**Abstract.** Adiga and Anitha [1] investigated the Ramanujan’s continued fraction (18) to present many interesting identities. Motivated by this work, by using known formulas, we also investigate the Ramanujan’s continued fraction (18) to give certain relationships between the Ramanujan’s continued fraction and the combinatorial partition identities given by Andrews et al. [3].

1. Introduction and Preliminaries

Throughout this paper, \( \mathbb{N}, \mathbb{Z}, \) and \( \mathbb{C} \) denote the sets of positive integers, integers, and complex numbers, respectively, and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). The following \( q \)-notations are recalled (see, e.g., [8, Chapter 6]): The \( q \)-shifted factorial \( (a; q)_n \) is defined by

\[
(a; q)_n := \begin{cases} 
1 & (n = 0) \\
\prod_{k=0}^{n-1} \left( 1 - a q^k \right) & (n \in \mathbb{N}),
\end{cases}
\]

where \( a, q \in \mathbb{C} \) and it is assumed that \( a \neq q^{-m} \) (\( m \in \mathbb{N}_0 \)). We also write

\[
(a; q)_\infty := \prod_{k=0}^{\infty} \left( 1 - a q^k \right) \quad (a, q \in \mathbb{C}; |q| < 1).
\]
It is noted that, when \(a \neq 0\) and \(|q| \geq 1\), the infinite product in (2) diverges. So, whenever \((a; q)_\infty\) is involved in a given formula, the constraint \(|q| < 1\) will be tacitly assumed.

The following notations are also frequently used:

(3) \[(a_1, a_2, \cdots, a_m; q)_n := (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n\]

and

(4) \[(a_1, a_2, \cdots, a_m; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty.\]

Ramanujan defined the general theta function \(f(a, b)\) as follows (see, for details, [4, p. 31, Eq.(18.1)]; see also [2]):

\[
f(a, b) = 1 + \sum_{n=1}^{\infty} (ab)^{\frac{n(n-1)}{2}} (a^n + b^n)
= \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = f(b, a) \quad (|ab| < 1).
\]

We find from (5) that

\[
f(a, b) = f(a, b)(ab)^n, b(ab)^{-n} = f(b, a) \quad (n \in \mathbb{Z}).
\]

Ramanujan also rediscovered the Jacobi’s famous triple-product identity (see [4, p. 35, Entry 19]):

\[
f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty,
\]

which was first proved by Gauss.

Several \(q\)-series identities emerging from Jacobi’s triple-product identity (7) are worthy of note here (see [4, pp. 36-37, Entry 22]):

\[
\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2 \sum_{n=1}^{\infty} q^n = \frac{(-q; -q)_\infty}{(q; -q)_\infty} = \{(-q; q^2)_\infty\}^2 (q^2; q^2)_\infty = \frac{(-q; q^2)_\infty (q^2; q^2)_\infty}{(q; q^2)_\infty (-q^2; q^2)_\infty}.
\]

(9) \[
\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty};
\]
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\[
f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} \\
= \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} = (q; q)_\infty.
\]

Equation (10) is known as Euler’s Pentagonal Number Theorem. The following \(q\)-series identity:

\[
(-q; q)_\infty = \frac{1}{(q; q^2)_\infty} = \frac{1}{\chi(-q)}
\]
provides the analytic equivalence of Euler’s famous theorem: The number of partitions of a positive integer \(n\) into distinct parts is equal to the number of partitions of \(n\) into odd parts.

The following relationship between the functions \(\phi\) and \(\psi\) is recalled (see [2, p. 36, Entry 25 (iv)]):

\[
\phi(q) = \frac{\psi^2(q)}{\psi(q^2)} \quad \text{and} \quad \phi(q^4) = \frac{\psi^2(q^4)}{\psi(q^8)}.
\]

Andrews et al. [3] investigated new double summation hypergeometric \(q\)-series representations for several families of partitions and further explored the role of double series in combinatorial partition identities by introducing the following general family:

\[
R(s, t, l, u, v, w) := \sum_{n=0}^{\infty} q^{s(n)} r(l, u, v; n),
\]
where

\[
r(l, u, v; n) := \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{q^{uv(j) + (w-ul)j}}{(q; q)_{n-j}(q^{uv}; q^{uv})_j}.
\]

The following interesting special cases of (13) are recalled (see [3, p. 106, Theorem 3]; see also [7]):

\[
R(2, 1, 1, 1, 2, 2) = (-q; q^2)_\infty;
\]
\[
R(2, 2, 1, 1, 2, 2) = (-q^2; q^3)_\infty;
\]
\[
R(m, m, 1, 1, 1, 2) = \frac{(q^{2m}; q^{2m})_\infty}{(q^m; q^{2m})_\infty}.
\]
Ramanujan defined the following continued fraction (see [1, p. 489. Eq. (1.2))):

\[
c(q) := \frac{1}{1+\frac{2q}{1-q^2+\frac{q^2(1+q^2)^2}{1-q^6+\frac{q^4(1+q^4)^2}{1-q^{10}+\cdots}}} \quad (|q| < 1).
\]

Adiga and Anitha [1] investigated the Ramanujan’s continued fraction (18) to present many interesting identities including (for example) the following relation (see [1, Theorem 1.2]):

\[
c(q) = \frac{\phi(q^4)}{\phi(q)}.
\]

Motivated by the work of Adiga and Anitha [1], here, in this paper, we also investigate the Ramanujan’s continued fraction (18) to give certain relationships between the \(c(q)\) and the combinatorial partition identities (13) by using known formulas.

### 2. A Set of Preliminary Results

For our purpose, here, we recall some known identities for the \(c(q)\) (see [1, Theorem 3.1]):

\[
c^2(q) = \frac{\psi^4(q^4)}{\phi^2(q)\psi^2(q^8)};
\]

\[
c^2(q) - c^2(-q) = \frac{-8q\psi^6(q^4)}{\psi^2(q^8)\phi^4(-q^4)};
\]

\[
c^2(q) \cdot c^2(-q) = \frac{\phi^4(q^4)}{\phi^4(-q^4)};
\]

\[
\frac{c^{-1}(q) - c(q)}{c^{-1}(q) + c(q)} = \frac{\phi^2(q) - \phi^2(q^4)}{\phi^2(q) + \phi^2(q^4)}.
\]

### 3. Main Results

Here we present some identities which give certain relationships between the Ramanujan continued fraction and combinatorial partition identities as in the following theorem.
Theorem 3.1. Each of the following identities holds true:

\[(24) \quad c^2(q) = \frac{1}{\phi^2(q)} \cdot \frac{\{R(4, 4, 1, 1, 1, 2)\}^4}{\{R(8, 8, 1, 1, 1, 2)\}^2}; \]

\[(25) \quad c^2(q) = \frac{\{R(2, 2, 1, 1, 1, 2)\}^2 \cdot \{R(4, 4, 1, 1, 1, 2)\}^4}{\{R(1, 1, 1, 1, 1, 2)\}^4 \cdot \{R(8, 8, 1, 1, 1, 2)\}^2}; \]

\[(26) \quad c^2(q) - c^2(-q) = \frac{-8q}{\phi^2(-q^2)} \cdot \frac{\{R(4, 4, 1, 1, 1, 2)\}^6}{\{R(8, 8, 1, 1, 1, 2)\}^2}; \]

\[(27) \quad c^2(q) \cdot c^2(-q) = \frac{1}{\phi^2(-q^2)} \cdot \frac{\{R(4, 4, 1, 1, 1, 2)\}^8}{\{R(8, 8, 1, 1, 1, 2)\}^4}; \]

\[(28) \quad \frac{c^{-1}(q) - c(q)}{c^{-1}(q) + c(q)} = \frac{\mathcal{N}(1, 2, 4, 8)}{\mathcal{D}(1, 2, 4, 8)}, \]

where

\[\mathcal{N}(1, 2, 4, 8) := \{R(1, 1, 1, 1, 1, 2)\}^4 \cdot \{R(8, 8, 1, 1, 1, 2)\}^2 - \{R(4, 4, 1, 1, 1, 2)\}^4 \cdot \{R(2, 2, 1, 1, 1, 2)\}^2\]

and

\[\mathcal{D}(1, 2, 4, 8) := \{R(1, 1, 1, 1, 1, 2)\}^4 \cdot \{R(8, 8, 1, 1, 1, 2)\}^2 + \{R(4, 4, 1, 1, 1, 2)\}^4 \cdot \{R(2, 2, 1, 1, 1, 2)\}^2.\]

Proof. To prove (24), we find from (9) that

\[(29) \quad \psi^4(q^4) = \left\{ \frac{(q^8; q^8)_\infty}{(q^4; q^4)_\infty} \right\}^4 \quad \text{and} \quad \psi^2(q^8) = \left\{ \frac{(q^{16}; q^{16})_\infty}{(q^8; q^8)_\infty} \right\}^2, \]

which, in view of the identity (17) (with \(m = 4\) and \(m = 8\), respectively), yield the following relations:

\[(30) \quad \psi^4(q^4) = \{R(4, 4, 1, 1, 1, 2)\}^4 \quad \text{and} \quad \psi^2(q^8) = \{R(8, 8, 1, 1, 1, 2)\}^2. \]

Then, applying (30) to (20), we obtain the identity (24).

To prove (25), we find from the first identity in (12) that

\[(31) \quad \phi^2(q) = \frac{\psi^4(q)}{\psi^2(q^2)}. \]
Using the formula (9) in (31) and considering the identity (17) (with \(m = 1\) and \(m = 2\), respectively), we obtain

\[
\phi^2(q) = \frac{\{R(1,1,1,1,1,2)\}^4}{\{R(2,2,1,1,1,2)\}^2},
\]

which can be applied to the identity (24) to prove the identity (25).

To prove (26), we find from (9) that

\[
\psi^6(q^4) = \left\{ \frac{(q^8;q^8)_\infty}{(q^4;q^8)_\infty} \right\}^6 \quad \text{and} \quad \psi^2(q^8) = \left\{ \frac{(q^{16};q^{16})_\infty}{(q^8;q^{16})_\infty} \right\}^2.
\]

Then, using (17) in (33) and applying the resulting identity to (21), we get the identity (26).

For (27), applying (9) to the second identity in (12), we have

\[
\psi^8(q^4) = \left\{ \frac{(q^8;q^8)_\infty}{(q^4;q^8)_\infty} \right\}^8 \quad \text{and} \quad \psi^4(q^8) = \left\{ \frac{(q^{16};q^{16})_\infty}{(q^8;q^{16})_\infty} \right\}^4.
\]

Now, using (17) in (34) and applying the resulting identity to (22), we obtain (27).

To prove (28), using (12) and (17), after some rearrangement, we obtain the following identities:

\[
\phi^2(q) = \frac{\{R(1,1,1,1,1,1,2)\}^4}{\{R(2,2,1,1,1,1,2)\}^2} \quad \text{and} \quad \phi^2(q^4) = \frac{\{R(4,4,1,1,1,1,2)\}^4}{\{R(8,8,1,1,1,1,2)\}^2}.
\]

Now, applying (35) to the right-hand side of (23), after a suitable arrangement of the involved terms, we get (28). This completes the proof.

\[\square\]

References


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