with respect to \( f_1 \);
(C) \( 2^{\lambda_{f_1} (g_1)} \leq 1 \).

Then we have
\[
\lambda_{f_1} (g_1 \cdot g_2) = \lambda_{f_1} (g_1) = \lambda_{f_1} (g_2).
\]

(II) The following conditions are assumed to be satisfied:
(A) Either \( \tau_{f_1} (g_1) \neq \tau_{f_2} (g_2) \) or \( \tau_{f_1} (g_1) \neq \tau_{f_2} (g_1) \) holds;
(B) \( g_1 \) has the Property (A);
(C) \( 2^{\lambda_{f_1} (g_1)} \geq 1 \).

Then we have
\[
\lambda_{f_1} \cdot f_2 (g_1) = \lambda_{f_1} (g_1) = \lambda_{f_2} (g_1).
\]

Similar results for the above three cases hold for the quotient \( \frac{f_1}{f_2} \) provided \( \frac{f_1}{f_2} \) is entire.
3. Required Known Properties

Here we recall some known properties, which will be required in the next section, as in the following lemmas. For Lemmas 1 and 2, see [2]. For Lemma 3 and Lemma 4, one may refer to [12] and [7, p.18], respectively.

Lemma 1. Suppose $f$ be an entire function and $\alpha, \beta$ be such that $\alpha > 1$ and $0 < \beta < \alpha$. Then

$$M_f (\alpha r) > \beta M_f (r).$$

Lemma 2. Let $f$ be an entire function satisfying the Property (A). Then for any positive integer $n$ and for all sufficiently large $r$,

$$[M_f (r)]^n \leq M_f \left( r^\delta \right)$$

holds for $\delta > 1$.

Lemma 3. Every entire function $f$ satisfying the Property (A) is transcendental.

Lemma 4. Let $f$ be an entire function. Then, for all sufficiently large values of $r$, we have

$$T_f (r) \leq \log M_f (r) \leq 3T_f (2r).$$

4. Proofs

Here we prove our main results.

Proof of Theorem 1. From the definition of relative type and relative lower type of entire function, we have for all sufficiently large values of $r$ that

(1) \[ M_{g_k} (r) \leq M_{f_k} \left( \sigma_{f_k} (g_k) + \varepsilon \right) r^{\rho_{f_k} (g_k)}, \]

\[ M_{g_k} (r) \geq M_{f_k} \left( \sigma_{f_k} (g_k) - \varepsilon \right) r^{\rho_{f_k} (g_k)} \]

(2) \[ i.e., \quad M_{f_k} (r) \leq M_{g_k} \left( \frac{r}{(\sigma_{f_k} (g_k) - \varepsilon)} \right)^{\rho_{f_k} (g_k)}, \]

\[ \leq M_{g_k} \left( \frac{r}{(\sigma_{f_k} (g_k) + \varepsilon)} \right)^{\rho_{f_k} (g_k)}. \]
and also for a sequence \( \{ r_n \} \) of values of \( r \) tending to infinity, we get

\[
M_{g_k}(r) \geq M_{f_k} \left[ (\sigma_{f_k}(g_k) - \varepsilon) r_n^{\rho_{f_k}(g_k)} \right]
\]

(3)

i.e.,

\[
M_{f_k}(r) \leq M_{g_k} \left[ \frac{r_n}{(\sigma_{f_k}(g_k) - \varepsilon)} \right]^{\frac{1}{\rho_{f_k}(g_k)}},
\]

(4)

where \( \varepsilon > 0 \) is any arbitrary positive number and \( k = 1, 2 \).

**Case I.** Let \( \rho_{f_1}(g_k) < \rho_{f_1}(g_i) \) where \( k, i = 1, 2 \) with \( g_k \neq g_i \) for \( k \neq i \).

Now from (1) and (4) we get for a sequence \( \{ r_n \} \) of values of \( r \) tending to infinity that

\[
M_{g_1 \pm g_2}(r_n) < M_{g_1}(r_n) + M_{g_2}(r_n),
\]

which implies that

\[
M_{g_1 \pm g_2}(r_n) < M_{f_1} \left[ (\sigma_{f_1}(g_k) + \varepsilon) r_n^{\rho_{f_1}(g_k)} \right] + M_{f_1} \left[ (\sigma_{f_1}(g_i) + \varepsilon) r_n^{\rho_{f_1}(g_i)} \right].
\]

So we have

\[
M_{g_1 \pm g_2}(r_n) < M_{f_1} \left[ (\sigma_{f_1}(g_k) + \varepsilon) r_n^{\rho_{f_1}(g_k)} \right] \left[ 1 + \frac{M_{f_1} \left[ (\sigma_{f_1}(g_k) + \varepsilon) r_n^{\rho_{f_1}(g_k)} \right]}{M_{f_1} \left[ (\sigma_{f_1}(g_i) + \varepsilon) r_n^{\rho_{f_1}(g_i)} \right]} \right].
\]

Since \( \rho_{f_1}(g_k) < \rho_{f_1}(g_i) \), one can make the term

\[
\frac{M_{f_1} \left[ (\sigma_{f_1}(g_k) + \varepsilon) r_n^{\rho_{f_1}(g_k)} \right]}{M_{f_1} \left[ (\sigma_{f_1}(g_i) + \varepsilon) r_n^{\rho_{f_1}(g_i)} \right]}
\]

sufficiently small by taking \( n \) sufficiently large. Therefore in view of Lemma 1 and the above inequality, we get for a sequence \( \{ r_n \} \) of values of \( r \) tending to infinity that

\[
M_{g_1 \pm g_2}(r_n) < M_{f_1} \left[ (\sigma_{f_1}(g_i) + \varepsilon) r_n^{\rho_{f_1}(g_i)} \right] (1 + \varepsilon_1).
\]

That is,

\[
M_{g_1 \pm g_2}(r_n) < M_{f_1} \left[ \alpha (\sigma_{f_1}(g_i) + \varepsilon) r_n^{\rho_{f_1}(g_i)} \right],
\]

where \( \alpha > (1 + \varepsilon_1) \).
Now making $\alpha \to 1+$, we obtain from Theorem C for a sequence \( \{r_n\} \) of values of \( r \) tending to infinity that

\[
M_{f_1}^{-1}M_{g_1 \pm g_2} (r_n) < (\sigma f_i (g_i) + \varepsilon) r_{\rho f_1 (g_i)}^k
\]

and so

\[
\frac{M_{f_1}^{-1}M_{g_1 \pm g_2} (r_n)}{r_{\rho f_1 (g_1 \pm g_2)}} < (\sigma f_i (g_i) + \varepsilon).
\]

Since $\varepsilon > 0$ is arbitrary, we get

\[
\sigma_{f_1} (g_1 \pm g_2) \leq \sigma f_i (g_i).
\]

Further without any loss of generality, let $\rho f_i (g_1) < \rho f_i (g_2)$ and $g = g_1 \pm g_2$. Then $\sigma f_i (g) \leq \sigma f_i (g_2)$. Also let $g_2 = \pm (g - g_1)$ and in this case we obtain from Theorem C that $\rho f_i (g_1) < \rho f_i (g)$. So $\sigma f_i (g_2) \leq \sigma f_i (g)$. Hence $\sigma f_i (g) = \sigma f_i (g_2)$ ⇒ $\sigma f_i (g_1 \pm g_2) = \sigma f_i (g_2)$. Thus, $\sigma f_i (g_1 \pm g_2) = \sigma f_i (g_i) \ (i = 1, 2)$ where $\rho f_i (g_i) = \max \{\rho f_i (g_k) \mid k, i = 1, 2\}$ and $\rho f_i (g_1) \neq \rho f_i (g_2)$ which is the first part of the theorem.

**Case II.** Now suppose that $\rho f_i (g_1) < \rho f_k (g_1)$ where $k, i = 1, 2$ with $f_i \neq f_k \ (i \neq k)$ and $g_1$ is of regular relative growth with respect to at least any one of $f_1$ or $f_2$.

Therefore, in view of (2) and (3), we obtain for a sequence \( \{r_n\} \) of values of \( r \) tending to infinity that

\[
M_{f_1 \pm f_2} (r_n) < M_{f_1} (r_n) + M_{f_2} (r_n).
\]

Thus we have

\[
M_{f_1 \pm f_2} (r_n) < M_{g_1} \left( \frac{r_n}{(\sigma f_i (g_1) - \varepsilon)} \right)^{\frac{1}{\rho f_i (g_i)}} + M_{g_1} \left( \frac{r_n}{(\sigma f_k (g_1) - \varepsilon)} \right)^{\frac{1}{\rho f_k (g_1)}}
\]

and so

\[
M_{f_1 \pm f_2} (r_n) < M_{g_1} \left( \frac{r_n}{(\sigma f_i (g_1) - \varepsilon)} \right)^{\frac{1}{\rho f_i (g_i)}} \left[ 1 + \frac{M_{g_1} \left( \frac{r_n}{(\sigma f_k (g_1) - \varepsilon)} \right)^{\frac{1}{\rho f_k (g_1)}}}{M_{g_1} \left( \frac{r_n}{(\sigma f_i (g_1) - \varepsilon)} \right)^{\frac{1}{\rho f_i (g_i)}}} \right].
\]
Since $\rho_{f_1} (g_1) < \rho_{f_k} (g_1)$, we can make the term sufficiently small by taking $n$ sufficiently large. Hence in view of Lemma 1 and the above inequality we get for a sequence $\{r_n\}$ of values of $r$ tending to infinity that

$$M_{g_1} \left[ \left( \frac{r_n}{\sigma_{f_1} (g_1)} \right) \frac{1}{\sigma_{f_1} (g_1)} \right] < M_{g_1} \left[ \alpha \left( \frac{r_n}{\sigma_{f_1} (g_1)} \right) \frac{1}{\sigma_{f_1} (g_1)} \right],$$

where $\alpha > (1 + \varepsilon_1)$.

Hence, making $\alpha \to 1+$, we obtain the first part of Theorem G for a sequence $\{r_n\}$ of values of $r$ tending to infinity that

$$M_{f_1 \pm f_2} (r_n) < M_{g_1} \left[ \left( \frac{r_n}{\sigma_{f_1} (g_1)} \right) \frac{1}{\sigma_{f_1} (g_1)} \right] (1 + \varepsilon_1)$$

$$< M_{g_1} \left[ \alpha \left( \frac{r_n}{\sigma_{f_1} (g_1)} \right) \frac{1}{\sigma_{f_1} (g_1)} \right],$$

i.e.,

$$\sigma_{f_1 \pm f_2} (g_1) < M_{f_1 \pm f_2} M_{g_1} (r_n)$$

$$\sigma_{f_1 \pm f_2} (g_1) < \frac{M_{f_1 \pm f_2} (r_n)}{r_n},$$

Since $\varepsilon > 0$ is arbitrary, we find

$$\sigma_{f_1 \pm f_2} (g_1) \geq \sigma_{f_1} (g_1).$$

Now without loss of generality, we may consider that $\rho_{f_1} (g_1) < \rho_{f_2} (g_1)$ and $f = f_1 \pm f_2$. Then $\sigma_f (g_1) \geq \sigma_{f_1} (g_1)$. Further let $f_1 = (f \pm f_2)$. Therefore in view of the first part of Theorem G, $\rho_f (g_1) < \rho_{f_2} (g_1)$ and accordingly $\sigma_f (g_1) \geq \sigma_{f_1} (g_1)$. Hence $\sigma_f (g_1) = \sigma_{f_1} (g_1)$ and $\sigma_{f_1 \pm f_2} (g_1) = \sigma_{f_1} (g_1)$. So, $\sigma_{f_1 \pm f_2} (g_1) = \sigma_{f_1} (g_1) (i = 1, 2)$ where $\rho_{f_i} (g_1) = \min (\rho_{f_k} (g_1) \mid k, i = 1, 2)$ provided $\rho_{f_1} (g_1) \neq \rho_{f_2} (g_1)$ and $g_1$ is of regular relative growth with respect to at least any one of $f_1$ or $f_2$.

**Case III.** In this case, one can clearly assume that $\rho_{f_1} (g_1) < \rho_{f_k} (g_1)$ where $k, i = 1, 2$ with $f_i \neq f_k (i \neq k)$ and $g_1$ is of regular relative growth with respect to at least any one of $f_1$ or $f_2$.

Then, in view of (2), we obtain for all sufficiently large values of $r$ that

$$M_{f_1 \pm f_2} (r) < M_{f_1} (r) + M_{f_2} (r).$$
That is, we have
\[ M_{f_1 \pm f_2} (r) < M_{g_1} \left[ \left( \frac{r}{(\sigma_{f_i} (g_1) - \varepsilon)} \right)^{\frac{1}{\rho_{f_i} (g_1)}} \right] + M_{g_1} \left[ \left( \frac{r}{(\sigma_{f_k} (g_1) - \varepsilon)} \right)^{\frac{1}{\rho_{f_k} (g_1)}} \right]. \]

And so
\[ M_{f_1 \pm f_2} (r) < M_{g_1} \left[ \left( \frac{r}{(\sigma_{f_i} (g_1) - \varepsilon)} \right)^{\frac{1}{\rho_{f_i} (g_1)}} \right] \cdot \left[ 1 + \frac{M_{g_1} \left[ \left( \frac{r}{(\sigma_{f_k} (g_1) - \varepsilon)} \right)^{\frac{1}{\rho_{f_k} (g_1)}} \right]}{M_{g_1} \left[ \left( \frac{r}{(\sigma_{f_i} (g_1) - \varepsilon)} \right)^{\frac{1}{\rho_{f_i} (g_1)}} \right]} \right]. \]

As \( \rho_{f_1} (g_1) < \rho_{f_k} (g_1) \), we can make the term \( \frac{M_{g_1} \left[ \left( \frac{r}{(\sigma_{f_k} (g_1) - \varepsilon)} \right)^{\frac{1}{\rho_{f_k} (g_1)}} \right]}{M_{g_1} \left[ \left( \frac{r}{(\sigma_{f_i} (g_1) - \varepsilon)} \right)^{\frac{1}{\rho_{f_i} (g_1)}} \right]} \)
sufficiently small by taking \( r \) sufficiently large and therefore using the similar technique for all sufficiently large values of \( r \) as executed in the proof of Case II we get from (8) that \( \sigma_{f_1 \pm f_2} (g_1) = \sigma_{f_i} (g_1) \) (\( i = 1, 2 \)) where \( \rho_{f_i} (g_1) = \min \{ \rho_{f_k} (g_1) \mid k, i = 1, 2 \} \) provided \( \rho_{f_1} (g_1) \neq \rho_{f_2} (g_1) \) and \( g_1 \) is of regular relative growth with respect to at least any one of \( f_1 \) or \( f_2 \).

Thus combining Case II and Case III we obtain the second part of the theorem.

The third part of the theorem is a natural consequence of Theorem I (i), Theorem K and the first part and second part of the theorem. Hence its proof is omitted.

**Proof of Theorem 2.** For any arbitrary positive number \( \varepsilon > 0 \), we have from Definition 5 for all sufficiently large values of \( r \) that
\[ M_{g_k} (r) \leq M_{f_k} \left[ (\tau_{f_k} (g_k) + \varepsilon) r^{\lambda_{f_k} (g_k)} \right], \]
\[ M_{g_k} (r) \geq M_{f_k} \left[ (\tau_{f_k} (g_k) - \varepsilon) r^{\lambda_{f_k} (g_k)} \right], \]
\[ \text{i.e., } M_{f_k} (r) \leq M_{g_k} \left[ \left( \frac{r}{(\tau_{f_k} (g_k) - \varepsilon)} \right)^{\frac{1}{\lambda_{f_k} (g_k)}} \right]. \]
and for a sequence \( \{r_n\} \) of values of \( r \) tending to infinity we have

\[
M_{g_k}(r) \geq M_f \left[ (\tau_{f_k}(g_k) - \varepsilon) \right] \tau^{r_n} \lambda_{f_k}(g_k) 
\]

(i.e., \( M_{f_k}(r) \leq M_{g_k} \left( \frac{r_n}{(\tau_{f_k}(g_k) - \varepsilon)} \right)^{\frac{1}{\lambda_{f_k}(g_k)}} \)),

(11)

\[
M_{g_k}(r) \leq M_{f_k} \left[ (\tau_{f_k}(g_k) + \varepsilon) \right] \tau^{r_n} \lambda_{f_k}(g_k) 
\]

(12)

where \( k = 1, 2 \).

**Case I.** Let us consider \( \lambda_{f_1}(g_k) < \lambda_{f_1}(g_i) \) where \( k, i = 1, 2 \) with \( g_k \neq g_i \) \((k \neq i)\) and at least \( g_1 \) or \( g_2 \) is of regular relative growth with respect to \( f_1 \).

Therefore from (5), (9) and (12) we get for a sequence \( \{r_n\} \) of values of \( r \) tending to infinity that

\[
M_{g_1 \pm g_2}(r_n) < M_{f_1} \left[ (\tau_{f_1}(g_k) + \varepsilon) \right] \tau^{r_n} \lambda_{f_1}(g_k) + M_{f_1} \left[ (\tau_{f_1}(g_i) + \varepsilon) \right] \tau^{r_n} \lambda_{f_1}(g_i) 
\]

That is, we have

\[
M_{g_1 \pm g_2}(r_n) < M_{f_1} \left[ (\tau_{f_1}(g_k) + \varepsilon) \right] \tau^{r_n} \lambda_{f_1}(g_k) \left[ 1 + \frac{M_{f_1} \left[ (\tau_{f_1}(g_i) + \varepsilon) \right] \tau^{r_n} \lambda_{f_1}(g_i)}{M_{f_1} \left[ (\tau_{f_1}(g_k) + \varepsilon) \right] \tau^{r_n} \lambda_{f_1}(g_k)} \right] 
\]

(13)

Since \( \lambda_{f_1}(g_k) < \lambda_{f_1}(g_i) \), we can make the term \( \frac{M_{f_1} \left[ (\tau_{f_1}(g_i) + \varepsilon) \right] \tau^{r_n} \lambda_{f_1}(g_i)}{M_{f_1} \left[ (\tau_{f_1}(g_k) + \varepsilon) \right] \tau^{r_n} \lambda_{f_1}(g_k)} \) sufficiently small by taking \( n \) sufficiently large. So with the help of Lemma 1 and the second part of Theorem G and using the similar technique of Case I of Theorem 1, we get from (13) that

\[
\tau_{f_1}(g_1 \pm g_2) \leq \tau_{f_1}(g_i).
\]

Now without loss of generality, let us suppose that \( \lambda_{f_1}(g_1) < \lambda_{f_1}(g_2) \) and \( g = g_1 \pm g_2 \). So \( \tau_{f_1}(g) \leq \tau_{f_1}(g_2) \). Also let \( g_2 = \pm (g - g_1) \) and in this case we have from Theorem E that \( \lambda_{f_1}(g_1) < \lambda_{f_1}(g) \). Therefore \( \tau_{f_1}(g_2) \leq \tau_{f_1}(g) \). Hence \( \tau_{f_1}(g) = \tau_{f_1}(g_2) \Rightarrow \tau_{f_1}(g_1 \pm g_2) = \tau_{f_1}(g_2) \).

Thus, \( \tau_{f_1}(g_1 \pm g_2) = \tau_{f_1}(g_i) \) \((i = 1, 2)\) where \( \lambda_{f_1}(g_i) = \max \{\lambda_{f_1}(g_k) \mid k, i = 1, 2\} \) and \( \lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2) \).
Case II. Let us consider that $\lambda_{f_1}(g_k) < \lambda_{f_1}(g_i)$ where $k = i = 1, 2$ with $g_k \neq g_i$. Now, in view of (9), we get for all sufficiently large values of $r$ that

$$M_{g_1 \pm g_2}(r) < M_{g_1}(r) + M_{g_2}(r).$$

That is, we have

$$M_{g_1 \pm g_2}(r) < M_{f_1} \left( (\tau_{f_1}(g_k) + \varepsilon)r^{\lambda_{f_1}(g_k)} \right) + M_{f_1} \left( (\tau_{f_1}(g_i) + \varepsilon)r^{\lambda_{f_1}(g_i)} \right).$$

And so

$$M_{g_1 \pm g_2}(r_n) < M_{f_1} \left( (\tau_{f_1}(g_k) + \varepsilon)r^{\lambda_{f_1}(g_k)} \right) \left[ 1 + \frac{M_{f_1} \left( (\tau_{f_1}(g_k) + \varepsilon)r^{\lambda_{f_1}(g_k)} \right)}{M_{f_1} \left( (\tau_{f_1}(g_i) + \varepsilon)r^{\lambda_{f_1}(g_i)} \right)} \right].$$

As $\lambda_{f_1}(g_k) < \lambda_{f_1}(g_i)$, by taking $r$ sufficiently large one can make the term

$$\frac{M_{f_1} \left( (\tau_{f_1}(g_k) + \varepsilon)r^{\lambda_{f_1}(g_k)} \right)}{M_{f_1} \left( (\tau_{f_1}(g_i) + \varepsilon)r^{\lambda_{f_1}(g_i)} \right)}$$

sufficiently small and therefore for similar reasoning of Case-I we get that $\tau_{f_1}(g_1 \pm g_2) = \tau_{f_1}(g_i) \mid i = 1, 2$ where $\lambda_{f_1}(g_i) = \max \{ \lambda_{f_1}(g_k) \mid k = i = 1, 2 \}$ and $\lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2)$ and hence details of its proof are omitted.

Thus the first part of the theorem follows from Case I and Case II.

Case III. Now suppose that $\lambda_{f_i}(g_1) < \lambda_{f_k}(g_1)$ where $k = i = 1, 2$ with $f_i \neq f_k$.

Now in view of (7) and (10) we have for all sufficiently large values of $r$ that

$$M_{f_1 \pm f_2}(r) < M_{g_1} \left[ \left( \frac{r}{(\tau_{f_1}(g_1) - \varepsilon)} \right)^{\lambda_{f_1}(g_1)} \right] + M_{g_1} \left[ \left( \frac{r}{(\tau_{f_2}(g_1) - \varepsilon)} \right)^{\lambda_{f_2}(g_1)} \right].$$
We thus have

\[ M_{f_1 \pm f_2}(r) < M_{g_1} \left[ \left( \frac{r}{(\tau_{f_1}(g_1) - \varepsilon)} \right)^{-\frac{1}{\lambda_{f_1}(g_1)}} \right] + M_{g_1} \left[ \left( \frac{r}{(\tau_{f_1}(g_1) - \varepsilon)} \right)^{-\frac{1}{\lambda_{f_1}(g_1)}} \right]. \]

Since \( \lambda_{f_1}(g_1) < \lambda_{f_k}(g_1) \), one can make the term

\[ \frac{M_{g_1} \left[ \left( \frac{r}{(\tau_{f_k}(g_1) - \varepsilon)} \right)^{-\frac{1}{\lambda_{f_k}(g_1)}} \right]}{M_{g_1} \left[ \left( \frac{r}{(\tau_{f_1}(g_1) - \varepsilon)} \right)^{-\frac{1}{\lambda_{f_1}(g_1)}} \right]} \]
sufficiently small by taking \( r \) sufficiently large. Therefore using the similar technique as executed in the proof of Case III of Theorem 1, it follows from above arguments and Theorem E that

\[ \tau_{f_1 \pm f_2}(g_1) \geq \tau_{f_1}(g_1). \]

At this time without loss of generality, we may consider that \( \lambda_{f_1}(g_1) < \lambda_{f_2}(g_1) \) and \( f = f_1 \pm f_2 \). Then \( \tau_f(g_1) \geq \tau_{f_1}(g_1) \). Further let \( f_1 = (f \pm f_2) \). Therefore, in view of Theorem C, \( \lambda_f(g_1) < \lambda_{f_2}(g_1) \) and accordingly \( \tau_{f_i}(g_1) \geq \tau_f(g_1) \). Hence \( \tau_f(g_1) = \tau_{f_1}(g_1) \Rightarrow \tau_{f_1 \pm f_2}(g_1) = \tau_{f_1}(g_1) \). So, \( \tau_{f_1 \pm f_2}(g_1) = \tau_{f_i}(g_1) \mid i = 1, 2 \) where \( \lambda_{f_i}(g_1) = \min\{\lambda_{f_k}(g_1) \mid k = i = 1, 2\} \) provided \( \lambda_{f_1}(g_1) \neq \lambda_{f_2}(g_1) \).

**Case IV**. Now let us consider \( \lambda_{f_i}(g_1) < \lambda_{f_k}(g_1) \) where \( k = i = 1, 2 \) with \( f_i \neq f_k \). Therefore in view of (6), (10) and (11) we obtain for a sequence \( \{r_n\} \) of values of \( r \) tending to infinity that

\[ M_{f_1 \pm f_2}(r_n) < M_{g_1} \left[ \left( \frac{r_n}{(\tau_{f_1}(g_1) - \varepsilon)} \right)^{-\frac{1}{\lambda_{f_1}(g_1)}} \right] + M_{g_1} \left[ \left( \frac{r_n}{(\tau_{f_1}(g_1) - \varepsilon)} \right)^{-\frac{1}{\lambda_{f_1}(g_1)}} \right]. \]

We thus have

\[ M_{f_1 \pm f_2}(r_n) < M_{g_1} \left[ \left( \frac{r_n}{(\tau_{f_1}(g_1) - \varepsilon)} \right)^{-\frac{1}{\lambda_{f_1}(g_1)}} \right] + M_{g_1} \left[ \left( \frac{r_n}{(\tau_{f_1}(g_1) - \varepsilon)} \right)^{-\frac{1}{\lambda_{f_1}(g_1)}} \right]. \]

(15)
We thus have
\begin{equation}
M_{g_1} \left( \frac{\tau_{f_k(g_1)}}{r_{f_k(g_1)-\delta}} \right)^{\lambda_{f_k}(g_1)}
\end{equation}

sufficiently small by taking \( n \) sufficiently large. Therefore using the similar technique of Case II of Theorem 1, we obtain the conclusion that
\( \mathfrak{f}_{f_1} \cdot \mathfrak{f}_{f_2}(g_1) = \mathfrak{f}_{f_1}(g_1) \) \( i = 1, 2 \) where \( \lambda_{f_1}(g_1) = \min \{ \lambda_{f_k}(g_1) \mid k = i = 1, 2 \} \) provided \( \lambda_{f_1}(g_1) \neq \lambda_{f_2}(g_1) \) from (15).

So the second part of the theorem follows from Case III and Case IV.

The proof of the third part of the theorem is omitted as it can be carried out in view of Theorem I (ii) and the above cases.

**Proof of Theorem 3.**  **Case I.** By Lemma 3, \( f_1 \) is transcendental. Suppose that \( \rho_{f_1}(g_k) < \rho_{f_1}(g_i) \) where \( k = i = 1, 2 \) with \( g_k \neq g_i \). Now for any arbitrary \( \varepsilon > 0 \), we have from (1) for all sufficiently large values of \( r \) that
\begin{equation}
M_{g_1 \cdot g_2}(r) \leq M_{g_1}(r) \cdot M_{g_2}(r).
\end{equation}

We thus have
\begin{equation}
M_{g_1 \cdot g_2}(r) \leq M_{f_1} \left( \left( \sigma_{f_1}(g_k) + \frac{\varepsilon}{2} \right) r^{\rho_{f_1}(g_k)} \right) \cdot M_{f_1} \left( \left( \sigma_{f_1}(g_i) + \frac{\varepsilon}{2} \right) r^{\rho_{f_1}(g_i)} \right).
\end{equation}

Since \( \rho_{f_1}(g_k) < \rho_{f_1}(g_i) \), we get for all sufficiently large values of \( r \) that \( (\sigma_{f_1}(g_k) + \varepsilon) r^{\rho_{f_1}(g_k)} > (\sigma_{f_1}(g_k) + \varepsilon) r^{\rho_{f_1}(g_i)} \). Therefore \( M_{f_1}[(\sigma_{f_1}(g_k) + \varepsilon) r^{\rho_{f_1}(g_k)}] > M_{f_1}[(\sigma_{f_1}(g_k) + \varepsilon) r^{\rho_{f_1}(g_i)}] \) and from above arguments it follows for all sufficiently large values of \( r \) that
\begin{equation}
M_{g_1 \cdot g_2}(r) < M_{f_1} \left( \left( \sigma_{f_1}(g_k) + \frac{\varepsilon}{2} \right) r^{\rho_{f_1}(g_k)} \right)^2.
\end{equation}

Let us observe that
\( \delta_1 := \frac{\sigma_{f_1}(g_k) + \varepsilon}{\sigma_{f_1}(g_i) + \varepsilon} > 1 \)
\begin{align}
\Rightarrow & \quad \log (\sigma_{f_1}(g_k) + \varepsilon) r^{\rho_{f_1}(g_k)} > \log \left( \sigma_{f_1}(g_k) + \frac{\varepsilon}{2} \right) r^{\rho_{f_1}(g_k)} \\
\Rightarrow & \quad \frac{\log (\sigma_{f_1}(g_k) + \varepsilon) r^{\rho_{f_1}(g_k)}}{\log (\sigma_{f_1}(g_k) + \frac{\varepsilon}{2}) r^{\rho_{f_1}(g_k)}} = \delta \text{ (say) } > 1 \\
\Rightarrow & \quad \log (\sigma_{f_1}(g_k) + \varepsilon) r^{\rho_{f_1}(g_k)} = \delta \log \left( \sigma_{f_1}(g_k) + \frac{\varepsilon}{2} \right) r^{\rho_{f_1}(g_k)}.
\end{align}
Since $f_1$ has the Property (A), in view of Lemma 2, Theorem D and (18) we obtain from (17) for all sufficiently large values of $r$ that

$$M_{g_1\cdot g_2}(r) < M_{f_1}\left[\left(\left(\sigma_{f_1}(g_i) + \frac{\varepsilon}{2}\right) r^{\rho_{f_1}(g_i)}\right)^{\delta}\right]$$

i.e., $M_{g_1\cdot g_2}(r) < M_{f_1}\left[(\sigma_{f_1}(g_i) + \varepsilon) r^{\rho_{f_1}(g_i)}\right]$. That is, we have

$$\frac{M_{f_1}^{-1}M_{g_1\cdot g_2}(r)}{r^{\rho_{f_1}(g_i)}} < (\sigma_{f_1}(g_i) + \varepsilon)$$

i.e., $\frac{M_{f_1}^{-1}M_{g_1\cdot g_2}(r)}{r^{\rho_{f_1}(g_1\cdot g_2)}} < (\sigma_{f_1}(g_i) + \varepsilon)$

(19)

i.e., $\sigma_{f_1}(g_1 \cdot g_2) \leq \sigma_{f_1}(g_i)$.

In order to establish the equality of (19), let us restrict the functions $f_1$ and $g_i$ with the property $2^{\rho_{f_1}(g_i)} \leq 1$ ($i = 1, 2$). Now let $h, h_1, h_2$ and $k$ be any four entire functions such that $h = \frac{h_1}{h_2}$ and $\rho_k(h_1) < \rho_k(h_2)$. So $T_{h}(r) = T_{h_1}(r) \leq T_{h_2}(r) + T_{h_1}(r) + O(1)$. Now, in view of Lemma 4, and as in the line of the procedure of the above proof, it follows that $\sigma_k(h) = \sigma_k\left(\frac{h_1}{h_2}\right) \leq 2^{\rho_k(h_2)}\sigma_k(h_2)$.

Further without loss of any generality, let $g = g_1 \cdot g_2$ and $\rho_{f_1}(g_i) < \rho_{f_1}(g_2)$. Then $\sigma_{f_1}(g) \leq \sigma_{f_1}(g_2)$. Also let $g_2 = \frac{g_1}{g_2}$ and in this case we obtain from above arguments that $\sigma_{f_1}(g_2) \leq \sigma_{f_1}(g)$. Hence $\sigma_{f_1}(g) = \sigma_{f_1}(g_2) \Rightarrow \sigma_{f_1}(g_1 \cdot g_2) = \sigma_{f_1}(g_2)$. Thus, $\sigma_{f_1}(g_1 \cdot g_2) = \sigma_{f_1}(g_i) \mid i = 1, 2$ where $\rho_{f_1}(g_i) = \max\{\rho_{f_1}(g_k) \mid k = i, 1, 2\}$ and $\rho_{f_1}(g_1) \neq \rho_{f_1}(g_2)$.

Next we may suppose that $g = \frac{g_1}{g_2}$ with $g_1, g_2, g$ are all entire functions and also suppose that $\rho_{f_1}(g_2) < \rho_{f_1}(g_1)$. We have $g_1 = g \cdot g_2$. Therefore $\sigma_{f_1}(g_1) = \sigma_{f_1}(g)$ as $\rho_{f_1}(g) > \rho_{f_1}(g_2)$ and $2^{\rho_{f_1}(g_1)} \leq 1$.

**Case II.** In view of Lemma 3, $f_1$ is transcendental. Now let $\rho_{f_1}(g_k) < \rho_{f_1}(g_i)$ where $k, i = 1, 2$ with $g_k \neq g_i$. Therefore from (1) and (4) it follows for a sequence $\{r_n\}$ of values of $r$ tending to infinity that

$$M_{g_1, g_2}(r_n) \leq M_{g_1}(r_n) \cdot M_{g_2}(r_n).$$

That is, we have

$$M_{g_1, g_2}(r_n) \leq M_{f_1}\left[(\sigma_{f_1}(g_k) + \frac{\varepsilon}{2}) r^{\rho_{f_1}(g_k)}\right] \cdot M_{f_1}\left[(\sigma_{f_1}(g_i) + \frac{\varepsilon}{2}) r^{\rho_{f_1}(g_i)}\right].$$

(21)
Since $\rho_{f_1}(g_k) < \rho_{f_1}(g_i)$, so for a sequence of values of $r$ tending to infinity
\[ M_{f_1} \left[ \left( \sigma_{f_1}(g_i) + \frac{\varepsilon}{2} \right) r_n^{\rho_{f_1}(g_i)} \right] > M_{f_1} \left[ \left( \sigma_{f_1}(g_k) + \frac{\varepsilon}{2} \right) r_n^{\rho_{f_1}(g_k)} \right] \]
holds. Therefore, from (21), we have
\[ (22) \quad M_{g_1, g_2}(r_n) < M_{f_1} \left[ \left( \sigma_{f_1}(g_i) + \frac{\varepsilon}{2} \right) r_n^{\rho_{f_1}(g_i)} \right]^2. \]

Now using the similar technique for a sequence of values of $r$ tending to infinity as explored in the proof of Case I, the second part of Theorem 3 I(ii) follows from (22).

Therefore the first part of theorem follows Case I and case II.

Case III. By Lemma 3, $g_1$ is transcendental. Suppose that $\rho_{f_i}(g_1) < \rho_{f_k}(g_1)$ ($k, i = 1, 2$) with $f_i \neq f_k$ and $g_1$ is of regular relative growth with respect to at least any one of $f_1$ or $f_2$.

Therefore in view of (2) and (3), we obtain for a sequence $\{r_n\}$ of values of $r$ tending to infinity that
\[ (23) \quad M_{f_1, f_2}(r_n) \leq M_{f_1}(r_n) \cdot M_{f_2}(r_n). \]
That is, we have
\[ M_{f_1, f_2}(r_n) \leq M_{g_1} \left[ \left( \frac{r_n}{\sigma_{f_1}(g_1) - \frac{\varepsilon}{2}} \right)^{\frac{1}{\sigma_{f_1}(g_1)}} \right] \cdot M_{g_1} \left[ \left( \frac{r_n}{\sigma_{f_k}(g_1) - \frac{\varepsilon}{2}} \right)^{\frac{1}{\sigma_{f_k}(g_1)}} \right]. \]

Now $M_{g_1} \left[ \left( \frac{r_n}{\sigma_{f_1}(g_1) - \frac{\varepsilon}{2}} \right)^{\frac{1}{\sigma_{f_1}(g_1)}} \right] > M_{g_1} \left[ \left( \frac{r_n}{\sigma_{f_k}(g_1) - \frac{\varepsilon}{2}} \right)^{\frac{1}{\sigma_{f_k}(g_1)}} \right]$ because for all sufficiently large values of $n$ and $\rho_{f_i}(g_1) < \rho_{f_k}(g_1)$,
\( \left( \frac{r_n}{\sigma_{f_1}(g_1) - \frac{\varepsilon}{2}} \right)^{\frac{1}{\sigma_{f_1}(g_1)}} > \left( \frac{r_n}{\sigma_{f_k}(g_1) - \frac{\varepsilon}{2}} \right)^{\frac{1}{\sigma_{f_k}(g_1)}} \) hold. Therefore from above arguments, it follows for a sequence of values of $r$ tending to infinity that
\[ (24) \quad M_{f_1, f_2}(r_n) < M_{g_1} \left[ \left( \frac{r_n}{\sigma_{f_1}(g_1) - \frac{\varepsilon}{2}} \right)^{\frac{1}{\sigma_{f_1}(g_1)}} \right]^2. \]

Now we observe that
\[ \delta_1 := \frac{\sigma_{f_1}(g_1) - \frac{\varepsilon}{2}}{\sigma_{f_1}(g_1) - \varepsilon} > 1 \]
\[ \Rightarrow \log \left( \frac{r_n}{(\sigma_{f_i}(g_1) - \varepsilon)} \right)^{1/(\sigma_{f_i}(g_1))} > \log \left( \frac{r_n}{(\sigma_{f_i}(g_1) - \frac{\varepsilon}{2})} \right)^{1/(\sigma_{f_i}(g_1))} \]

\[ \Rightarrow \log \left( \frac{r_n}{(\sigma_{f_i}(g_1) - \varepsilon)} \right)^{1/(\sigma_{f_i}(g_1))} = \delta \text{ (say)} > 1 \]

(25) \[ \Rightarrow \log \left( \frac{r_n}{(\sigma_{f_i}(g_1) - \varepsilon)} \right)^{1/(\sigma_{f_i}(g_1))} = \delta \log \left( \frac{r_n}{(\sigma_{f_i}(g_1) - \varepsilon)} \right)^{1/(\sigma_{f_i}(g_1))}. \]

Since \( g_1 \) has the Property (A), in view of Lemma 2, the first part of Theorem H and (25) we obtain from (24) for a sequence \( \{r_n\} \) of values of \( r \) tending to infinity that

\[ M_{f_1 \cdot f_2} (r_n) < M_{g_1} \left[ \left( \frac{r_n}{(\sigma_{f_i}(g_1) - \varepsilon)} \right)^{\frac{\varepsilon}{\sigma_{f_i}(g_1)}} \right] \]

\[ < M_{g_1} \left[ \left( \frac{r_n}{(\sigma_{f_i}(g_1) - \varepsilon)} \right)^{\frac{1}{\sigma_{f_i}(g_1)}} \right]. \]

That is, we have

\[ (\sigma_{f_i}(g_1) - \varepsilon)^{\rho_{f_i}(g_1)} r_n^{\rho_{f_i}(g_1)} < M_{f_1 \cdot f_2}^{-1} M_{g_1} (r_n) \]

i.e., \( (\sigma_{f_i}(g_1) - \varepsilon) < M_{f_1 \cdot f_2}^{-1} M_{g_1} (r_n) \).

Since \( \varepsilon > 0 \) is arbitrary, it follows from above arguments that

(26) \[ \sigma_{f_1 \cdot f_2} (g_1) \geq \sigma_{f_i} (g_1). \]

In order to establish the equality of (26), let us restrict the functions \( f_i \) and \( g_1 \) with the property \( 2^{\rho_{f_i}(g_1)} \geq 1 \) (\( i = 1, 2 \)). Now let \( h, h_1, h_2 \) and \( k \) be any four entire functions such that \( h = h_1 \leq h_2 \) and \( \rho_{h_1}(k) < \rho_{h_2}(k) \).

So \( T_h (r) = T_{h_1} (r) \leq T_{h_2} (r) \) and \( T_{h_2} (r) + O(1) \). Now in view of Lemma 4 and as in the line of procedure of the above proof, it follows that \( \sigma_{h_1(k)} \leq \sigma_{h} (k) = \sigma_{h_1} (k) \).

Further without loss of any generality, let \( f = f_1 \cdot f_2 \) and \( \rho_{f_i}(g_1) = \rho_{f}(g_1) < \rho_{f_2}(g_1) \). Then \( \sigma_f (g_1) \geq \sigma_{f_1} (g_1) \). Also let \( f_1 = \frac{f_1}{f_2} \) and in
this case we obtain from above arguments that $\sigma_f (g_1) \geq \frac{\sigma_f (g_1)}{2^{2^M g_1}}$. Hence $\sigma_f (g_1) = \sigma_f (g_1)$ implies that $\sigma_{f_1, f_2} (g_1) = \sigma_f (g_1)$. Thus, $\sigma_{f_1, f_2} (g_1) = \sigma_f (g_1)$ (i = 1, 2), where $\rho_f (g_1) = \min \{ \rho_{f_k} (g_1) \} (k = 1, 2), \rho_f (g_1)$ is transcendental. Suppose $\rho_f (g_1) > \rho_{f_k} (g_1)$ and $2^{\rho_f (g_1)} \geq 1 (i = 1, 2)$.

Next one may suppose that $f = \frac{f_1}{f_2}$ with $f_1, f_2, f$ are all entire and $\rho_{f_2} (g_1) < \rho_f (g_1)$. We have $f_2 = f \cdot f_1$. Therefore $\sigma_{f_2} (g_1) = \sigma_f (g_1)$ as $\rho_f (g_1) > \rho_f (g_1)$ and $2^{\rho_f (g_1)} \geq 1 (i = 1, 2)$.

**Case IV.** By Lemma 3, $g_1$ is transcendental. Suppose $\rho_f (g_1) < \rho_{f_k} (g_1)$ (k, i = 1, 2) where $f_i \neq f_k (i \neq k)$ and $g_1$ is of regular relative growth with respect to at least any one of $f_1$ or $f_2$.

Therefore in view of (2) we obtain for all sufficiently large values of $r$ that

$$M_{f_1, f_2} (r) \leq M_{f_1} (r) \cdot M_{f_2} (r).$$

That is, we have

$$M_{f_1, f_2} (r) \leq M_{g_1} \left[ \left( \frac{r}{(\sigma_{f_1} (g_1) - \frac{\rho_{f_1}}{2})} \right)^{\frac{1}{\rho_{f_1} (g_1)}} \right] \cdot M_{g_1} \left[ \left( \frac{r}{(\sigma_{f_2} (g_1) - \frac{\rho_{f_2}}{2})} \right)^{\frac{1}{\rho_{f_2} (g_1)}} \right].$$

Therefore $M_{g_1} \left[ \left( \frac{r}{(\sigma_{f_1} (g_1) - \frac{\rho_{f_1}}{2})} \right)^{\frac{1}{\rho_{f_1} (g_1)}} \right] > M_{g_1} \left[ \left( \frac{r}{(\sigma_{f_2} (g_1) - \frac{\rho_{f_2}}{2})} \right)^{\frac{1}{\rho_{f_2} (g_1)}} \right]$ as $\rho_f (g_1) < \rho_{f_k} (g_1)$ and from above arguments it follows all sufficiently large values of $r$ that

$$M_{f_1, f_2} (r) < M_{g_1} \left[ \left( \frac{r}{(\sigma_{f_1} (g_1) - \frac{\rho_{f_1}}{2})} \right)^{\frac{1}{\rho_{f_1} (g_1)}} \right]^2.$$

Therefore, using the similar technique as in the proof of Case III, for all sufficiently large values of $r$, Theorem 3 II (ii) follows from (28).

Thus the second part of the theorem follows from Case III and Case IV.

Proof of the third part of the theorem is omitted as it can be carried out in view of Theorem J (ii) and the above cases.

**Proof of Theorem 4. Case I.** By Lemma 3, $f_1$ is transcendental. Suppose that $\lambda_{f_1} (g_k) < \lambda_{f_1} (g_i)$ (k, i = 1, 2) with $g_k \neq g_i (k \neq i)$ and at least $g_1$ or $g_2$ is of regular relative growth with respect to $f_1$. Now for
any arbitrary $\varepsilon > 0$, from (9), (12) and (20), we obtain for a sequence \{${r_n}$\} of values of $r$ tending to infinity that

\[ M_{g_1, g_2} (r_n) \leq M_{f_1} \left[ \left( \tau f_1 (g_k) + \frac{\varepsilon}{2} \right) r_{f_1} (g_k) \right] \cdot M_{f_1} \left[ \left( \tau f_1 (g_i) + \frac{\varepsilon}{2} \right) r_{f_1} (g_i) \right]. \]

As $\lambda f_1 (g_k) < \lambda f_1 (g_i)$, we get from above arguments for a sequence \{${r_n}$\} of values of $r$ tending to infinity that

\[ (29) \quad M_{g_1, g_2} (r_n) \leq M_{f_1} \left[ \left( \tau f_1 (g_i) + \frac{\varepsilon}{2} \right) r_{f_1} (g_i) \right]^2. \]

Now using the similar technique as explored in the proof of Case II of Theorem 3, we have from (29) and the second part of Theorem H that

\[ (30) \quad \tau f_1 (g_1 \cdot g_2) \leq \tau f_1 (g_i). \]

In order to establish the equality of (30), let us restrict the functions $f_1$ and $g_i$ with the property $2^{\lambda f_1 (g_i)} \leq 1$ ($i = 1, 2$). Now let $h, h_1, h_2$ and $k$ be any four entire functions such that $h = \frac{\lambda k}{h_1}$ and $\lambda k (h_1) < \lambda k (h_2)$. So $T_h (r) = T_{h_1} (r) \leq T_{h_2} (r) + T_{h_2} (r) + O(1)$. Now in view of Lemma 4 and as in the line of procedure of the above proof, it follows that

\[ \tau_k (h) = \tau_k \left( \frac{\lambda k}{h_1} \right) \leq \tau k (h_2) \sigma_k (h_2). \]

Further without loss of generality, let $g = g_1 \cdot g_2$ and $\lambda f_1 (g_1) < \lambda f_1 (g_2) = \lambda f_1 (g)$. Then $\tau f_1 (g) \leq \tau f_1 (g_2)$. Also let $g_2 = \frac{\lambda k}{g_1}$ and in this case we obtain from above arguments that $\tau f_1 (g_2) \leq 2^{\lambda f_1 (g_1)} \tau f_1 (g) \leq \sigma f_1 (g_2)$. Hence $\tau f_1 (g) = \tau f_1 (g_2) \Rightarrow \tau f_1 (g_1 \cdot g_2) = \tau f_1 (g_2)$. Thus, $\tau f_1 (g_1 \cdot g_2) = \sigma f_1 (g_1) | \ s f_1 (g_2) | i = 1, 2$ where $\lambda f_1 (g_i) = \max \{\lambda f_1 (g_k)\}$ ($k, i = 1, 2$).

Next we may suppose that $g = \frac{\lambda k}{g_1}$ with $g_1, g_2, g$ all entire functions and also suppose that $\lambda f_1 (g_2) < \lambda f_1 (g_1)$. We have $g_1 = g \cdot g_2$. Therefore $\lambda f_1 (g_1) = \lambda f_1 (g)$ as $\lambda f_1 (g) > \lambda f_1 (g_2)$ and $2^{\lambda f_1 (g_2)} \leq 1$.

**Case II.** In view of Lemma 3, $f_1$ is transcendental. Now let $\lambda f_1 (g_k) < \lambda f_1 (g_i)$ ($k, i = 1, 2$) with $g_k \neq g_i$ ($k \neq i$) and at least $g_1$ or $g_2$ is of regular relative growth with respect to $f_1$. Therefore from (16) and (9) it follows for all sufficiently large values of $r$ that

\[ (31) \quad M_{g_1, g_2} (r) \leq M_{f_1} \left[ \left( \tau f_1 (g_k) + \frac{\varepsilon}{2} \right) r_{f_1} (g_k) \right] \cdot M_{f_1} \left[ \left( \tau f_1 (g_i) + \frac{\varepsilon}{2} \right) r_{f_1} (g_i) \right]. \]

Since $\lambda f_1 (g_k) < \lambda f_1 (g_i)$, so for all sufficiently large values of $r$,

\[ M_{f_1} \left[ \left( \tau f_1 (g_k) + \frac{\varepsilon}{2} \right) r_{f_1} (g_k) \right] > M_{f_1} \left[ \left( \tau f_1 (g_i) + \frac{\varepsilon}{2} \right) r_{f_1} (g_i) \right]. \]
holds and therefore from (31) we get for all sufficiently large values of \( r \) that

\[
(32) \quad M_{g_1 \cdot g_2} (r) < M_{f_1} \left( \left( \frac{\tau_{f_1} (g_1) + \frac{\varepsilon}{2}}{2} \right) r^{\lambda_{f_1} (g_1)} \right)^2.
\]

Now using the similar technique of Case I of Theorem 3, Theorem 4 I (i) follows from (32).

Therefore combining Case I and Case II, the first part of the theorem follows.

**Case III.** By Lemma 3, \( g_1 \) is transcendental. Suppose that \( \lambda_{f_i} (g_1) < \lambda_{f_k} (g_1) \) \((k, i = 1, 2)\) with \( f_i \neq f_k \) \((i \neq k)\).

Therefore, in view of (10), we obtain from (27) for all sufficiently large values of \( r \) that

\[
M_{f_1 \cdot f_2} (r) \leq M_{g_1} \left[ \left( \frac{r}{(\tau_{f_1} (g_1) - \frac{\varepsilon}{2})} \right)^{\frac{1}{\lambda_{f_1} (g_1)}} \right] \cdot M_{g_1} \left[ \left( \frac{r}{(\tau_{f_k} (g_1) - \frac{\varepsilon}{2})} \right)^{\frac{1}{\lambda_{f_k} (g_1)}} \right].
\]

As \( \lambda_{f_1} (g_1) < \lambda_{f_k} (g_1) \), we find from above arguments that, for all sufficiently large values of \( r \),

\[
(33) \quad M_{f_1 \cdot f_2} (r) < M_{g_1} \left[ \left( \frac{r^{\gamma}}{(\tau_{f_1} (g_1) - \frac{\varepsilon}{2})} \right)^{\frac{1}{\lambda_{f_1} (g_1)}} \right]^2.
\]

Further using the similar technique as explored in the proof of case II in Theorem 3, we have from (33) and Theorem F that

\[
(34) \quad \tau_{f_1 \cdot f_2} (g_1) \geq \tau_{f_k} (g_1).
\]

In order to establish the equality of (34), let us restrict the functions \( f_i \) and \( g_1 \) with the property \( 2^{\lambda_{f_i} (g_1)} \geq 1 \) \((i = 1, 2)\). Now let \( h, h_1, h_2 \) and \( k \) be any four entire functions such that \( h = \frac{h_1}{h_2} \) and \( \lambda_{h_1} (k) < \lambda_{h_2} (k) \).

So \( T_h (r) = T_{h_1} (r) \leq T_{h_2} (r) + O(1) \). Now in view of Lemma 4 and as in the line of procedure of the above proof, it follows that \( \frac{\tau_{h_1} (k)}{2^{\lambda_{h_2} (k)}} \leq \tau_h (k) = \tau_{h_1} (k) \).

Further without loss of generality, let \( f = f_1 \cdot f_2 \) and \( \lambda_{f_1} (g_1) = \lambda_f (g_1) < \lambda_{f_2} (g_1) \). Then \( \tau_f (g_1) \geq \tau_{f_1} (g_1) \).

Also let \( f_1 = \frac{f}{f_2} \) and in this case we obtain from above arguments that \( \tau_{f_1} (g_1) \geq \frac{\tau_f (g_1)}{2^{\lambda_f (g_1)}} \). Hence \( \tau_f (g_1) = \tau_{f_1} (g_1) \) implies that \( \tau_{f_1 \cdot f_2} (g_1) = \tau_{f_1} (g_1) \). Thus, \( \tau_{f_1 \cdot f_2} (g_1) = \tau_{f_1 \cdot f_2} (g_1) \).
\( \tau_{f_1}(g_1) \ (i = 1, 2) \) where \( \lambda_{f_i}(g_1) = \min \{ \lambda_{f_k}(g_1) \} \ (k = 1, 2) \), \( \lambda_{f_i}(g_1) \neq \lambda_{f_2}(g_1) \) and \( 2^{\lambda_{f_i}(g_1)} \geq 1 \ (i = 1, 2) \).

Next one may suppose that \( f = \frac{f_1}{f_2} \) with \( f_1, f_2, f \) are all entire and \( \lambda_{f_2}(g_1) < \lambda_{f_1}(g_1) \). We have \( f_2 = f \cdot f_1 \). Therefore \( \tau_{f_2}(g_1) = \tau_{f_1}(g_1) \) as \( \lambda_{f_1}(g_1) > \lambda_{f_1}(g_1) \) and \( 2^{\lambda_{f_1}(g_1)} \geq 1 \ (i = 1, 2) \).

**Case IV.** By Lemma 3, \( g_1 \) is transcendental. Suppose \( \lambda_{f_i}(g_1) < \lambda_{f_k}(g_1) \) \( (k, i = 1, 2) \) with \( f_i \neq f_k \) \( (i \neq k) \).

Therefore, in view of (35), (10) and (11), we obtain that, for a sequence \( \{ r_n \} \) of values of \( r \) tending to infinity,

\[
M_{f_1, f_2}(r_n) \leq M_{g_1} \left[ \left( \frac{r_n}{(\tau_{f_1}(g_1)-\frac{1}{2})} \right)^{\frac{1}{\lambda_{f_1}(g_1)}} \right] \cdot M_{g_1} \left[ \left( \frac{r_n}{(\tau_{f_2}(g_1)-\frac{1}{2})} \right)^{\frac{1}{\lambda_{f_1}(g_1)}} \right].
\]

Therefore \( M_{g_1} \left[ \left( \frac{r_n}{(\tau_{f_1}(g_1)-\frac{1}{2})} \right)^{\frac{1}{\lambda_{f_1}(g_1)}} \right] > M_{g_1} \left[ \left( \frac{r_n}{(\tau_{f_2}(g_1)-\frac{1}{2})} \right)^{\frac{1}{\lambda_{f_1}(g_1)}} \right] \) as \( \lambda_{f_1}(g_1) < \lambda_{f_2}(g_1) \) and from above arguments it follows that, for a sequence \( \{ r_n \} \) of values of \( r \) tending to infinity,

\[
M_{f_1, f_2}(r_n) < M_{g_1} \left[ \left( \frac{r_n}{(\tau_{f_1}(g_1)-\frac{1}{2})} \right)^{\frac{1}{\lambda_{f_1}(g_1)}} \right]^2.
\]

Therefore using the similar technique, for all sufficiently large values of \( r \), as in the proof of Case III, the second part of Theorem 4 II (ii) follows from (35).

Thus the second part of the theorem follows from Case III and Case IV.

Proof of the third part of the theorem is omitted as it can be carried out in view of Theorem J (ii) and the above cases.

**Proof of Theorem 5. Case I.** Suppose that \( \rho_{f_1}(g_1) = \rho_{f_2}(g_2) \) \( (0 < \rho_{f_1}(g_1), \rho_{f_2}(g_2) < \infty) \). Now in view of Theorem C it is easy to see that \( \rho_{f_1}(g_1 \pm g_2) \leq \rho_{f_1}(g_1) = \rho_{f_2}(g_2) \). If possible let

\[
\rho_{f_1}(g_1 \pm g_2) < \rho_{f_1}(g_1) = \rho_{f_2}(g_2).
\]

Let \( \sigma_{f_1}(g_1) \neq \sigma_{f_1}(g_2) \). Then in view of Theorem K and (36) we obtain that \( \sigma_{f_1}(g_1) = \sigma_{f_1}(g_1 \pm g_2) = \sigma_{f_2}(g_2) \) which is a contradiction. Hence \( \rho_{f_1}(g_1 \pm g_2) = \rho_{f_1}(g_1) = \rho_{f_2}(g_2) \). Similarly with the help of the first part of Theorem 1, one can obtain the same conclusion under the hypothesis \( \sigma_{f_1}(g_1) \neq \sigma_{f_1}(g_2) \). This proves the first part of the theorem.
Case II. Let us consider that $\rho_{f_1}(g_1) = \rho_{f_2}(g_1)$ \(0 < \rho_{f_1}(g_1), \rho_{f_2}(g_1) < \infty\) and $g_1$ is of regular relative growth with respect to at least any one of $f_1$ or $f_2$.
Therefore in view of the first part of Theorem G, it follows that $\rho_{f_1 \pm f_2}(g_1) \geq \rho_{f_1}(g_1) = \rho_{f_2}(g_1)$ and if possible let
\begin{equation}
\rho_{f_1 \pm f_2}(g_1) > \rho_{f_1}(g_1) = \rho_{f_2}(g_1).
\end{equation}

Let us consider that $\sigma_{f_1}(g_1) \neq \sigma_{f_2}(g_1)$. Then, in view of the Theorem 1 II (i) and (37) we obtain that $\sigma_{f_1}(g_1) = \sigma_{f_1 \pm f_2}(g_1) = \sigma_{f_2}(g_1)$ which is a contradiction. Hence $\rho_{f_1 \pm f_2}(g_1) = \rho_{f_1}(g_1) = \rho_{f_2}(g_1)$. Also in view of Theorem 1 II (ii) one can derive the same conclusion for the condition $\overline{\sigma}_{f_1}(g_1) \neq \overline{\sigma}_{f_2}(g_1)$ and therefore the second part of the theorem is established.

Proof of Theorem 6. Case I. Let $\lambda_{f_1}(g_1) = \lambda_{f_1}(g_2)$ \(0 < \lambda_{f_1}(g_1), \lambda_{f_1}(g_2) < \infty\) and at least $g_1$ or $g_2$ is of regular relative growth with respect to $f_1$. Now, in view of Theorem G(ii), it is easy to see that $\lambda_{f_1}(g_1 \pm g_2) \leq \lambda_{f_1}(g_1) = \lambda_{f_1}(g_2)$. If possible let
\begin{equation}
\lambda_{f_1}(g_1 \pm g_2) < \lambda_{f_1}(g_1) = \lambda_{f_1}(g_2).
\end{equation}

Let $\tau_{f_1}(g_1) \neq \tau_{f_1}(g_2)$. Then in view of Theorem 2 I (i) and (38) we obtain that $\tau_{f_1}(g_1) = \tau_{f_1}(g_1 \pm g_2) = \tau_{f_1}(g_2)$ which is a contradiction. Hence $\lambda_{f_1}(g_1 \pm g_2) = \lambda_{f_1}(g_1) = \lambda_{f_1}(g_2)$. Similarly with the help of Theorem 2 I (ii) one can establish the same conclusion under the hypothesis $\overline{\tau}_{f_1}(g_1) \neq \overline{\tau}_{f_1}(g_2)$. This prove the first part of the theorem.

Case II. Let us consider that $\lambda_{f_1}(g_1) = \lambda_{f_2}(g_1)$ \(0 < \lambda_{f_1}(g_1), \lambda_{f_2}(g_1) \infty\). Therefore in view of Theorem E it follows that $\lambda_{f_1 \pm f_2}(g_1) \geq \lambda_{f_1}(g_1) = \lambda_{f_2}(g_1)$ and if possible let
\begin{equation}
\lambda_{f_1 \pm f_2}(g_1) > \lambda_{f_1}(g_1) = \lambda_{f_2}(g_1).
\end{equation}

Suppose $\tau_{f_1}(g_1) \neq \tau_{f_2}(g_1)$. Then in view of Theorem 2 II (i) and (39) we obtain that $\tau_{f_1}(g_1) = \tau_{f_1 \pm f_2}(g_1) = \tau_{f_2}(g_1)$ which is a contradiction. Hence $\lambda_{f_1 \pm f_2}(g_1) = \lambda_{f_1}(g_1) = \lambda_{f_2}(g_1)$. Analogously with the help of Theorem 2 II (ii), the same conclusion can also be derived under the condition $\underline{\tau}_{f_1}(g_1) \neq \underline{\tau}_{f_2}(g_1)$ and therefore the second part of the theorem is established.

Proof of Theorem 7. Case I. Suppose that $\rho_{f_1}(g_1) = \rho_{f_1}(g_2)$ \(0 < \rho_{f_1}(g_1), \rho_{f_1}(g_2) < \infty\). Now in view of Theorem D it is easy to see
that \( \rho_{f_1}(g_1 \cdot g_2) \leq \rho_{f_1}(g_1) = \rho_{f_1}(g_2) \). If possible let
\[
(40) \quad \rho_{f_1}(g_1 \cdot g_2) < \rho_{f_1}(g_1) = \rho_{f_1}(g_2).
\]

Let \( \sigma_{f_1}(g_1) \neq \sigma_{f_1}(g_2) \). Now in view of Theorem 3 I (i) and (40) we obtain that \( \sigma_{f_1}(g_1) = \sigma_{f_2}\left(\frac{g_1 - g_2}{g_2}\right) = \sigma_{f_1}(g_2) \) which is a contradiction. Hence \( \rho_{f_1}(g_1 \cdot g_2) = \rho_{f_1}(g_1) = \rho_{f_1}(g_2) \). Similarly with the help of Theorem 3 II (ii), one can obtain the same conclusion under the hypothesis \( \overline{\sigma}_{f_1}(g_1) \neq \overline{\sigma}_{f_1}(g_2) \). This proves the first part of the theorem.

**Case II.** Let us consider that \( \rho_{f_1}(g_1) = \rho_{f_2}(g_1) \) \((0 < \rho_{f_1}(g_1), \rho_{f_2}(g_1) < \infty)\) and \( g_1 \) is of regular relative growth with respect to at least any one of \( f_1 \) or \( f_2 \). Therefore in view of the first part of Theorem H, it follows that \( \rho_{f_1 \cdot f_2}(g_1) \geq \rho_{f_1}(g_1) = \rho_{f_2}(g_1) \) and if possible let
\[
(41) \quad \rho_{f_1 \cdot f_2}(g_1) > \rho_{f_1}(g_1) = \rho_{f_2}(g_1).
\]
Further suppose that \( \sigma_{f_1}(g_1) \neq \sigma_{f_2}(g_1) \). Therefore in view of the first part of Theorem 3 II (i) and (37), we obtain that \( \sigma_{f_1}(g_1) = \sigma_{f_2}(g_1) = \sigma_{f_2}(g_1) \) which is a contradiction. Hence \( \rho_{f_1 \cdot f_2}(g_1) = \rho_{f_1}(g_1) = \rho_{f_2}(g_1) \). Likewise with the help of Theorem 3 II (ii), one can obtain the same conclusion under the hypothesis \( \overline{\sigma}_{f_1}(g_1) \neq \overline{\sigma}_{f_2}(g_1) \). This proves the second part of the theorem.

We omit the proof for quotient as it is an easy consequence of the above two cases.

**Proof of Theorem 8. Case I.** Let \( \lambda_{f_1}(g_1) = \lambda_{f_1}(g_2) \) \((0 < \lambda_{f_1}(g_1), \lambda_{f_1}(g_2) < \infty)\) and at least \( g_1 \) or \( g_2 \) is of regular relative growth with respect to \( f_1 \). Now in view of Theorem H (ii) it is easy to see that \( \lambda_{f_1}(g_1 \cdot g_2) \leq \lambda_{f_1}(g_1) = \lambda_{f_1}(g_2) \). If possible let
\[
(42) \quad \lambda_{f_1}(g_1 \cdot g_2) < \lambda_{f_1}(g_1) = \lambda_{f_1}(g_2).
\]

Also let \( \tau_{f_1}(g_1) \neq \tau_{f_1}(g_2) \). Then in view of Theorem 4 I (i) and (42), we obtain that \( \tau_{f_1}(g_1) = \tau_{f_1}\left(\frac{g_1 - g_2}{g_2}\right) = \tau_{f_1}(g_2) \) which is a contradiction. Hence \( \lambda_{f_1}(g_1 \cdot g_2) = \lambda_{f_1}(g_1) = \lambda_{f_1}(g_2) \). Analogously with the help of Theorem 4 I (ii), the same conclusion can also be derived under the condition \( \overline{\tau}_{f_1}(g_1) \neq \overline{\tau}_{f_1}(g_2) \). Hence the first part of the theorem is established.

**Case II.** Let us consider that \( \lambda_{f_1}(g_1) = \lambda_{f_2}(g_1) \) \((0 < \lambda_{f_1}(g_1), \lambda_{f_2}(g_1) < \infty)\). Therefore in view of Theorem F it follows that \( \lambda_{f_1 \cdot f_2}(g_1) \geq \lambda_{f_1}(g_1) = \lambda_{f_2}(g_1) \).
λ_{f_2}(g_1) and if possible let

(43) \quad \lambda_{f_1, f_2}(g_1) > \lambda_{f_1}(g_1) = \lambda_{f_2}(g_1).

Further let $\tau_{f_1}(g_1) \neq \tau_{f_2}(g_1)$. Then in view of the second part of Theorem 4 II (i) and (43) we obtain that $\tau_{f_1}(g_1) = \tau_{f_1, f_2}(g_1) = \tau_{f_2}(g_1)$ which is a contradiction. Hence $\lambda_{f_1, f_2}(g_1) = \lambda_{f_1}(g_1) = \lambda_{f_2}(g_1)$. Similarly by Theorem 4 II (ii), we get the same conclusion when $\tau_{f_1}(g_1) \neq \tau_{f_2}(g_1)$ and therefore the second part of the theorem follows.

We omit the proof for quotient as it is an easy consequence of the above two cases.

5. Concluding Remarks

In this paper, we investigate certain properties of relative type \( (\text{relative lower type}) \) and relative weak type of entire functions. Here we actually prove Theorem 1 to Theorem 4 under some different conditions stated in Theorem A to Theorem J, respectively. Moreover, the treatment of these notions may also be extended for meromorphic functions, in the field of slowly changing functions and also in case of entire or meromorphic functions of several complex variables. Further some natural questions may arise about the sum and product properties for relative type \( (\text{relative lower type}) \) and relative weak type of entire functions when the conditions of Theorem 5 to Theorem 8 are, respectively, provided. Answers of these last questions are left to the interested readers or the involved authors for future study in this research subject.

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