INTERVAL-VALUED FUZZY GROUP CONGRUENCES

JEONG GON LEE, KUL HUR AND PYUNG KI LIM∗

Abstract. We introduce the concepts of interval-valued fuzzy complete inner-unitary subsemigroups and interval-valued fuzzy group congruences on a semigroup. And we investigate some of their properties. Also, we prove that there is a one to one correspondence between the interval-valued fuzzy complete inner-unitary subsemigroups and the interval-valued fuzzy group congruences on a regular semigroup.

1. Introduction


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∗Corresponding author.

In this paper, we introduce the concepts of interval-valued fuzzy complete inner-unitary subsemigroups and interval-valued fuzzy group congruences on a semigroup. And we investigate some of their properties. Also, we prove that there is a one to one correspondence between the interval-valued fuzzy complete inner-unitary subsemigroups and the interval-valued fuzzy group congruences on a regular semigroup.

2. Preliminaries

In this section, we list some basic concepts and well-known results which are needed in the later sections.

Throughout this paper, we will denote the unit interval [0, 1] as $I$. For any ordinary relation $R$ on a set $X$, we will denote the characteristic function of $R$ as $\chi_R$.

Let $D(I)$ be the set of all closed subintervals of the unit interval [0, 1]. The elements of $D(I)$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M^L, M^U]$, where $M^L$ and $M^U$ are the lower and the upper end points respectively. Especially, we denote $\mathbf{0} = [0, 0], \mathbf{1} = [1, 1], \text{and } a = [a, a]$ for every $a \in (0, 1)$. We also note that

(i) $\forall M, N \in D(I)$ ($M = N \Leftrightarrow M^L = N^L, M^U = N^U)$,
(ii) $\forall M, N \in D(I)$ ($M \leq N \Leftrightarrow M^L \leq N^L, M^U \leq N^U$).

For every $M \in D(I)$, the complement of $M$, denoted by $M^C$, is defined by $M^C = 1 - M = [1 - M^U, 1 - M^L]$ (See[12]).

Definition 2.1[4,15]. A mapping $A : X \rightarrow D(I)$ is called an interval-valued fuzzy set (in short, IVFS) in $X$, denoted by $A = [A^L, A^U]$, if $A^L, A^U \in I^X$ such that $A^L \leq A^U$, i.e., $A^L(x) \leq A^U(x)$ for each $x \in X$, where $A^L(x)[\text{resp } A^U(x)]$ is called the lower[resp upper] end point of $x$ to $A$. For any $[a, b] \in D(I)$, the interval-valued fuzzy set $A$ in $X$ defined by $A(x) = [A^L(x), A^U(x)] = [a, b]$ for each $x \in X$ is denoted by $\tilde{a}, \tilde{b}$ and if $a = b$, then the IVFS $[a, b]$ is denoted by simply $\tilde{a}$. In particular, $\tilde{0}$ and $\tilde{1}$ denote the interval-valued fuzzy empty set and the interval-valued
fuzzy whole set in $X$, respectively.

We will denote the set of all IVFSs in $X$ as $D(I)^X$. It is clear that set $A = [A, A] \in D(I)^X$ for each $A \in I^X$.

**Definition 2.2** [12]. Let $A, B \in D(I)^X$ and let $\{A_\alpha\}_{\alpha \in \Gamma} \subset D(I)^X$. Then

(i) $A \subset B$ iff $A^L \leq B^L$ and $A^U \leq B^U$.

(ii) $A = B$ iff $A \subset B$ and $B \subset A$.


(iv) $A \cup B = [A^L \lor B^L, A^U \lor B^U]$.

(v) $A \cap B = [A^L \land B^L, A^U \land B^U]$.

**Result 2.1**[12, Theorem 1]. Let $A, B, C \in D(I)^X$ and let $\{A_\alpha\}_{\alpha \in \Gamma} \subset D(I)^X$. Then

(a) $\tilde{0} \subset A \subset \tilde{1}$.

(b) $A \cup B = B \cup A$, $A \cap B = B \cap A$.

(c) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$.

(d) $A, B \subset A \cup B$, $A \cap B \subset A, B$.

(e) $A \cap \left( \bigcup_{\alpha \in \Gamma} A_\alpha \right) = \bigcup_{\alpha \in \Gamma} (A \cap A_\alpha)$.

(f) $A \cup \left( \bigcap_{\alpha \in \Gamma} A_\alpha \right) = \bigcap_{\alpha \in \Gamma} (A \cup A_\alpha)$.

(g) $(\tilde{0})^c = \tilde{1}$, $(\tilde{1})^c = \tilde{0}$.

(h) $(A^c)^c = A$.

(i) $(\bigcup_{\alpha \in \Gamma} A_\alpha)^c = \bigcap_{\alpha \in \Gamma} A_\alpha^c$, $(\bigcap_{\alpha \in \Gamma} A_\alpha)^c = \bigcup_{\alpha \in \Gamma} A_\alpha^c$.

**Definition 2.3**[14]. Let $X$ be a set. Then a mapping $R = [R^L, R^U] : X \prod X \to D(I)$ is called an interval-valued fuzzy relation (in short, IVFR) on $X$.

We will denote the set of all IVFRs on $X$ as $IVR(X)$.

**Definition 2.4**[13]. Let $R \in IVR(X)$. Then the inverse of $R$, $R^{-1}$ is defined by $R^{-1}(x, y) = R(y, x)$, for each $x, y \in X$. 
Definition 2.5[7]. Let \( X \) be a set and let \( R, Q \in \text{IVR}(X) \). Then the composition of \( R \) and \( Q \), \( Q \circ R \), is defined as follows: For any \( x, y \in X \),

\[
(Q \circ R)^L(x, y) = \bigvee_{z \in X} [R^L(x, z) \land Q^L(z, y)]
\]

and

\[
(Q \circ R)^U(x, y) = \bigvee_{z \in X} [R^U(x, z) \land Q^U(z, y)].
\]

Result 2.B[7, Proposition 3.4]. Let \( X \) be a set and let \( R, R_1, R_2, R_3, Q_1, Q_2 \in \text{IVR}(X) \). Then

(a) \( (R_1 \circ R_2) \circ R_3 = R_1 \circ (R_2 \circ R_3) \).

(b) If \( R_1 \subseteq R_2 \) and \( Q_1 \subseteq Q_2 \), then \( R_1 \circ Q_1 \subseteq R_2 \circ Q_2 \). In particular, if \( Q_1 \subseteq Q_2 \), then \( R_1 \circ Q_1 \subseteq R_1 \circ Q_2 \).

(c) \( R_1 \circ (R_2 \cup R_3) = (R_1 \circ R_2) \cup (R_1 \circ R_3) \), \( R_1 \circ (R_2 \cap R_3) = (R_1 \circ R_2) \cap (R_1 \circ R_3) \).

(d) If \( R_1 \subseteq R_2 \), then \( R_1^{-1} \subseteq R_2^{-1} \).

(e) \( (R^{-1})^{-1} = R \), \( (R_1 \circ R_2)^{-1} = R_2^{-1} \circ R_1^{-1} \).

(f) \( (R_1 \cup R_2)^{-1} = R_1^{-1} \cup R_2^{-1} \), \( (R_1 \cap R_2)^{-1} = R_1^{-1} \cap R_2^{-1} \).

Definition 2.6[7]. An IVFR \( R \) on a set \( X \) is called an interval-valued fuzzy equivalence relation (in short, \( \text{IVFER} \)) on \( X \) if it satisfies the following conditions:

(i) it is interval-valued fuzzy reflexive, i.e., \( R(x, x) = [1, 1] \), for each \( x \in X \),

(ii) it is interval-valued fuzzy symmetric, i.e., \( R^{-1} = R \),

(iii) it is interval-valued fuzzy transitive, i.e., \( R \circ R \subseteq R \).

We will denote the set of all IVFERs on \( X \) as \( \text{IVE}(X) \).

Let \( R \) be an IVFER on a set \( X \) and let \( a \in X \). We define a mapping \( Ra : X \rightarrow D(I) \) as follows: For each \( a \in X \),

\[
Ra(x) = R(a, x).
\]

Then clearly \( Ra \in D(I)^X \). In this case, \( Ra \) is called the interval-valued fuzzy equivalence class of \( R \) containing \( a \in X \). The set \( \{Ra : a \in X\} \) is called the interval-valued fuzzy quotient set of \( X \) by \( R \) and denoted by \( X/R \).
Result 2.3 [7, Proposition 3.10]. Let $R$ be an IVFER on a set $X$. Then the following hold:

(a) $Ra = Rb$ if and only if $R(a, b) = [1, 1]$, for any $a, b \in X$.
(b) $R(a, b) = [0, 0]$ if and only if $Ra \cap Rb = \tilde{0}$, for any $a, b \in X$.
(c) $\bigcup_{a \in X} Ra = \tilde{1}$.
(d) There exists the surjection $\pi : X \to X/R$ defined by $\pi(x) = Rx$ for each $x \in X$.

Definition 2.7 [6]. A relation $R$ on a groupoid $S$ is said to be:

(i) left compatible if $(a, b) \in R$ implies $(xa, xb) \in R$, for any $a, b, x \in S$,
(ii) right compatible if $(a, b) \in R$ implies $(ax, bx) \in R$, for any $a, b, x \in S$,
(iii) compatible if $(a, b) \in R$ and $(c, d) \in R$ imply $(ab, cd) \in R$, for any $a, b, c, d \in S$,
(iv) a left [resp. right] congruence on $S$ if it is a left [resp. right] compatible equivalence relation.
(v) a congruence on $S$ if it is both a left and a right congruence on $S$.

It is well-known [6, Proposition I.5.1] that a relation $R$ on a groupoid $S$ is congruence if and only if it is both a left and a right congruence on $S$. We will denote the set of all ordinary congruences on $S$ as $C(S)$.

Now we will introduce the concept of interval-valued fuzzy compatible relation on a groupoid.

Definition 2.8 [11]. An IVFR $R$ on a groupoid $S$ is said to be:

(i) interval-valued fuzzy left compatible if for any $x, y, z \in G$,
\[ R^L(x, y) \leq R^L(zx, zy) \quad \text{and} \quad R^U(x, y) \leq R^U(zx, zy), \]
(ii) interval-valued fuzzy right compatible if for any $x, y, z \in G$,
\[ R^L(x, y) \leq R^L(xz, yz) \quad \text{and} \quad R^U(x, y) \leq R^U(xz, yz), \]
(iii) interval-valued fuzzy compatible if for any $x, y, z, t \in G$,
\[ R^L(x, y) \wedge R^L(z, t) \leq R^L(xz, yz) \quad \text{and} \quad R^U(x, y) \wedge R^U(z, t) \leq R^U(xz, yz). \]
Definition 2.9[11]. An IVFER \( R \) on a groupoid \( S \) is called an:

(i) \emph{interval-valued fuzzy left congruence} (in short, \emph{IVLC}) if it is interval-valued fuzzy left compatible,

(ii) \emph{interval-valued fuzzy right congruence} (in short, \emph{IVRC}) if it is interval-valued fuzzy right compatible,

(iii) \emph{interval-valued fuzzy congruence} (in short, \emph{IVC}) if it is interval-valued fuzzy compatible.

We will denote the set of all IVCs [resp. IVLCs and IVRCs] on \( S \) as \( \text{IVC}(S) \) [resp. \( \text{IVLC}(S) \) and \( \text{IVRC}(S) \)].

Result 2.D[11, Theorem 3.9]. Let \( R \) be a relation on a groupoid \( S \). Then \( R \in \text{C}(S) \) if and only if \( [\chi_R, \chi_R] \in \text{IVC}(S) \).

Let \( R \) be an IVC on a groupoid \( S \) and let \( a \in S \). Then \( Ra \in D(I)^S \) is called an \emph{interval-valued fuzzy congruence class of} \( R \) containing \( a \in S \) and we will denote the set of all interval-valued fuzzy congruence classes of \( R \) as \( S/R \).

3. Interval-valued fuzzy inner-unitary subsemigroups

A (ordinary) subsemigroup \( H \) of a semigroup \( S \) is said to be \emph{inner-unitary} if \( xay, xy \in H \) implies \( a \in H \) for any \( a, x, y \in S \) (See [17]).

Definition 3.1. Let \( S \) be a semigroup and let \( A \in D(I)^S \). Then \( A \) is called an \emph{interval-valued fuzzy inner-unitary subset} of \( S \), if it satisfies the followings: For any \( a, x, y \in S \),
\[
A^L(a) \geq A^L(xay) \land A^L(xy) \quad \text{and} \quad A^U(a) \geq A^U(xay) \land A^U(xy).
\]

Proposition 3.2. Let \( S \) be a semigroup with \( \text{Reg}(S) = \{ a \in S : a = axa \text{ for some } x \in S \} \neq \emptyset \) and let be \( A \) an interval-valued fuzzy inner-unitary subset of \( S \). Then for any \( a, b \in \text{Reg}(S) \) and any \( a', a^* \in V(a), b' \in V(b) \),
\[
(a) \ A^L(ab) \leq A^L(a'a) \land A^L(bb') \quad \text{and} \quad A^U(ab) \leq A^U(a'a) \land A^U(bb'). \\
(b) \ A(a'a) = A(aa'). \\
(c) \ A^L(a') \geq A^L(a) \land A^L(a^2) \quad \text{and} \quad A^U(a') \geq A^U(a) \land A^U(a^2). \\
(d) \ A(aa') = A(aa^*).
\]

Here \( V(a) = \{ x \in S : a = axa \text{ and } x = xax \} \).

Proof. (a) Since \( a = aa'a \), \( A(ab) = A(aa'ab) \). Then
\[
A^L(ab) = A^L(aa'ab) \land A^L(ab) \leq A^L(a'a)
\]
and
\[ A^U(ab) = A^U(aa'ab) \land A^U(ab) \leq A^U(a'a) \]
By the similar arguments, we have
\[ A^L(ab) \leq A^L(bb') \] and \[ A^U(ab) \leq A^U(bb') \].
So (a) hold.
(b) It is clear that \( a' \in \text{Reg}(S) \) and \( a \in V(a') \). By (a),
\[ A^L(a'a) \leq A^L(aa') \land A^L(aa') = A^L(aa') \]
and
\[ A^U(a'a) \leq A^U(aa') \land A^U(aa') = A^U(aa') \].
Similarly, we have
\[ A^L(a'a) \geq A^L(aa') \] and \[ A^U(a'a) \geq A^U(aa') \].
Thus (b) holds.
(c) By Definition 3.1, the proofs are obvious.
(d) It is clear that \( aa^* \in V(aa') \) and \( aa^* \in V(aa^*) \). Then
\[ A^L(aa^*) \geq A^L(aa') \land A^L((aa')^2) \geq A^L(aa') \]
\[ = A^L(aa') \land A^L(aa') = A^L(aa') \]
and
\[ A^U(aa^*) \geq A^U(aa') \land A^U((aa')^2) \geq A^U(aa') \]
\[ = A^U(aa') \land A^U(aa') = A^U(aa') \].
By the similar arguments, we have
\[ A^L(aa') \geq A^L(aa^*) \] and \[ A^U(aa') \geq A^U(aa^*) \].
Thus \( A^L(aa') = A^L(aa^*) \) and \( A^U(aa') = A^U(aa^*) \). So (d) holds. This completes the proof. □

**Definition 3.3[9].** Let \( S \) be a semigroup. Then \( A \in D(I)^S \) is called an **interval-valued fuzzy subsemigroup** (in short, **IVSG**) of \( S \) if for any \( x, y \in S \),
\[ A^L(xy) \geq A^L(x) \land A^L(y) \] and \[ A^U(xy) \geq A^U(x) \lor A^U(y) \].
We will denote the set of all IVSGs of \( S \) as IVSG(\( S \)).

**Definition 3.4[9].** Let \( A \) be an IVFS in a set \( X \) and let \( \lambda, \mu \in I \) with \( \lambda \leq \mu \). Then the set \( A^{[\lambda, \mu]} = \{ x \in X : A^L(x) \geq \lambda \text{ and } A^U(x) \geq \mu \} \) is called a **[\lambda, \mu]-level subset** of \( A \).
It is well-known (See Proposition 4.16 and Proposition 4.17 in [9]) that $A \in \text{IVSG}(S)$ if and only if $A^{[\lambda, \mu]}$ is a subsemigroup of $S$ for each $[\lambda, \mu] \in \text{Im}A$.

**Definition 3.5.** Let $S$ be a semigroup and let $A \in D(I)^S$. Then $A$ is called an interval-valued fuzzy inner-unitary subsemigroup of $S$ if it is an IVSG of $S$ and an interval-valued fuzzy inner-unitary subset of $S$.

**Example 3.6.** Let $S$ be the semigroup of natural numbers with respect to multiplication and let $p$ be a prime number. Let $A : S \to D(I)^S$ be the mapping defined as follows: For each $x \in S$,

$$A(x) = \begin{cases} 
[1,1] & \text{if } x = 1, \\
[0.3,0.6] & \text{if } x = p^t, t = 1, 2, \cdots, \\
[0,0] & \text{otherwise}.
\end{cases}$$

Then $A$ is an interval-valued fuzzy inner-unitary subsemigroup of $S$. □

**Theorem 3.7.** Let $S$ be a semigroup. Then $A$ is an interval-valued fuzzy inner-unitary subsemigroup of $S$ if and only if $A^{[\lambda, \mu]}$ is an inner-unitary subsemigroup of $S$ for each $[\lambda, \mu] \in \text{Im}A$.

**Proof** ($\Rightarrow$) : Suppose $A$ is an interval-valued fuzzy inner-unitary subsemigroup of $S$. Let $[\lambda, \mu] \in \text{Im}A$ and let $x, y \in A^{[\lambda, \mu]}$. Then

$$A^L(x) \geq \lambda, \ A^U(x) \geq \mu \text{ and } A^L(y) \geq \lambda, \ A^U(y) \geq \mu.$$ Since $A \in \text{IVSG}(S)$,

$$A^L(xy) \geq A^L(x) \land A^L(y) \geq \lambda \text{ and } A^U(xy) \geq A^U(x) \land A^U(y) \geq \mu.$$ Thus $A^L(xy) \geq \lambda$ and $A^U(y) \geq \mu$, i.e., $xy \in A^{[\lambda, \mu]}$. So $A^{[\lambda, \mu]}$ is a subsemigroup of $S$.

For any $a, x, y \in S$, suppose $xay \in A^{[\lambda, \mu]}$ and $xy \in A^{[\lambda, \mu]}$. Then

$$A^L(xay) \geq \lambda, \ A^U(xay) \geq \mu \text{ and } A^L(xy) \geq \lambda, \ A^U(xy) \geq \mu.$$ Since $A$ is an interval-valued fuzzy inner-unitary subsemigroup of $S$,

$$A^L(a) \geq A^L(xay) \land A^L(xy) \geq \lambda$$

and

$$A^U(a) \geq A^U(xay) \land A^U(xy) \geq \mu.$$
Then $a \in A^{[\lambda, \mu]}$. So $A^{[\lambda, \mu]}$ is an inner-unitary subset of $S$. Hence $A^{[\lambda, \mu]}$ is an inner-unitary subsemigroup of $S$.

$(\Leftarrow)$: Suppose the necessary condition holds. For any $x, y, a \in S$, let $A(x) = [t_1, s_1]$, $A(y) = [t_2, s_2]$ and let $\lambda = t_1 \wedge t_2$, $\mu = s_1 \wedge s_2$. Then

$$A^L(x) = t_1 \geq t_1 \wedge t_2 = \lambda, \quad A^U(x) = s_1 \geq s_1 \wedge s_2 = \mu$$

and

$$A^L(y) = t_2 \geq t_1 \wedge t_2 = \lambda, \quad A^U(y) = s_2 \geq s_1 \wedge s_2 = \mu.$$ 

Thus $x, y \in A^{[\lambda, \mu]}$. Since $A^{[\lambda, \mu]}$ is a subsemigroup of $S$, $xy \in A^{[\lambda, \mu]}$. So

$$A^L(xy) \geq \lambda = t_1 \wedge t_2 = A^L(x) \wedge A^L(y)$$

and

$$A^U(xy) \geq \mu = s_1 \wedge s_2 = A^U(x) \wedge A^U(y).$$

Hence $A \in \IVSG(S)$.

Now let $A(xy) = [t_1, s_1]$, $A(xay) = [t_2, s_2]$ and let $\lambda = t_1 \wedge t_2$, $\mu = s_1 \wedge s_2$. Then

$$A^L(xy) \geq \lambda, \quad A^U(xy) \geq \mu$$

and

$$A^L(xay) \geq \lambda, \quad A^U(xay) \geq \mu.$$ 

Thus $xy \in A^{[\lambda, \mu]}$ and $xay \in A^{[\lambda, \mu]}$. Since $A^{[\lambda, \mu]}$ is inner-unitary, $a \in A^{[\lambda, \mu]}$. So

$$A^L(a) \geq \lambda = A^L(xay) \wedge A^L(xy)$$

and

$$A^U(a) \geq \mu = A^U(xay) \wedge A^U(xy).$$

Hence $A$ is an interval-valued fuzzy inner-unitary subsemigroup of $S$. Therefore $A$ is an interval-valued fuzzy inner-unitary subsemigroup of $S$. □

**Proposition 3.8.** Let $A$ be an interval-valued fuzzy inner-unitary subsemigroup of a semigroup $S$. Then

(a) $A(xy) = A(yx)$ for any $x, y \in S$.

(b) $A^L(a) \geq A^L(ax) \wedge A^L(x)$ [resp. $A^L(a) \geq A^L(xa) \wedge A^L(x)$] and

$$A^U(a) \geq A^U(ax) \wedge A^U(x)$$ [resp. $A^U(a) \geq A^U(xa) \wedge A^U(x)$], for any $a, x \in S$.

(c) If $E_S \neq \phi$, then $A(ef) = [A^L(e) \wedge A^L(f), A^U(e) \wedge A^U(f)]$, for any $e, f \in E_S$, where $E_S = \{x \in S : x^2 = x\}$. 

(d) If Reg(S) ≠ φ, then A(a') = A(a) for each a ∈ Reg(S) and each a' ∈ V(a).

**Proof.** (a) Let x, y ∈ S. Then, by Definition 3.1,

\[ A_L(xy) ≥ A_L(xyx) \land A_L(yx) \]

and

\[ A_U(xy) ≥ A_U(xyx) \land A_U(yx). \]

Since \( A \in \text{IVSG}(S) \),

\[ A_L(xyx) = A_L((yx)^2) ≥ A_L(yx) \land A_L(yx) = A_L(yx) \]

and

\[ A_U(xyx) = A_U((yx)^2) ≥ A_U(yx) \land A_U(yx) = A_U(yx). \]

Thus \( A_L(xy) ≥ A_L(yx) \) and \( A_U(xy) ≥ A_U(yx) \). By the similar arguments, we have

\[ A_L(xy) ≤ A_L(yx) \text{ and } A_U(xy) ≤ A_U(yx). \]

So \( A(xy) = A(yx) \).

(b) It is obvious from Definition 3.4.

(c) Let \( e, f \in ES \). Then clearly \( e \in V(e) \) and \( f \in V(f) \). Then, by Proposition 3.2(a),

\[ A_L(ef) ≤ A_L(ee) \land A_L(ff) = A_L(e) \land A_L(f) \]

and

\[ A_U(ef) ≤ A_U(ee) \land A_U(ff) = A_U(e) \land A_U(f). \]

Since \( A \in \text{IVSG}(S) \),

\[ A_L(ef) ≥ A_L(e) \land A_L(f) \text{ and } A_U(ef) ≥ A_U(e) \land A_U(f). \]

Thus \( A_L(ef) = A_L(e) \land A_L(f) \) and \( A_U(ef) = A_U(e) \land A_U(f) \).

(d) Let \( a ∈ \text{Reg}(S) \) and let \( a' ∈ V(a) \). Then, by Proposition 3.2(c),

\[ A_L(a') ≥ A_L(a) \land A_L(a^2) \text{ and } A_U(a') ≥ A_U(a) \land A_U(a^2). \]

Since \( A \in \text{IVSG}(S) \), \( A_L(a^2) ≥ A_L(a) \) and \( A_U(a^2) ≥ A_U(a) \). Thus \( A_L(a') ≥ A_L(a) \) and \( A_U(a') ≥ A_U(a) \). On the other hand,

\[ A_L(a) ≥ A_L(a'aa') \land A_L(a'a') ≥ A_L(a') \]

and

\[ A_U(a) ≥ A_U(a'aa') \land A_U(a'a') ≥ A_U(a'). \]

So \( A(a') = A(a) \). This completes the proof. □

**Theorem 3.9.** Let \( A \) be an IVSG of a semigroup \( S \). Then \( A \) is interval-valued fuzzy inner-unitary if and only if
(a) $A(xy) = A(yx)$ for any $x, y \in S$.
(b) $A^L(a) \geq A^L(ax) \land A^L(x)$, $A^U(a) \geq A^U(ax) \land A^U(x)$
or
$A^L(a) \geq A^L(xa) \land A^L(x)$, $A^U(a) \geq A^U(xa) \land A^U(x)$,
for any $a, x \in S$.

**Proof.** $(\Rightarrow)$: It is obvious from Proposition 3.8.
$(\Leftarrow)$: Suppose the necessary conditions hold. Let $a, x, y \in S$. Then

\[ A^L(a) \geq A^L(a(ey)) \land A^L(xy) \] [By (b)]
\[ = A^L((ay)x) \land A^L(yx) \]
\[ = A^L(xay) \land A^L(xy) \] [By (a)]

and

\[ A^U(a) \geq A^U(a(yx)) \land A^U(yx) \] [By (b)]
\[ = A^U((ay)x) \land A^U(yx) \]
\[ = A^U(xay) \land A^U(xy) \] [By (a)]

Hence $A$ is an interval-valued fuzzy inner-unitary subset of $S$. □

For a semigroup $S$, $S^1$ is the monoid defined as follows:

\[
S^1 = \begin{cases} 
S & \text{if } S \text{ has the identity,} \\
S \cup \{1\} & \text{otherwise} \end{cases}
\]

**Proposition 3.10.** Let $A$ be an interval-valued fuzzy inner-unitary subsemigroup of a semigroup $S$. Then

(a) If $e \in ES$ and $a \in S$ such that $aRe$ (or $aLe$), then $A^L(a) \leq A^L(e)$ and $A^U(a) \leq A^U(e)$, where $aRe$ if and only if $aS^1 = eS^1$ and $aLe$ if and only if $S^1a = S^1e$.

(b) If $a \in \text{Reg}(S)$ and $a' \in V(a)$, then $A^L(a) \leq A^L(aa')$ and $A^U(a) \leq A^U(aa')$.

**Proof.** (a) Suppose $a = e$. Then $A^L(a) = A^L(e)$ and $A^U(a) = A^U(e)$.

Suppose $a \neq e$. Then there exist $x, y \in S$ such that $e = ax$ and $a = ey$. Thus

\[ A^L(e) \geq A^L(ea) \land A^L(a) \] [By Theorem 3.8(b)]
\[ = A^L(eey) \land A^L(a) \]
\[ = A^L(ey) \land A^L(a) \] [Since $e \in ES$]
\[ = A^L(a) \land A^L(a) = A^L(a). \]
Similarly, we have $A^U(e) \geq_U (a)$. So (a) holds.

(b) Let $a \in \text{Reg}(S)$ and let $a' \in V(a)$. Then clearly, $aRa' - aLa'a$. Moreover, $aa', a'a \in E_S$. Thus, by (a),

$$A^L(a) \leq A^L(aa'), A^U(a) \leq A^U(aa')$$

and

$$A^L(a) \leq A^L(a'a), A^U(a) \leq A^U(a'a).$$

This completes the proof. □

In [17], Zhang has defined the complete inner-unitary subsemigroup of a semigroup $S$ as follows: Let $S$ be a semigroup with $E_S \neq \phi$. An inner-unitary subsemigroup $H$ of $S$ is said to be complete if $E_S \subseteq H$.

Similarly, we will give the concept of interval-valued fuzzy complete inner-unitary subsemigroup of a semigroup $S$.

**Definition 3.11.** Let $S$ be a semigroup with $E_S \neq \phi$ and let $A$ be an interval-valued fuzzy inner-unitary subsemigroup of $S$. Then $A$ is said to be complete if $A(e) = [1, 1]$ for each $e \in E_S$.

**Theorem 3.12.** Let $S$ be a semigroup with $E_S \neq \phi$ and let $A \in D(I)^S$. Then $A$ is an interval-valued fuzzy complete inner-unitary subsemigroup of $S$ if and only if $A^{[\lambda, \mu]}$ is a complete inner-unitary subsemigroup of $S$ for each $[\lambda, \mu] \in D(I)$.

**Proof.** ($\Rightarrow$) : Suppose $A$ is an interval-valued fuzzy complete inner-unitary subsemigroup of $S$. Let $[\lambda, \mu] \in D(I)$. Then, by Theorem 3.7, $A^{[\lambda, \mu]}$ is an inner-unitary subsemigroup of $S$. Let $e \in E_S$. Since $A$ is interval-valued fuzzy complete, $A(e) = [1, 1]$. Then $A^L(e) = 1 \geq \lambda$ and $A^U(e) = 1 \geq \mu$. Thus $e \in A^{[\lambda, \mu]}$. So $E_S \subseteq A^{[\lambda, \mu]}$, i.e., $A^{[\lambda, \mu]}$ is complete. So $A^{[\lambda, \mu]}$ is a complete inner-unitary subsemigroup of $S$.

($\Leftarrow$) : Suppose the necessary condition holds. Then, by Theorem 3.7, $A$ is an interval-valued fuzzy inner-unitary subsemigroup of $S$. Let $e \in E_S$. Since $A^{[1,1]}$ is complete, $e \in A^{[1,1]}$. Then $A^L(e) \geq 1$ and $A^U(e) \geq 1$. Thus $A(e) = [1, 1]$. So $A$ is interval-valued fuzzy complete. Hence $A$ is an interval-valued fuzzy complete inner-unitary subsemigroup of $S$. This completes the proof. □

**Theorem 3.13.** Let $S$ be a regular semigroup and let $A$ be an interval-valued fuzzy inner-unitary subsemigroup of $S$. Then $A$ is interval-valued fuzzy complete if and only if

(a) $A(\varepsilon a) = A(a)$, for each $e \in E_S$ and each $a \in S$. 

(b) For each $a \in S$, there is an $x \in S$ such that $A(ax) = [1, 1]$.

**Proof.** ($\Rightarrow$): Suppose $A$ is interval-valued fuzzy complete. Let $e \in ES$ and let $a \in S$. Then

$$A^L(ea) \geq A^L(e) \land A^L(a) \quad [\text{Since } A \in IVSG(S)]$$

$$= 1 \land A^L(a) \quad [\text{Since } A(e) = [1, 1]]$$

$$= A^L(a).$$

Similarly, we have $A^U(ea) \geq A^U(a)$. Since $A$ is an interval-valued fuzzy inner-unitary subsemigroup of $S$, by Theorem 3.9(b),

$$A^L(a) \geq A^L(ea) \land A^L(e)$$

$$= A^L(ea) \land 1 \quad [\text{Since } A(e) = [1, 1]]$$

$$= A^L(ea).$$

By the similar arguments, we have $A^U(ea) \geq A^U(a)$. Thus $A(ea) = A(a)$. So the condition (a) holds.

Since $S$ is regular, $V(a) \neq \emptyset$ for each $a \in S$. Let $x \in V(a)$. Then clearly $ax \in ES$. Since $A$ is interval-valued fuzzy complete, $A(ax) = [1, 1]$. So the condition (b) holds.

($\Leftarrow$): Suppose the necessary conditions hold. Let $e \in ES$. Then, by the condition (b), there exists an $x \in S$ such that $A(ex) = [1, 1]$. Since $A$ is an interval-valued fuzzy inner-unitary subsemigroup of $S$, by Theorem 3.9(b),

$$A^L(e) \geq A^L(ex) \land A^L(x) = A^L(x)$$

and

$$A^U(e) \geq A^U(ex) \land A^U(x) = A^U(x)$$

Also

$$A^L(x) \geq A^L(ex) \land A^L(x) = A^L(e)$$

and

$$A^U(x) \geq A^U(ex) \land A^U(x) = A^U(e).$$

Thus $A(e) = A(x)$. On the other hand, by the condition (a), $A(ex) = A(x)$. So $A(e) = [1, 1]$. Hence $A$ is interval-valued fuzzy complete. This completes the proof.$\square$

**Proposition 3.14.** Let $S$ be a regular semigroup and let $A$ an interval-valued fuzzy complete inner-unitary subsemigroup of $S$. Then $A(a'b) =$
A(a*b) for any a, b ∈ S and a', a* ∈ V(a).

**Proof.** Let a, b ∈ S and let a', a* ∈ V(a), b' ∈ V(b). Then

\[ A^L(a'b) \geq A^L(a(a'b)b') \wedge A^L(ab') \]

[Since A is interval-valued fuzzy inner-unitary ]

\[ = A^L((aa')(bb')) \wedge A^L(ab') \]

\[ = A^L(aa') \wedge A^L(bb') \wedge A^L(ab') \]

[Since aa', bb' ∈ Es, by Proposition 3.8(c)]

\[ = 1 \wedge 1 \wedge A^L(ab') \]

[Since A interval-valued fuzzy complete]

\[ = A^L(ab') \]

\[ \geq A^L(a^*(ab')b) \wedge A^L(a^*b) \]

[Since A is interval-valued fuzzy inner-unitary ]

\[ = A^L(a^*b). \]

Similarly, we have \( A^U(a'b) \geq A^U(a^*b) \). By the similar arguments, we can show that

\( A^L(a^*b) \geq A^L(a'b) \) and \( A^U(a^*b) \geq A^U(a'b) \).

So \( A(a'b) = A(a^*b) \). This completes the proof. □

### 4. Interval-valued fuzzy group congruences

Let S be a semigroup. Then it is easy to show that an interval-valued fuzzy relation R on S is an IVC if and only if \( R^{[\lambda, \mu]} = \{(a, b) ∈ S × S : R^L(a, b) ≥ \lambda \text{ and } R^U(a, b) ≥ \mu \} \) is a congruence on S for each \([\lambda, \mu] ∈ D(I)\).

**Definition 4.1.** Let R be an interval-valued fuzzy congruence on a semigroup S. Then R is called an interval-valued fuzzy group congruence (in short, IVGC) if \( R^{[\lambda, \mu]} \) is a group congruence, i.e, \( S/R^{[\lambda, \mu]} \) is a group, for each \([\lambda, \mu] ∈ D(I)\).

We will denote the set of all IVGCs on S as IVGC(S).

**Theorem 4.2.** Let R be an interval-valued fuzzy congruence on a regular semigroup S. Then R ∈ IVGC(S) if and only if \( R(e, f) = [1, 1] \), for any \( e, f ∈ E_S \).
Proof. ($\Rightarrow$): Suppose $R \in \text{IVGC}(S)$. Then, by Definition 4.1, $S/R^{[\lambda, \mu]}$ is a group for each $[\lambda, \mu] \in D(I)$. Let $e \in E_S$. Then clearly $eR^{[\lambda, \mu]} \in S/R^{[\lambda, \mu]}$. Moreover, $(eR^{[\lambda, \mu]})(eR^{[\lambda, \mu]}) = eR^{[\lambda, \mu]}$. Then $eR^{[\lambda, \mu]} \in E_{S/R^{[\lambda, \mu]}}$. Since $S/R^{[\lambda, \mu]}$ is a group, $S/R^{[\lambda, \mu]}$ has only one idempotent element which is the identity of $S/R^{[\lambda, \mu]}$. Thus $eR^{[\lambda, \mu]} = fR^{[\lambda, \mu]}$ for any $e, f \in E_S$. Thus $(e, f) \in R^{[\lambda, \mu]}$ for any $e, f \in E_S$. In particular, $(e, f) \in R^{[1, 1]}$, i.e., $R^L(e, f) \geq 1$ and $R^L(e, f) \geq 1$ for any $e, f \in E_S$. So $R(e, f) = [1, 1]$ for any $e, f \in E_S$.

($\Leftarrow$): Suppose the necessary condition holds. Let $[\lambda, \mu] \in D(I)$. Then clearly $R^{[\lambda, \mu]}$ is a congruence on $S$. Since $S$ is a semigroup, $S/R^{[\lambda, \mu]}$ is a semigroup. Let $e, f \in E_S$. Then, by the hypothesis, $R(e, f) = [1, 1]$. Then $(e, f) \in R^{[\lambda, \mu]}$. So $eR^{[\lambda, \mu]} = fR^{[\lambda, \mu]}$. Let $aR^{[\lambda, \mu]} \in S/R^{[\lambda, \mu]}$ and let $a' \in V(a)$. Since $aa' \in E_S$, $R(aa', e) = [1, 1]$. Then $(aa', e) \in R^{[\lambda, \mu]}$, i.e., $aa'R^{[\lambda, \mu]} = eR^{[\lambda, \mu]}$. Similarly, $a'aR^{[\lambda, \mu]} = eR^{[\lambda, \mu]}$. Thus $aa'R^{[\lambda, \mu]} = a'aR^{[\lambda, \mu]} = eR^{[\lambda, \mu]}$. So,

$$(eR^{[\lambda, \mu]})(aR^{[\lambda, \mu]}) = (aa'R^{[\lambda, \mu]})(aR^{[\lambda, \mu]}) = aa'aR^{[\lambda, \mu]} = aRa^{[\lambda, \mu]}.$$ 

By the similar arguments, we have that 

$$(aR^{[\lambda, \mu]})(eR^{[\lambda, \mu]}) = aR^{[\lambda, \mu]}.$$ 

Hence $eR^{[\lambda, \mu]}$ is the identity of $S/R^{[\lambda, \mu]}$. On the other hand, 

$$(aR^{[\lambda, \mu]})(a'R^{[\lambda, \mu]}) = aa'R^{[\lambda, \mu]} = eR^{[\lambda, \mu]}$$ 

and

$$(a'R^{[\lambda, \mu]})(aR^{[\lambda, \mu]}) = a'aR^{[\lambda, \mu]} = eR^{[\lambda, \mu]}.$$ 

Thus $a'R^{[\lambda, \mu]}$ is the inverse of $aR^{[\lambda, \mu]} \in S/R^{[\lambda, \mu]}$. So $S/R^{[\lambda, \mu]}$ is a group for each $[\lambda, \mu] \in D(I)$. Therefore $R \in \text{IVGC}(S)$. This completes the proof. $\square$

Proposition 4.3. Let $S$ be a regular semigroup. If $R \in \text{IVC}(S)$, then $R(aa', a) = R(a', aa')$ for each $a \in S$ and each $a' \in V(a)$.

Proof. Let $a \in S$ and let $a' \in V(a)$. Then

$$R^L(aa', a) = R^L(a', aa') \geq R^L(a', a')$$ [Since $R$ is interval-valued fuzzy compatible and reflexive]

$$= R^L(a', a')$$ [Since $R$ is interval-valued fuzzy symmetric]

$$= R^L(aa', a)$$ [Since $R$ is interval-valued fuzzy symmetric].
Similarly, we have that
\[ R^U(aa', a) \geq R^U(a'a, a') \geq R^U(aa', a). \]
Hence \( R(aa', a) = R(a'a, a'). \) □

**Proposition 4.4.** Let \( S \) be a regular semigroup. If \( R \in IVGC(S) \), then \( R(aa', a) = R(aa^*, a) \) for each \( a \in S \) and any \( a', a^* \in V(a) \).

**Proof.** Let \( a \in S \) and let \( a', a^* \in V(a) \). Then
\[
R^L(aa', a) = R^L(aa*aa', aa'a) \\
\geq R^L(aa*aa', aad') \land R^L(aa'a, aa'a) \\
[\text{Since } R \text{ is interval-valued fuzzy transitive}] \\
\geq R^L(aa^*, a) \land R^L(aa', a'a) \\
[\text{Since } R \text{ is interval-valued fuzzy compatible and reflexive}] \\
= R^L(aa^*, a) \land 1 \text{ [Since } aa', a'a \in E_S, \text{ by Theorem 4.2]} \\
= R^L(aa^*, a).
\]
Similarly, we have \( R^U(aa', a) \geq R^U(aa^*, a) \).

Also, by the similar arguments, we have that
\[
R^L(aa^*, a) \geq R^L(aa', a) \text{ and } R^U(aa^*, a) \geq R^U(aa', a).
\]
Hence \( R(aa', a) = R(aa^*, a) \). This completes the proof. □

**Proposition 4.5.** Let \( R \) be an IVGC on a regular semigroup \( S \). Then
\( R(a, b) = R(a, be) = R(a, eb) = R(ae, b) = R(ce, b), \) \( R(a, bc) = R(ac', b) = R(b'a, c), \)
\( R(ab, cd) = R(c'a, db'), \) for any \( a, b, c, d \in S, e \in E_S \) and \( a' \in V(a), b' \in V(b), c' \in V(c) \).

**Proof.** (a) Let \( a, b \in S \), let \( e \in E_S \) and let \( a' \in V(a), b' \in V(b) \). Then
\[
R^L(a, be) = R^L(aa'a, be) \\
\geq R^L(aa'a, ba'a) \land R^L(ba'a, be) \text{ [Since } R \text{ is transitive}] \\
\geq R^L(a, b) \land R^L(a'a, e) \\
[\text{Since } R \text{ is interval-valued fuzzy compatible and intuitionistic fuzzy reflexive}] \\
= R^L(a, b). \text{ [Since } a'a, e \in E_S, R^L(a'a, e) = 1]\]
Similarly, we have \( R^U(a, be) \geq R^U(a, b) \). Also, by the similar arguments, we have that
\[
R^L(a, b) \geq R^L(a, be) \text{ and } R^U(a, b) \geq R^U(a, be).
\]
So $R(a, b) = R(a, be)$.

By the similar arguments, we have that

$$R(a, b) = R(a, eb) = R(ea, b) = R(a, e).$$

(b) Let $a, b, c \in S$ and let $a' \in V(a), b' \in V(b), c' \in V(c)$. Then

$$R^L(a, bc) = R^L(bb'a, bc) \quad [\text{Since } bb' \in E_S, \ \text{by}(a)]$$

$$\geq R^L(b'a, c) \quad [\text{Since } R \text{ is interval-valued fuzzy compatible and reflexive } ]$$

$$= R^L(b'a, b'c) \quad [\text{by}(a)]$$

$$\geq R^L(a, bc). \quad [\text{Since } R \text{ is interval-valued fuzzy compatible and reflexive } ]$$

Similarly, we have that

$$R^U(a, bc) \geq R^U(b'a, c) \geq R^U(a, bc).$$

So $R(a, bc) = R(b'a, c)$. Also, by the similar arguments, we have that

$$R(a, bc) = R(ac', b).$$

(c) It follows immediately from (b). This completes the proof. $\square$

5. Correspondence between interval-valued fuzzy complete inner-unitary subsemigroups and interval-valued fuzzy group congruences on a regular semigroup

In this section, $S$ always denotes a regular semigroup. We shall show that each interval-valued fuzzy complete inner-unitary subsemigroup of $S$ determines one interval-valued fuzzy group congruence on $S$ and the converse.

Lemma 5.1. Let $A$ be any interval-valued fuzzy complete inner-unitary subsemigroup of $S$. Define a mapping $R_A = [R^L_A, R^U_A] : S \times S \to D(I)$ as follows: For any $a, b \in S$,

$$R^L_A(a, b) = A^L(a'b) \quad \text{and} \quad R^U_A(a, b) = A^U(a'b),$$

where $a' \in V(a)$. Then $R_A \in \text{IVGC}(S)$.

Proof. Let $a, b \in S$ and let $a', a^* \in V(a)$. Then, by Proposition 3.14, $A(a'b) = A(a^*b)$. Thus $R_A$ is well-defined and $R_A \in \text{IVRC}(S)$. In order to show that $R_A \in \text{IVGC}(S)$, we need only to show that $R_A^{[\lambda, \mu]}$ is a group congruence on $S$ for each $[\lambda, \mu] \in D(I)$. By Theorem 3.12, $A^{[\lambda, \mu]}$
is a complete inner-unitary subsemigroup of $S$. Then it is sufficient to show that
\[(a, b) \in R_A^{[\lambda, \mu]} \iff \{ x \in S : ax \in A^{[\lambda, \mu]} \} = \{ x \in S : bx \in A^{[\lambda, \mu]} \}\]
for any $a, b \in S$, by Theorem 3.1 in [17]. Since $R_A(a, b) = A(a'b)$ for any $a, b \in S$,
\[(a, b) \in R_A^{[\lambda, \mu]} \iff a'b \in A^{[\lambda, \mu]}.

Thus we need only to prove that
\[a'b \in A^{[\lambda, \mu]} \iff \{ x \in S : ax \in A^{[\lambda, \mu]} \} = \{ x \in S : bx \in A^{[\lambda, \mu]} \}.\]

Let $a, b \in S, a' \in V(a)$ and let $[\lambda, \mu] \in D(I)$. Suppose $a'b \in A^{[\lambda, \mu]}$. Let $ax \in A^{[\lambda, \mu]}$. Then, by Proposition 2.6 in [17], $xa \in A^{[\lambda, \mu]}$. Since $A^{[\lambda, \mu]}$ is a subsemigroup of $S$, $a'bx \in A^{[\lambda, \mu]}$. Since $A$ is interval-valued fuzzy complete, $A(a'a) = [1, 1]$. Thus $a'a \in A^{[\lambda, \mu]}$. Since $A^{[\lambda, \mu]}$ is inner-unitary, $bx \in A^{[\lambda, \mu]}$. By the similar arguments, we show that $bx \in A^{[\lambda, \mu]}$ implies $ax \in A^{[\lambda, \mu]}$. So \(\{ x \in S : ax \in A^{[\lambda, \mu]} \} = \{ x \in S : bx \in A^{[\lambda, \mu]} \}\).

Conversely, suppose \(\{ x \in S : ax \in A^{[\lambda, \mu]} \} = \{ x \in S : bx \in A^{[\lambda, \mu]} \}\). It is clear that $aa' \in Es \subseteq A^{[\lambda, \mu]}$. Then, by the hypothesis, $ba' \in A^{[\lambda, \mu]}$. Thus $aa'ba' \in A^{[\lambda, \mu]}$. So $a'b \in A^{[\lambda, \mu]}$. This completes the proof. □

**Lemma 5.2.** Let $R \in IVGC(S)$. We define a mapping $A_R : S \to D(I)$ as follows: For each $a \in S$,
\[A_R(a) = R(aa', a),\]
where $a' \in V(a)$. Then $A_R$ is an interval-valued fuzzy complete inner-unitary subsemigroup of $S$.

**Proof.** Let $a', a^* \in V(a)$. Then, by Proposition 4.4, $R(aa', a) = R(aa^*, a)$. Thus $A_R$ is well-defined. Moreover, $A_R \in D(I)^S$. Since $S$ is a regular semigroup, $E_S \neq \emptyset$. Then, by Theorem 3.12, it is enough to show that $A_R^{[\lambda, \mu]}$ is a complete inner-unitary subsemigroup of $S$ for each $[\lambda, \mu] \in D(I)$. It is clear that $R^{[\lambda, \mu]}$ is a group congruence on $S$. Thus we only need to prove that
\[A_R^{[\lambda, \mu]} = \{ a \in S : aR^{[\lambda, \mu]} \text{ is the identity of the group } S/R^{[\lambda, \mu]} \}\]
by Theorem 3.2 in [17]. Since $A_R(a) = R(aa', a), a \in A^{[\lambda, \mu]}$ if and only if $(aa', a) \in R^{[\lambda, \mu]}$. Then $aR^{[\lambda, \mu]} = aa'R^{[\lambda, \mu]}$. Since $aa' \in Es$, it is clear that $aa'R^{[\lambda, \mu]}$ is the identity of the group $S/R^{[\lambda, \mu]}$. So $A_R^{[\lambda, \mu]} = \{ a \in S : aR^{[\lambda, \mu]} \text{ is the identity of the group } S/R^{[\lambda, \mu]} \}$. Hence $A_R$ is an interval-valued fuzzy complete inner-unitary subsemigroup of
S. □

Theorem 5.3. Let $IVCIG(S)$ denote the collection of all interval-valued fuzzy complete inner-unitary subsemigroups of $S$. Define two mappings:

$$\Phi : IVCIG(S) \longrightarrow IVGC(S)$$

and

$$\Psi : IVGC(S) \longrightarrow IVCIG(S).$$

as follows, respectively: For each $A \in IVCIG(S)$ and each $R \in IVGC(S),$

$$\Phi(A) = R_A$$ and $$\Psi(R) = A_R,$$

where $R_A$ and $A_R$ are defined as in Lemma 5.1 and Lemma 5.2. Then $\Psi \circ \Phi = id_{IVCIG(S)}$ and $\Phi \circ \Psi = id_{IVGC(S)}.$

Proof. By Lemmas 5.1 and 5.2, it is easy to see that $\Phi$ and $\Psi$ are well-defined.

First, we will show that $\Psi \circ \Phi = id_{IVCIG(S)}.$ Let $A \in IVCIG(S).$ Then $(\Psi \circ \Phi)(A) = \Psi(\Phi(A)) = \Psi(R_A) = A_R.$ Let $a' \in S$ and let $a' \in V(a).$ Then clearly $aa' \in V(aa').$ Thus

$$A_{R_A}(a) = R_A(aa', a) = A((aa')'a) = A(aa'a) = A(a).$$

So $[(\Psi \circ \Phi)(A)](a) = A(a)$ for each $a \in S$ and each $A \in IVCIG(S),$ i.e., $(\Psi \circ \Phi)(A) = A$ for each $A \in IVCIG(S).$ Hence $\Psi \circ \Phi = id_{IVCIG(S)}.$

Finally, we will show that $\Psi \circ \Phi = id_{IVGC(S)}.$ Let $R \in IVGC(S).$ Then $(\Psi \circ \Phi)(R) = \Phi(\Psi(R)) = \Phi(A_R) = R_{A_R}.$ Let $a, b \in S$ and let $a' \in V(a), b' \in V(b)$ and $(a'b')' \in V(a'b).$ Then, by proposition 4.5 (a) and (b),

$$R_{A_R}(a, b) = A_R(a'b) = R(a'b(a'b)', a'b) = R(a'b, a'b) = R(b, ba'b) = R(b, a, a'b) = R(ab'b, b) = R(a, b).$$

Thus $[(\Phi \circ \Psi)(R)](a, b) = R(a, b)$ for each $(a, b) \in S \times S$ and each $R \in IVGC(S).$ So $A_R \circ \Psi = R$ for each $R \in IVGC(S).$ Hence $\Phi \circ \Psi = id_{IVGC(S)}.$ This completes the proof. □

References

Jeong Gon Lee  
Division of Mathematics and Informational Statistics, and Nanoscale Science and Technology Institute, Wonkwang University, Iksan City Jeonbuk 54538, Korea.  
E-mail: jukolee@wku.ac.kr

Kul Hur  
Division of Mathematics and Informational Statistics, and Nanoscale Science and Technology Institute, Wonkwang University, Iksan City Jeonbuk 54538, Korea.  
E-mail: kulhur@wku.ac.kr

Pyung Ki Lim  
Division of Mathematics and Informational Statistics, and Nanoscale Science and Technology Institute, Wonkwang University, Iksan City Jeonbuk 54538, Korea.  
E-mail: pklim@wku.ac.kr