Generalized topologies on finite sets

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Abstract. The number of topologies on a finite set is a famous open problem. In the present paper we discuss a method of obtaining the number of generalized topologies on finite sets.

1. Introduction

The number $T(n)$ of topologies on a finite set of cardinal $n$ is an open question [6]. There is no known simple formula to compute $T(n)$ for arbitrary $n$. The online Encyclopedia of Integer Sequences presently lists $T(n)$ for $n \leq 18$. Recall that a subset $\mu$ of the power set $\exp X$ of a set $X$ is a generalized topology (briefly GT) in $X$ iff $G_i \in \mu$ $(i \in I)$ implies $\cup_{i \in I} G_i \in \mu$ (in particular, $I$ can be empty so that the definition implies $\emptyset \in \mu$). The pair $(X, \mu)$ is called a generalized topological space (briefly GTS). A member of $\mu$ is called open and a subset $F$ of $X$ is called closed if $X \setminus F \in \mu$. Sets which are simultaneously open and closed are called open-closed sets. The theory of generalized topological spaces, which was introduced by Á. Császár [5], is one of the most important development of general topology in recent years. A GT on $X$ is a join-sublattice $(\exp X, \subseteq)$ with the minimum element $\emptyset$, denoted by $0$. Some important counterexamples in topological spaces or GTS can be found in the finite forms (see for example, [1, 3]).

Let $X$ be an $n$-element set. Then the number $GT(n)$ of generalized topologies on $X$ is exactly the number of join-sublattices of $(\exp X, \subseteq)$ with 0. There is no known formula giving $GT(n)$. Let $gt(n, k)$ be the set of all labeled generalized topologies on $X$ having $k$ open sets and $GT(n, k) = |gt(n, k)|$. Thus $GT(n) = \sum_{k=1}^{2^n} GT(n, k)$.

Definition 1.1. [7] Let $\mu$ be a GT on $X$ and $\mu'$ a GT on $X'$. A mapping $f : X \rightarrow X'$ is $(\mu, \mu')$-continuous iff $M' \in \mu'$ implies...
If \( f \) is bijective and \((\mu, \mu')\)-continuous, moreover \( f^{-1} \) is \((\mu', \mu)\)-continuous, then we say that \( f \) is a \((\mu, \mu')\)-homeomorphism and \((X, \mu)\) and \((X', \mu')\) are said to be equivalent.

**Example 1.2.** There is a unique GT on the empty set. Likewise there are two distinct generalized topologies on \( \{a\} \): \( g\tau = \{\emptyset\} \) and \( g\tau' = \{\emptyset, \{a\}\} \). Let \( X = \{a, b\} \) be a two-element set. There are 7 distinct generalized topologies on \( X \) but only 5 inequivalent generalized topologies: \( g_1 = \{\emptyset\} \), \( g_2 = \{\emptyset, \{a\}\} \), \( g_3 = \{\emptyset, X\} \), \( g_4 = \{\emptyset, \{a\}, X\} \) and \( g_5 = \{\emptyset, \{a\}, \{b\}, X\} \).

2. A GTS \((X, \mu)\), Where \(|\mu| \geq 2^{|X|} - 7\)

Recall that a GTS \((X, \mu)\) is said to be a \(\mu\)-\(T_1\) space \([9]\) if for any pair of distinct points \(x\) and \(y\) of \(X\), there exists a \(U \in \mu\) such that \(x \notin U\) and \(y \in U\). As \([9]\) a GTS \((X, \mu)\) is called \(\mu\)-\(T_2\) if for every distinct points \(x, y \in X\), there exist disjoint open sets \(U_x\) and \(U_y\) such that \(x \in U_x\) and \(y \in U_y\). It is well known that a finite Hausdorff topological space, i.e., a finite \(\mu\)-\(T_2\) topological space, is discrete, but in \([2]\) there is a non-discrete \(\mu\)-\(T_2\) GTS which is finite. Here, we give another example.

**Example 2.1.** \(\mu = \{\emptyset, \{a\}, \{b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}\) is a \(\mu\)-\(T_2\) GT on \(X = \{a, b, c\}\), that is not discrete.

**Definition 2.2.** If a GTS \((X, \mu)\) which is \(T_1\) has a base consisting of open-and-closed sets, then it is called zero-dimensional. A GTS \((X, \mu)\) which is \(T_1\) is called completely regular if for every \(x \in X\) and every closed subset \(F \subset X\) such that \(x \notin F\) there exists a continuous function \(f\) from \(X\) to \(\mathbb{R}\) such that \(f(x) = 0\) and \(f(y) = 1\) for \(y \in F\). If the cardinality of every nonempty member of \(\mu\) is greater than one, then we say that \((X, \mu)\) is crowded.

We give an example of a crowded completely regular GTS which is finite.

**Example 2.3.** Let \(X = \{a, b, c, d\}\) and \(\mu = \exp X \setminus \{\{a\}, \{b\}, \{c\}, \{d\}\}\). Then \((X, \mu)\) is a zero-dimensional space since

\[
\beta = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}\n\]

is a base for the GTS such that every member of \(\beta\) is closed. Therefore the generalized topological space is completely regular.
Remark 2.4. For \( n \geq 2 \), the calculations of \( GT(n, k) \) are complicated. In the following proposition we give some results about them.

Proposition 2.5. For \( n \geq 3 \) we have the following.

(I) \( GT(n, 1) = GT(n, 2^n) = 1 \);

(II) \( GT(n, 2) = 2^n - 1 \);

(III) \( GT(n, 2^n - 1) = n \) for \( n \geq 2 \);

(IV) \( GT(n, 2^n - 2) = \frac{3n(n-1)}{2} \);

(V) \( GT(n, 2^n - 3) = \binom{n}{6} + \binom{n}{2}(2n - 3) + \binom{n}{1}\binom{n-1}{2} \);

(VI) \( GT(n, 2^n - 4) = \binom{n}{6} + \binom{n}{3}3n - 6 + \binom{n}{3}(2n - 3) + \frac{n^2(n-1)(n-2)}{6} \);

(VII) \( GT(n, 2^n - 5) = \binom{n}{6} + \binom{n}{4}4n - 10 + \binom{n}{3}(3n - 6) + \binom{n}{2}(2n - 3) + 2\binom{n-1}{2} + \binom{n}{1}\binom{n-1}{3} + \binom{n}{0} \);

(VIII) \( GT(n, 2^n - 6) = \binom{n}{6} + (5n - 15)\binom{n}{5} + \binom{n}{4}\binom{4n-10}{2} + \binom{n}{3}\binom{3n-6}{2} + \binom{n}{2}\binom{2n-3}{2} + \binom{n}{1}\binom{n-1}{3} + \binom{n}{0} \);

(IX) \( GT(n, 2^n - 7) = \binom{n}{6} + (6n - 21)\binom{n}{5} + \binom{n}{4}\binom{5n-15}{2} + \binom{n}{3}\binom{4n-10}{2} + 4\binom{n-1}{2} + \binom{n}{3}\binom{3n-6}{2} + \binom{n}{2}\binom{2n-3}{2} + \binom{n}{1}\binom{n-1}{3} + \binom{n}{0} \).

Proof. Let \( \mu \) be a GT in \( X \). (I) and (II). The proofs of (I) and (II) are clear. (III). If \( |\mu| = 2^n - 1 \), then \( \mu = \exp X \setminus \{x\} \) for some \( x \in X \).

(IV). If \( |\mu| = 2^n - 2 \), then there exist two elements \( x, y \) of \( X \) such that \( \mu = \exp X \setminus \{x, y\} \) or \( \mu = \exp X \setminus \{x\} \). (V). Let \( |\mu| = 2^n - 3 \). Then there exist three elements \( a, b, x \) of \( X \) such that \( \mu = \exp X \setminus \{a, b, x\} \), \( \mu = \exp X \setminus \{a, y\} \), or \( \mu = \exp X \setminus \{a, x, y\} \). (VI). Let \( |\mu| = 2^n - 4 \). Then \( \mu \) has one of the following forms:

Case (1): Let \( \mu = \exp X \setminus A \); where \( A = \{(a, b), (c, d)\} \subset \exp X \). The set \( A = \{(a, b), (c, d)\} \) is chosen in \( \binom{3}{2} \) ways.

Case (2): Let \( \mu = \exp X \setminus \{(a, b), (c, D)\} \); where \( a, b, c \in X \) and \( D \subset X \). Then \( D \) is a two-element set and \( D \cap \{a, b, c\} \neq \emptyset \). \( D \) is chosen in \( 3n - 6 \) different ways and the subsets \( \{a, b\} \), \( \{a, c\} \), and \( \{b, c\} \) in \( \binom{3}{2} \) different ways.

Case (3): Let \( \mu = \exp X \setminus \{(a, b), C, D\} \); where \( a, b \in X \) and \( C, D \subset X \). Then \( D \) and \( C \) intersect \( \{a, b, c\} \) and \( |C| = |D| = 2 \).

Case (4): Let \( \mu = \exp X \setminus \{a, B, C, D\} \); where \( a \in X \) and \( B, C, D \subset X \). Then \( a \in B \cap C \cap D \) and so \( |B| = |C| = |D| = 2 \) or \( |B| = |C| \) and \( |D| = B \cup C \). \( \{a, b, c\} \) is chosen in \( n \) different ways and \( B, C, D \) are chosen in \( \frac{n(n-1)(n-2)}{6} \) different ways. Similarly, (VII), (VIII) and (IX) hold. \( \square \)
Example 2.6. Let $X = \{a, b, c\}$. Then by the above proposition $GT(3, 7) = 3$, $GT(3, 6) = 9$, $GT(3, 5) = 13$, $GT(3, 4) = 15$, $GT(3, 3) = 12$ and $GT(3, 2) = 7$ $GT(3, 1) = 1 = GT(3, 8)$. Thus, the total number of generalized topologies on $X$ is

$$GT(3) = \sum_{i=1}^{8} GT(3, i) = 61.$$ 

Example 2.7. Let $X = \{a, b, c, d\}$. Then by the above proposition $GT(4, 15) = 4$, $GT(4, 14) = 18$, $GT(4, 13) = 46$, $GT(4, 12) = 51$, $GT(4, 11) = 174$, $GT(4, 10) = 221$, $GT(4, 9) = 196$, $GT(4, 2) = 15$ and $GT(4, 16) = 1 = GT(4, 1)$.

3. A GT with less than seven open sets

Recall that a chain topology on a finite set $X$, is a topology whose open sets are totally ordered by inclusion. For generalized topological spaces, we have the following definition.

Definition 3.1. A GT-chain on $X$, is a generalized topology whose open sets are totally ordered by inclusion.

Proposition 3.2. [4, 8] Let $C(n, k)$ be the number of chain topologies on $X$ having $k$ open sets. Then,

$$C(n, k) = \sum_{i=1}^{n-1} \binom{n}{i} C(i, k-1) = \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} (k-1-i)^n.$$ 

Corollary 3.3. Let $CGT(n)$ be the total number of GT-chains on $X$, where $|X| \geq 3$. Then,

$$CGT(n) = 1 + \sum_{i=1}^{n} \sum_{k=2}^{i+1} \sum_{j=0}^{k-1} (-1)^j \binom{n}{i} \binom{k-1}{j} (k-1-j)^i.$$ 

Proof. If $\mu$ is a GT-chain on $X$, then $\mu$ is a chain topology on $A = \cup \mu$. It is clear that any chain topology on a subset $B$ of $X$ is a GT-chain on $X$. Thus, there is a bijective correspondence between GT-chains on $X$ and chain topologies on subsets of $X$. Therefore,

$$CGT(n) = 1 + \sum_{i=1}^{n} \binom{n}{i} \sum_{k=2}^{i+1} C(i, k),$$

and so by the above proposition we are done. \qed
Corollary 3.4. Let $CGT(n, m)$ be the total number of GT-chains on $X$ with $m$ open sets, where $|X| = n$ and $m \leq n + 1$. Then,

$$CGT(n, m) = \binom{m-1}{0} m^n - \binom{m-1}{1} (m-1)^n + \cdots + (-1)^{m-1} \binom{m-1}{m-1} 1^n.$$  

Proof. Let $\mu$ be a GT-chain on $X$ with $m$ open sets. If $X \notin \mu$, then $\mu \cup \{X\}$ is a chain topology on $X$ with $m + 1$ open sets. Otherwise, $\mu$ is a chain topology on $X$ with $m$ open sets. Thus by Proposition 3.2,

$$CGT(n, m) = C(n, m + 1) + C(n, m) = \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} (m - i)^n$$

$$+ \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} (m - 1 - i)^n = \binom{m}{0} m^n$$

$$- \binom{m}{1} - \binom{m-1}{0} (m-1)^n + \cdots + (-1)^{m-1} \binom{m}{m-1} m^n - \binom{m-1}{1} (m - 1)^n$$

$$+ \cdots + (-1)^{m-1} \binom{m-1}{m-1} 1^n.$$

Example 3.5. Let $X$ be a set and $|X| = n$ where $n \in \mathbb{N}$. Then by the above corollary, the total number of GT-chains on $X$ with three open sets is $CGT(n, 3) = 3^n - 2^{n+1} + 1$. Similarly $CGT(n, 4) = 4^n - 3^{n+1} + 3.2^n - 1$, $CGT(n, 5) = 5^n - 4^{n+1} + 6.3^n - 4.2^n + 1$, $CGT(n, 6) = 6^n - 5^{n+1} + 10.4^n - 10.3^n + 5.2^n - 1$, and $CGT(n, 7) = 7^n - 6^{n+1} + 15.5^n - 20.4^n + 15.3^n - 6.2^n + 1$.

Proposition 3.6. For every $2 \leq n \in \mathbb{N}$ we have

(a) $GT(n, 3) = 3^n - 2^{n+1} + 1$;

(b) $GT(n, 4) = 4^n - 3^{n+1} + 3.2^n - 1 + \frac{1}{2} \sum_{m=2}^{n} \sum_{i=0}^{m-1} \binom{n}{m} \binom{m}{i} (2^i - 1)$.

Proof. (a) For every $\mu \in gt(n, 3)$ there are a subset $B$ of $X$ and a subset $A$ of $B$ such that $\emptyset \notin A \subseteq B \subset X$ and $\mu = \{\emptyset, A, B\}$. Thus by Example 3.5, $GT(n, 3) = 3^n - 2^{n+1} + 1$.

(b) If $\mu = \{\emptyset, A, B, C\} \in gt(n, 4)$, then either $A \subset B \subset C$ or $C = A \cup B$. The two cases are disjoint.

Case (1) : This is the number of GT-chains on $X$ having 4 open sets; so by Example 3.5, the total number of generalized topologies in this case is $4^n - 3^{n+1} + 3.2^n - 1$. 
Case (2) : Let $C$ be a subset of $X$ such that 
\[ 2 \leq m = |C| \leq n \]
and $A$ be proper and non-empty subset of $C$. Then, $1 \leq |A| = i < m$ and so $B = (C \setminus A) \cup B_1$, where $B_1 \subset A$. Therefore, the total number of generalized topologies in this case is
\[
\frac{1}{2} \sum_{m=2}^{n} \sum_{i=1}^{m-1} \binom{n}{m} \binom{m}{i} (2^i - 1).
\]

Proposition 3.7. For every $n = |X| \geq 3$ we have $GT(n, 5) =$
\[
5^n - 4^{n+1} + 6.3^n - 4.2^n + 1 + \frac{1}{2} \sum_{m=2}^{n} \sum_{i=1}^{m-1} \binom{n}{m} \binom{m}{i} (2^i - 1)(2^{n-m} - 1)
\]
\[
+ \frac{5}{3} \sum_{m=3}^{n} \sum_{i=2}^{m-1} \sum_{j=1}^{i-1} \binom{n}{m} \binom{m}{i} \binom{i}{j} (2^j - 1).
\]

Proof. Let $\mu \in gt(n, 5)$. If $\mu$ is a GT-chain, then by Example 3.5, the total number of GT-chains on $X$ is $5^n - 4^{n+1} + 6.3^n - 4.2^n + 1$. If $\mu$ is not a GT-chain, then there are $A, B \in \mu$ such that $A \nsubseteq B$ and $B \nsubseteq A$, and so $C = A \cup B \in \mu$. Thus, the non-empty members of $\mu$ has one of the following forms:

(1) $A \cup B = C \subset D$.

(2) $\emptyset \neq D \subset C = A \cup B$.

The two cases are disjoint.

Case (1) : $D$ can be chosen in $2^{n-m} - 1$ ways, where
\[ 2 \leq m = |C| \leq n - 1. \]

Thus by the proof of the above proposition, the total number of generalized topologies in this case is
\[
\frac{1}{2} \sum_{m=2}^{n} \sum_{i=1}^{m-1} \binom{n}{m} \binom{m}{i} (2^i - 1)(2^{n-m} - 1).
\]

Case (2) : Since in this case $3 \leq m = |C| \leq n$, we have $2 \leq i = |A| \leq m - 1$.

In this case, $D \cap (A \cap B) \neq \emptyset$ and so
\[ 1 \leq |A \cap B| = j \leq i - 1. \]

Thus the total number of generalized topologies in this case is $n_1 + n_2 + n_3$, where $n_1, n_2$ and $n_3$ can be computed as follows:
(2I) If \( D \) is a nonempty subset of \( A \cap B \). Then, by the proof of the above proposition

\[
n_1 = \frac{1}{2} \sum_{m=3}^{n} \sum_{i=2}^{m-1} \sum_{j=1}^{i-1} \binom{n}{m} \binom{m}{i} \binom{i}{j} (2^j - 1).
\]

(2II) If \( D = (A \setminus B) \cup (B \setminus A) \cup D_1 \), where \( D_1 \) is a proper subset of \( A \cap B \). Then, \( n_2 = \frac{1}{2} n_1 \).

(2II) If \( D = (A \setminus B) \cup D_1 \), where \( D_1 \) is a proper subset of \( A \cap B \). Then \( n_3 = 2n_1 \). The proof is complete.

\[ \Box \]

Notation 3.8. Let \( Y \) be a subset of \( A \cup B \). Then \( Y \) can be written as \( Y = Y' \cup Y_1 \cup Y'' \), where \( Y' \subset A \setminus B \), \( Y_1 \subset A \cap B \) and \( Y'' \subset B \setminus A \).

Theorem 3.9. Let \( \mu = \{\emptyset, A, B, A \cup B, D, E\} \) be in \( gt(n, 6) \), such that \( A \nsubseteq B \) and \( B \nsubseteq A \) and \( D, E \nsubseteq A \cup B \). If two of the sets \( D', D'' \) and \( D_1 \) are empty and one of them is non-empty, then \( \mu \) has one of the following forms:

1. \( \mu = \{\emptyset, A, B, A \cup B, D_1, ((A \cap B) \setminus I) \cup E_2\} \), where \( I \) is a non-empty subset of \( D_1 \) and \( E_2 = A \setminus B \) or \( E_2 = (A \setminus B) \cup (B \setminus A) \).
2. \( \mu = \{\emptyset, A, B, A \cup B, D_1, E\} \), where \( \emptyset \neq D_1 \subset E_1 = E \subset A \cap B \), \( E = E_1 \cup (B \setminus A) \cup (A \setminus B) \) or \( E = E_1 \cup (B \setminus A) \), where \( \emptyset \neq D_1 \subset E_1 \).\n3. \( \mu = \{\emptyset, A, B, A \cup B, D_1, A \cup E''\} \), where \( \emptyset \nsubseteq E'' \subseteq B \setminus A \).
4. \( \mu = \{\emptyset, A, B, A \cup B, D', D' \setminus B\} \), where \( \emptyset \neq D' \subset A \setminus B \).
5. \( \mu = \{\emptyset, A, B, A \cup B, A \setminus B, B \cup E'\} \), where \( \emptyset \neq E' \subset A \setminus B \).
6. \( \mu = \{\emptyset, A, B, A \cup B, A \setminus B, E_1 \cup E_2\} \), where \( E_2 = (A \setminus B) \cup (B \setminus A) \) or \( E_2 = A \setminus B \).

Proof. Let \( C = A \cup B \). If \( D = D_1 \), then \( D \cup E = E' \cup (D_1 \cup E_1) \cup E'' \) and so \( |\mu| = 6 \) implies that there are the following cases which are disjoint:

1. If \( D \cup E = A \), then \( E'' = \emptyset \), and so \( A = E' \cup (D_1 \cup E_1) \), i.e., \( D_1 \cup E_1 = A \cap B \) and \( E' = A \setminus B \). Thus \( E_1 = (A \cap B) \setminus I \), where \( I \) is a non-empty subset of \( D_1 \). We note that if \( D \cup E = B \), then the set of generalized topologies in this case is coincided with the set of generalized topologies in the case \( D \cup E = A \).
2. If \( D \cup E = C \), then \( (D_1 \cup E_1) \cup E' \cup E'' = C \). Thus, \( A \setminus B = E', B \setminus A = E'' \) and \( D_1 \cup E_1 = A \cap B \), i.e., \( E = (A \setminus B) \cup (B \setminus A) \cup E_1 \) and \( E_1 = (A \cap B) \setminus I \), where \( I \) is a non-empty subset of \( D_1 \).
(3) If $E = D \cup E$, then $E = (D_1 \cup E_1) \cup E' \cup E''$, and so $D_1 \subset E_1$. We note that $E \cup A = E'' \cup A \in \mu$ can not be equal to $B$ or $D = D_1$ so there are the following cases which are disjoint.

(3I) If $E \cup A = E'' \cup A = A$, then $E'' = \emptyset$ and $E \subset A$. But $B \cup E = B \cup E' \in \mu$ implies that $E' = \emptyset$ or $E' = A \setminus B$. Thus $D = D_1 \not\subset E = E_1 \subset A \cap B$ or $E = E_1 \cup (A \setminus B)$, where $D_1 \subset E_1 \not\subset A \cap B$.

(3II) If $E \cup A = E'' \cup A = C$, then $E'' = B \setminus A$ and $B \cup E' = B \cup E \in \mu$ implies that $E' = \emptyset$ or $E' = A \setminus B$. Thus $E = E_1 \cup (A \setminus B) \cup (B \setminus A)$ or $E = E_1 \cup (B \setminus A)$. We note that if $E = E_1 \cup (B \setminus A)$, then the set of generalized topologies in this case is coincided with the set of generalized topologies in the case $E = E_1 \cup (A \setminus B)$.

(3III) If $E \cup A = E'' \cup A = E$, then $A \subset E$. Thus $E' = A \setminus B$, $E_1 = A \cap B$ and $E''$ is a non-empty and proper subset of $B \setminus A$.

Let $D_1 = D'' = \emptyset$ and $\emptyset \neq D'$. Then $D \cup A = A$ and $D' \cup B = D \cup B$ is equal to $E$ or $C$. Thus there are the following cases which are disjoint.

(4) If $D' \cup B = E$, then $D'$ is a non-empty and proper subset of $A \setminus B$.

(5) If $D' \cup B = C$, then $D' = D = A \setminus B$. Since $E \cup D = E_1 \cup (A \setminus B) \cup E'' \in \mu$, we have the following cases which are disjoint:

(5I) If $E_1 \cup (A \setminus B) \cup E'' = E \cup D = A$. Then $E'' = \emptyset$ and $E_1 = A \cap B$ and so $E = E' \cup (A \setminus B)$. Thus $E \cup B = E' \cup B$ implies that $E' = \emptyset$ and so $E = A \setminus B$; which is not a new GT.

(5II) If $E_1 \cup (A \setminus B) \cup E'' = E \cup D = C$, then $E = B \cup E'$. Since $E \neq B$ and $E \neq C$, we have $\emptyset \neq E' \subset A \setminus B$.

If $E_1 \cup (A \setminus B) \cup E'' = E \cup D = E$, then $E' = A \setminus B = D$ and so $E = (A \setminus B) \cup E_1 \cup E''$. Since $A \cup E = A \cup E''$ there are two cases which are disjoint:

(5III) If $A \cup E'' = C$, then $E'' = B \setminus A$ and so $E = (A \setminus B) \cup E_1 \cup (B \setminus A)$.

(5IV) If $A \cup E'' = A$, then $E'' = \emptyset$ and so $E = (A \setminus B) \cup E_1$.

We note that if $D_1 = \emptyset = D'$ and $D'' \neq \emptyset$, then the set of generalized topologies in this case is coincided with the set of generalized topologies in the case $D_1 = \emptyset = D''$ and $D' \neq \emptyset$.

Theorem 3.10. Let $\mu = \{\emptyset, A, B, A \cup B, D, E\}$ be in $gt(n, 6)$, such that $A \not\subset B$ and $B \not\subset A$ and $B \not\subset A \cup B$. If one of the sets $D'$, $D''$ and $D_1$ is empty and two of them are non-empty, then $\mu$ has one of the following forms:
Thus we have the following cases which are disjoint:

(1) $\mu = \{\emptyset, A, B, A \cup B, D_1 \cup D', D' \cup B\}$, where $D_1$ is non-empty and $\emptyset \neq D' \subseteq A \setminus B$.

(2) $\mu = \{\emptyset, A, B, A \cup B, (A \setminus B) \cup D_1, (A \setminus B) \cup (A \cap B) \setminus I\}$, where $I$ is a non-empty subset of $A \cap B$ or $\mu = \{\emptyset, A, B, A \cup B, (A \setminus B) \cup D_1, (A \cap B) \setminus I\}$, where $I \subset A \cap B$.

(3) $\mu = \{\emptyset, A, B, A \cup B, (A \setminus B) \cup D_1, E_2 \cup E_1\}$, where $E_2 = A \setminus B$ or $E_2 = (A \setminus B) \cup (B \setminus A)$ and $D_1 \not\subseteq E_1 \subseteq A \cap B$.

(4) $\mu = \{\emptyset, A, B, A \cup B, (A \setminus B) \cup D_1, E_2 \cup ((A \cap B) \setminus I)\}$, where $E_2$ is $B \setminus A$ or $(B \setminus A) \cup (A \setminus B)$ and $I$ is a non-empty subset of $D_1$.

(5) $\mu = \{\emptyset, A, B, A \cup B, (A \setminus B) \cup D'', A \cup D''\}$, where $\emptyset \neq D'' \subseteq A \setminus B$.

Proof. Let $C = A \cup B$. If $D_1$ and $D'$ are non-empty and $D'' = \emptyset$. Then $D = D_1 \cup D', D \cup A = A$ and $D \cup B = D' \cup B$ is equal to $E$ or $C$.

Thus we have the following cases which are disjoint:

(1) If $D' \cup B = D \cup B = E$. Then $\emptyset \neq D'$ is a proper subset of $A \setminus B$ since $E \neq C$. Thus the general form of $\mu$ in this case is

$$\mu = \{\emptyset, A, B, A \cup B, D_1 \cup D', D' \cup B\}.$$ 

(2) If $D \cup B = C$, then $D' = A \setminus B$ and $D = (A \setminus B) \cup D_1$. Thus $\emptyset \neq D_1$ is a proper subset of $A \cap B$. Since $D \cup E = (D_1 \cup E_1) \cup (A \setminus B) \cup E'' \in \mu$, we have the following cases which are disjoint:

(2I) If $D \cup E = A$, then $D_1 \cup E_1 = (A \setminus B) \cup E'' \subseteq A$ and $E'' = \emptyset$, $D_1 \cup E_1 = A \cap B$ and $E = E_1 \cup E'$. If $E' \subseteq A \setminus B$, then $E' = \emptyset$.

Thus $\mu$ has the following form:

$$\mu = \{\emptyset, A, B, A \cup B, (A \setminus B) \cup D_1, (A \cap B) \setminus I\},$$

where $I$ is a subset of $D_1$. If $E' = A \setminus B$, then $E_1 = (A \cap B) \setminus I$, where $I$ is a non-empty subset of $D_1$. Thus $\mu$ has the following form:

$$\mu = \{\emptyset, A, B, A \cup B, (A \setminus B) \cup D_1, (A \setminus B) \cup ((A \cap B) \setminus I)\}.$$ 

(3) If $D_1 \cup (A \setminus B) \cup E = D \cup E = E$, then $E' = A \setminus B$. $A \cup E = A \cup E''$ implies that $E''$ is equal to $\emptyset$ or $B \setminus A$ and so $E = (A \setminus B) \cup E_1$ or $E = (A \setminus B) \cup (B \setminus A) \cup E_1$. Thus the general form of $\mu$ in this case is

$$\mu = \{\emptyset, A, B, A \cup B, (A \setminus B) \cup D_1, E_1 \cup E_2\},$$

where $E_2 = (A \setminus B)$ or $E_2 = (A \setminus B) \cup (B \setminus A)$ and $D_1 \not\subseteq E_1 \subseteq A \cap B$.

(4) If $D_1 \cup (A \setminus B) \cup E = D \cup E = C$, then $D_1 \cup E_1 = A \cap B$ and $E'' = B \setminus A$. Thus $E \cup B = E' \cup B$ implies that $E' = \emptyset$ or $E' = A \setminus B$.
and $E_1 = (A \cap B) \setminus I$, where $I$ is a non-empty subset of $D_1$ and so the general form of $\mu$ in this case is

$$\mu = \{\emptyset, A, B, A \cup B, (A \setminus B) \cup D_1, E_2 \cup ((A \cap B) \setminus I)\},$$

where $E_2$ is $B \setminus A$ or $(B \setminus A) \cup (A \setminus B)$, and $I$ is a non-empty subset of $D_1$. We note that if $D_1$ and $D''$ are nonempty and $D' = \emptyset$. Then the set of generalized topologies in this case is coincided with the set of generalized topologies in the case $D_1 \neq \emptyset \neq D''$.

Let $D'$ and $D''$ be non-empty and $D_1 = \emptyset$. Then $D = D' \cup D''$ and so $B \cup D = B \cup D' \in \mu$ and $A \cup D = A \cup D'' \in \mu$. Since $A \cup D''$ is equal to $E$ or $C$, the following cases are disjoint:

(5) If $A \cup D'' = E$, then $B \cup D' = C$ and so $D' = A \setminus B$. Thus the general form of $\mu$ in this case is

$$\mu = \{\emptyset, A, B, A \cup B, (A \setminus B) \cup D_1 \cup D'' \cup A \cup D''\},$$

where $\emptyset \neq D'' \subsetneq B \setminus A$. We note that if $A \cup D'' = C$, then $D'' = B \setminus A$. Thus $D = D' \cup (B \setminus A)$ and $D \cup B = D' \cup B$, and so the set of generalized topologies in this case is coincided with the set of generalized topologies in the case $A \cup D'' = E$.

\[ \square \]

**Theorem 3.11.** Let $\mu = \{\emptyset, A, B, A \cup B, D, E\}$ be in $gt(n, 6)$, such that $A \not\subset B$ and $B \not\subset A$, and each of the sets $D_1, D'$ and $D''$ are non-empty. Then $\mu = \{\emptyset, A, B, A \cup B, (A \setminus B) \cup D_1 \cup D'', A \cup D''\}$, where $\emptyset \neq D_1 \subsetneq A \cap B$; and $\emptyset \neq D'' \subsetneq B \setminus A$.

**Proof.** Let $D_1, D'$ and $D''$ be non-empty, then $A \cup D = A \cup D'' \in \mu$ implies that $A \cup D$ is equal to $C$ or $E$. If $A \cup D = E$, then from $B \cup D = B \cup D'$ we conclude that $D' = A \setminus B$. Thus the general form of $\mu$ in this case is

$$\mu = \{\emptyset, A, B, A \cup B, (A \setminus B) \cup D_1 \cup D'', A \cup D''\},$$

where $\emptyset \neq D_1 \subsetneq A \cap B$; and $\emptyset \neq D'' \subsetneq B \setminus A$.

We note that if $A \cup D = E$, then the set of all generalized topologies obtained in this case is equal to the set of all generalized topologies obtained in the case $A \cup D = E$. \[ \square \]

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References


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