DERIVATIONS OF $MV$-ALGEBRAS FROM HYPER $MV$-ALGEBRAS

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Abstract. In this paper, we investigate some new results in $MV$-algebras and (strong) hyper $MV$-algebras. We show that for any infinite countable set $M$, we can construct an $MV$-algebra and a strong hyper $MV$-algebra on $M$. Specially, for any infinite totally bounded set, we can construct a strong hyper $MV$-algebra on it. Then by considering the concept of fundamental relation on hyper $MV$-algebras, we define the notion of fundamental $MV$-algebra and prove that any $MV$-algebra is a fundamental $MV$-algebra. In practical, we show that any infinite countable $MV$-algebra is a fundamental $MV$-algebra of itself, but it is not correct for finite $MV$-algebras.

1. Introduction

$MV$-algebras introduced by C. C. Chang [2] in 1958 provide an algebraic proof of completeness theorem of infinite valued Lukasewicz propositional calculus. The hyper structure theory was introduced by F. Marty [12] at the 8th congress of Scandinavian Mathematicians in 1934. Since then many researches have worked in this areas. Recently in [5], Sh. Ghorbani, et al. applied the hyperstructure to $MV$-algebras and introduced the concept of a hyper $MV$-algebra which is a generalization of an $MV$-algebra and investigated some related results. Based on [6, 7], L. Torkzadeh, et al. [15], discussed hyper $MV$-ideals in hyper $MV$-algebras. In [13, 14], Davvaz et al. are defined the concept of fundamental relation on hyper $MV$-algebras. Now, in this paper, we prove that any $MV$-algebra is a fundamental $MV$-algebra. But, we show that any finite $MV$-algebra is not a fundamental $MV$-algebra of itself.

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2. Preliminaries

Definition 2.1. [3, 13] Let \( M \) be a set with a binary operation "\( \oplus \)" , a unary operation "\( * \)" and a constant "\( 0 \)". Then, \( (M, \oplus, *, 0) \) is called an \( \text{MV-} \)algebra if it satisfies the conditions (MV1): \( x \oplus (y \oplus z) = (x \oplus y) \oplus z \) ,
(MV2): \( x \oplus y = y \oplus x \) , (MV3): \( x \oplus 0 = x \) , (MV4): \( (x^*)^* = x \) , (MV5):
\( x \oplus 0^* = 0^* \) , (MV6): \( (x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x \) . Let \( (M, \oplus, *, 0) \) be
an \( \text{MV-} \)algebra. For any two nonempty subsets of \( M \), the hyperproduct of \( a \) and \( b \)
is called a hypergroupoid. For any two nonempty subsets \( A \) and \( B \) of hypergropoid
\( M \), the hyperproduct is defined by \( x \preceq y \iff x^* \oplus y = 0^* \) is a partial order and is called the natural order (See [3]). We call \( (M, \oplus, *, 0, \preceq) \) an \( \text{MV-} \)natural partial ordered
and an \( \text{MV-} \)natural total ordered is an \( \text{MV-} \)chain. Let \( (M, \oplus, *, 0) \) and
\( (M', \oplus', *, 0') \) be two \( \text{MV-} \)algebras. A mapping \( f : M \rightarrow M' \) is called
a homomorphism from \( M \) into \( M' \), if for any \( x, y \in X \), \( f(x \oplus y) = f(x) \oplus' f(y) \), \( f(0) = 0' \) and \( f(x^*) = (f(x))^* \). The homomorphism \( f \), is
called an isomorphism, if it is onto and one to one.

Definition 2.2. [4] Let \( H \) be a nonempty set and \( P^*(H) \) be the family of all nonempty subsets of \( H \). Functions \( *_H : H \times H \rightarrow P^*(H) \),
where \( i \in \{1, 2, \ldots, n\} \), are called binary hyperoperations. For all \( x, y, z \in H \), \( *_H(x, y) \) is called the hyperproduct of \( x \) and \( y \) and structure \( (H, *_H) \)
is called a hypergroupoid. For any two nonempty subsets \( A \) and \( B \) of hypergropoid \( H \), we define \( A *_H B = \bigcup_{a \in A} a *_H b \),
\( A *_H x = \bigcup_{a \in A} a *_H x \) and \( x *_H B = \bigcup_{b \in B} x *_H b \).

Definition 2.3. [13, 14] Let \( M \) be a non-empty set, endowed with a
binary hyperoperation "\( \oplus \)" , a unary operation "\( * \)" and a constant "\( 0 \)".
Then, \( (M, \oplus, *, 0) \) is called a hyper \( \text{MV-} \)algebra if it satisfies the following axioms,
\( (HMV1) : x \oplus (y \oplus z) = (x \oplus y) \oplus z \) , \( (HMV2) : x \oplus y = y \oplus x \) ,
\( (HMV3) : (x^*)^* = x \) , \( (HMV4) : (x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x \) , \( (HMV5) : 0^* \in x \oplus 0^* \) , \( (HMV6) : x \in x \oplus 0 \) , and we say that hyper \( \text{MV-} \)algebra
\( M \), is a strong hyper \( \text{MV-} \)algebra, if it satisfies the axiom \( (HMV7) : x \ll y \) and \( y \ll x \), then \( x = y \), for all \( x, y, z \in M \), where \( x \ll y \) is defined by \( 0^* \in x^* \oplus y \). For every subsets \( A \) and \( B \) of \( M \), we define
\( A \ll B \iff \exists a \in A \text{ and } \exists b \in B \text{ such that } a \ll b \text{ and } A^* = \{a^* | a \in A\} \).
Let \( (M, \oplus, *, 0) \) be a hyper \( \text{MV-} \)algebra and \( R \) be an equivalence relation on \( M \). If \( A \) and \( B \) are nonempty subsets of \( M \), then \( A \mathcal{R} B \) means that for all \( a \in A \), there exists \( b \in B \) such that \( aRb \) and for all \( b \in B \), there exists \( a' \in A \) such that \( b'Ra' \). \( \mathcal{A} \mathcal{R} B \) means that for all \( a \in A \), and \( b \in B \), we have \( aRb \), \( R \) is called regular on the right (on the left) if for
all $x \in M$, from $aRb$, it follows that $(a \circ x)\overline{R}(b \circ x) \iff ((x \circ a)\overline{R}(x \circ b))$, $R$ is called strongly regular on the right (on the left) if for all $x \in M$, from $aRb$, it follows that $(a \circ x)\overline{R}(b \circ x) \iff ((x \circ a)\overline{R}(x \circ b))$, $R$ is called regular (strongly regular) if it is regular (strongly regular) on the right and on the left, $R$ is called good if $(a \circ b)R0$ and $(b \circ a)R0$ imply $aRb$, for all $a, b \in M$.

A totally ordered set $(X, 0)$ is said to be well ordered (or have a well-founded order) if every nonempty subset of $X$, has a least element. Every finite totally ordered set is well ordered.

**Theorem 2.4.** [10] (Zermelo’s Well-Ordering Theorem) Every set can be well-ordered.

**Lemma 2.5.** [8] Let $X$ be an infinite set. Then for any set $\{a, b\}$, we have $|X \times \{a, b\}| = |X|$.

**Theorem 2.6.** [1] Let $X$ and $Y$ be two sets such that $|X| = |Y|$. If $(Y, \leq, 0)$ is a well-ordered set, then there exists a binary order relation $\sim$ on $X$ and $x_0 \in X$, such that $(X, \leq, x_0)$ is a well-ordered set.

### 3. Constructing of Some $MV$-algebras

In this section, we get some results that we need in the next sections. Specially, we construct an $MV$-algebra and a strong hyper $MV$-algebra from a nonempty countable set and any totally ordered set with maximum element. We show that the $MV$-algebras and the hyper $MV$-algebras with the same cardinal are isomorphism.

**Lemma 3.1.** Let $X$ and $Y$ be two sets such that $|X| = |Y|$. If $(X, \oplus_X, *, 0_X)$ is an $MV$-algebra, then there exist a binary operation $\oplus_Y$, a unary operation $*_{y^*}$ and constant $0_y$ on $Y$, such that $(Y, \oplus_Y, *_{y^*}, 0_y)$ is an $MV$-algebra and $(X, \oplus_X, *, 0_X) \cong (Y, \oplus_Y, *_{y^*}, 0_y)$.

**Proof.** Since $|X| = |Y|$, then there exists a bijection $\varphi : X \rightarrow Y$. For any $y_1, y_2 \in Y$, we define the binary operation $\oplus_Y$ on $Y$ by, $y_1 \oplus_Y y_2 = \varphi(x_1 \oplus_X x_2)$, where $y_1 = \varphi(x_1)$, $y_2 = \varphi(x_2)$ and $x_1, x_2 \in X$. It is easy to show that $\oplus_Y$ is well-defined. Moreover, for any $y \in Y$ we define the unary operation $*_{y^*}$ as $y^* = \varphi(x^*)$, where $x \in X, y = \varphi(x)$ and $0_y = \varphi(0_X)$. Since $\varphi$ is a bijection, then the unary operation $*$ is well-defined. Now, by some modification we can show that $(Y, \oplus_Y, *_{y^*}, 0_y)$ is an $MV$-algebra. In the follow, we define the map $\theta : (X, \oplus_X, *, 0_X) \rightarrow (Y, \oplus_Y, *_{y^*}, \varphi(0^*_X))$ by $\theta(x) = \varphi(x)$. Since $\varphi$ is a bijection then $\theta$ is a
bijection. Now, it is easy to see that \( \theta \) is a homomorphism and so it is an isomorphism.

**Lemma 3.2.** For any \( k \in \mathbb{N} \), we can construct an MV-algebra on \( \mathbb{W}_k = \{0, 1, 2, 3, \ldots, k-1\} \), which is a chain.

**Proof.** Let \( k \in \mathbb{N} \). We define the binary operation \( \circ \) and the unary operation \( \ast \), on \( \mathbb{W}_k \) as follows:

\[
x \circ y = \begin{cases} 
  k - 1 & \text{if } x + y \geq k - 1 \\
  x + y & \text{otherwise}
\end{cases}
\quad \text{and} \quad x^\ast = k - (x + 1)
\]

Clearly, 0 is the smallest element in \( \mathbb{W}_k \), \( k - 1 = \max(\mathbb{W}_k) \) and for any \( x \in \mathbb{W}_k \), \( (x^\ast)^\ast = x \). First, we show that \( \circ \) is well-defined on \( \mathbb{W}_k \).

Let \( x = x' \) and \( y = y' \). If \( x + y \geq k - 1 \) then \( x' + y' \geq k - 1 \) and so \( x \circ y = k - 1 = x' \circ y' \). Moreover, if \( x + y < k - 1 \) then \( x' + y' < k - 1 \) and so \( x \circ y = x + y = x' + y' = x' \circ y' \). Now, we show that \( (\mathbb{W}_k, \circ, \ast, 0) \) is an MV-algebra. Let \( x, y, z \in \mathbb{W}_k \). Then,

- **(MV1):** Case 1: \( x + y \geq k - 1 \). Then \( x + (y + z) = (x + y) + z \geq k - 1 \).
- **(MV2):** Since \( x + y = x + x \), then \( x \circ y = y \circ x \).
- **(MV3):** By hypothesis, \( x \circ 0 = x \).
- **(MV4):** By hypothesis, \( 0^\ast = k - 1, (k-1)^\ast = 0 \) and \( (x^\ast)^\ast = x \).
- **(MV5):** By hypothesis, \( x \circ 0^\ast = x \circ (k - 1) = k - 1 = 0^\ast \).
- **(MV6):** Case 1: \( y < x \). Then, \( x - (k + 1) + y < k - 1 \) and

\[
(x^\ast \circ y)^\ast \circ y = ((k - (x + 1)) \circ y)^\ast \circ y = (k - (x + 1) + y)^\ast \circ y = (x - y) \circ y = x
\]

Moreover, in this case we have \( k - (1 + y) + x \geq (k - 1) \) and so

\[
(y^\ast \circ x^\ast)^\ast \circ x = ((k - (1 + y)) \circ x)^\ast \circ x = (k - 1)^\ast \circ x = 0 \circ x = x
\]

Case 2: \( y > x \). Then, \( x - (k + 1) + y \geq k - 1 \) and

\[
(x^\ast \circ y)^\ast \circ y = ((k - (x + 1)) \circ y)^\ast \circ y = (k - 1)^\ast \circ y = 0 \circ y = y
\]

Moreover, in this case we have \( k - (y + 1) + x < k - 1 \) and so

\[
(y^\ast \circ x^\ast)^\ast \circ x = ((k - (y + 1)) \circ x)^\ast \circ x = (k - (y + 1) + x)^\ast \circ x = (y - x) \circ x = y
\]

Case 3: \( y = x \). Then, \( (x^\ast \circ y)^\ast \circ y = (y^\ast \circ x)^\ast \circ x \). Therefore, \( (\mathbb{W}_k, \circ, \ast, 0) \) is an MV-algebra.

Now, for any \( x, y \in \mathbb{W}_k \), \( x \leq y \) if and only if \( x^\ast \circ y = k - 1 \) if and only if \( (k - (x + 1)) \circ y = k - 1 \) if and only if \( (k - (x + 1)) + y \geq k - 1 \) if and only if \( x \leq y \). Therefore, \( (\mathbb{W}_k, \circ, \ast, 0, \leq) \) is an MV-chain.
Theorem 3.3. Let $X$ be a finite set. Then there exist a binary operation $\oplus_x$ and unary operation $\ast_x$ and constant $0_x$ on $X$, such that $(X, \oplus_x, \ast_x, 0_x)$, is an MV-algebra.

Proof. Let $X$ be a finite set. Then, there exists $k \in \mathbb{W}$ such that $|X| = |W_k|$. Now, since by Lemma 3.2, $(\mathbb{W}, \circ, \ast, 0)$ is an MV-algebra, then by Lemma 3.1, there exist a binary operation $\oplus_x$ and a unary operation $\ast_x$ and constant $0_x$ on $X$, such that $(X, \oplus_x, \ast_x, 0_x)$, is an MV-algebra.

Lemma 3.4. Let $1 < n \in \mathbb{Q}$. Then there exist a binary operation $\circ$ and a unary operation $\ast$ on $E = \mathbb{Q} \cap [1, n]$, such that $(E, \circ, \ast, 1)$ is an MV-algebra.

Proof. For any $1 < n \in E$, we define the binary operation $\circ$ and the unary operation $\ast$ on $E$ as follows:

$$x \circ y = \begin{cases} n, & \text{if } xy \geq n \\ xy, & \text{otherwise} \end{cases}$$

and $x^* = \frac{n}{x}$.

Then 1 is the smallest element in $E$, $n = \max(E)$ and for any $x \in E$, $(x^*)^* = x$. First, we show that $\circ$ is well-defined on $E$. Let $x = x_1$ and $y = y_1$. If $xy \geq n$ then $x_1y_1 \geq n$ and so $x \circ y = n = x_1 \circ y_1$. Moreover, if $xy < n$ then $x_1y_1 < n$ and so $x \circ y = xy = x_1y_1 = x_1 \circ y_1$. Clearly $\ast$ is well-defined. Now, we show that $(E, \circ, \ast, 1)$ is an MV-algebra. Let $x, y, z \in E$. Then,

(MV1): If $xy \geq n$, since $z \geq 1$, then $x(yz) = (xy)z \geq n$. Now, let $xy < n$. If $(xy)z < n$, then $x(yz) = (xy)z < n$ and if $(xy)z \geq n$, then $x(yz) = (xy)z \geq n$. Since in any cases, $(xy)z = x(yz)$, then $x \circ (y \circ z) = (x \circ y) \circ z = x \circ (y \circ z)$.

(MV2): Since $xy = yx$, then $x \circ y = y \circ x$.

(MV3): By hypothesis, $x \circ 1 = x$.

(MV4): By hypothesis, $1^* = \frac{n}{n} = n$, $n^* = \frac{n}{n} = 1$ and $(x^*)^* = x$.

(MV5): By hypothesis, $x \circ 1^* = x \circ n = n = 1^*$.

(MV6): If $y < x$, then $\frac{ny}{x} < n$ and $(x^* \circ y)^* \circ y = (\frac{n}{y} \circ x)^* \circ y = (\frac{ny}{x})^* \circ y = \frac{n}{x} \circ y = \frac{x}{y} \circ y = x$. Moreover, in this case we have $\frac{ny}{x} > n$ and so $(y^* \circ x)^* \circ x = (\frac{n}{y} \circ x)^* \circ x = n^* \circ x = 1 \circ x = x$. If $y > x$ then, $\frac{ny}{x} > n$ and $(x^* \circ y)^* \circ y = (\frac{n}{y} \circ y)^* \circ y = n^* \circ y = 1 \circ y = y$. 


Moreover, in this case we have \( \frac{n}{y} < n \) and so

\[
(y^\ast \circ x)^\ast \circ x = \left( \frac{n}{y} \circ x \right)^\ast \circ x = \left( \frac{n}{y} \right)^\ast x = \frac{n}{y} \circ x = y
\]

If \( y = x \), then clearly \( (x^\ast \circ y)^\ast \circ y = (y^\ast \circ x)^\ast \circ x \). Therefore, \((E, \circ, *, 1)\) is an MV-algebra.

**Theorem 3.5.** Let \( X \) be an infinite countable set. Then there exists a binary operation " \( \oplus \) " , a unary operation " \( * \) " and constant " \( 0 \) " on \( X \), such that \((X, \oplus, *, 0)\) is an MV-algebra.

**Proof.** Let \( X \) be an infinite countable set. Since \( E = \mathbb{Q} \cap [1, n] \) in Lemma 3.4, is an infinite countable MV-algebra, so \(|X| = |E|\). Now, by Theorem 2.6, there exist a bijection \( \psi : E \rightarrow X \), a binary relation " \( \preceq \) " and the smallest element \( 0 = \psi(1) \) on \( X \) such that \((X, \preceq, 0)\) is a totally ordered set and for any \( t, s \in E \) we have

\[
(1) \quad \psi(t) \preceq \psi(s) \text{ if and only if } t \leq s.
\]

Hence, for the largest element \( n \in E \) and for any \( x \in X \), we have, \( 0 = \psi(1) \preceq x \preceq \psi(n) \). For any \( x, y \in X \), since \( \psi \) is onto, there exist \( i, j \in E \) such that \( x = \psi(i) \) and \( y = \psi(j) \). Now, we define a binary operation " \( \oplus \) " and a unary operation " \( * \) " on \( X \) as follows:

\[
x \oplus y = \begin{cases} 
\psi(n) & \text{if } n \leq i \circ j \\
\psi(i \circ j) & \text{otherwise}
\end{cases}
\]

and \( x^\ast = \psi(i^\ast) = \psi\left(\frac{n}{i}\right) \).

that the operation " \( \circ \) " is defined in Lemma 3.4. First, we show that " \( \oplus \) " is well-defined. Let \( x = x_1 \) and \( y = y_1 \). Then there exist \( i, i_1, j, j_1 \in E \) such that \( x = \psi(i), x_1 = \psi(i_1), y = \psi(j), y_1 = \psi(j_1) \). Since, \( \psi \) is a bijection, then \( i = i_1 \) and \( j = j_1 \). Now, if \( i \circ j \geq n \) then \( i_1 \circ j_1 \geq n \) and so \( x \oplus y = \psi(n) = \psi(i_1 \circ j_1) = x_1 \oplus y_1 \). Moreover, if \( i \circ j < n \) then \( i_1 \circ j_1 < n \) and so \( x \oplus y = \psi(i \circ j) = \psi(i_1 \circ j_1) = x_1 \oplus y_1 \). Since, \( \psi \) is a bijection, then clearly the operation " \( * \) " is well-defined. Now, since \((E, \circ, *, 1)\) is an MV-algebra, then we show that \((X, \oplus, *, 0)\) is an MV-algebra. For this, let \( x = \psi(i), y = \psi(j), z = \psi(k) \in X \) where \( i, j, k \in E \).

(MV1): If \( i \circ j \geq n \), then by Lemma 3.4, for any \( k \in E \) we have, \( i \circ (j \circ k) = (i \circ j) \circ k \geq n \).

Now, let \( i \circ j < n \). If \( (i \circ j) \circ k < n \), then \( i \circ (j \circ k) = (i \circ j) \circ k < n \) and if \( (i \circ j) \circ k = n \), then \( i \circ (j \circ k) = (i \circ j) \circ k = n \). Since in any
cases, \((i \odot j) \odot k = i \odot (j \odot k)\), and \(\psi\) is a bijection, then \(\psi((i \odot j) \odot k) = \psi(i \odot (j \odot k))\) and so

\[
(x \oplus y) \odot z = \psi((i \odot j) \odot k) = \psi(i \odot (j \odot k))
\]

Therefore, \((X, \oplus, *, 0)\) is an MV-algebra.

\[
\square
\]
Corollary 3.6. For any nonempty countable set $X$, we can construct an $MV$-algebra on $X$.

Proof. Let $X$ be a nonempty countable set. Then, $|X| = |E|$, where $E = \mathbb{Q} \cap [1, n]$ is infinite countable set in Lemma 3.4, or there exists $k \in \mathbb{N}$ such that $|X| = |\mathbb{W}_k|$. Now, by the Theorems 3.3 and 3.5, the proof is straightforward. □

Theorem 3.7. Let $X$ be an infinite set. If $(X, \oplus_X, 0_X, \ast_X)$ is an $MV$-algebra, then for any set $\{a, b\}$, there exist a binary hyperoperation ”$\oplus$”, a unary operation ”$\ast$” and constant ”$0$” on $X$ such that $(X \times \{a, b\}, \oplus, \ast, 0)$ is an $MV$-algebra and $(X, \oplus_X, \ast_X, 0_X) \cong (X \times \{a, b\}, \oplus, \ast, 0)$

Proof. Since $X$ is an infinite set, then by Lemma 2.5, $|X \times \{a, b\}| = |X|$. Now, by Lemma 3.1, the proof is straightforward. □

4. Constructing of Some (Strong) Hyper $MV$-algebras

Theorem 4.1. Let $(M, \oplus_M, \ast_M, 0_M)$ and $(N, \oplus_N, \ast_N, 0_N)$ be two $MV$-algebras. Then there exist a binary hyperoperation ”$\oplus$”, a unary operation ”$\ast$” and constant ”$0$” on $M \times N$, such that $(M \times N, \oplus, \ast, 0)$ is a hyper $MV$-algebra.

Proof. Let $(M, \oplus_M, \ast_M, 0_M)$ and $(N, \oplus_N, \ast_N, 0_N)$ be two $MV$-algebras. For any $(m_1, n_1), (m_2, n_2) \in M \times N$, we define the binary hyperoperation ”$\oplus$” on $M \times N$ by, $(m_1, n_1) \oplus (m_2, n_2) = \{(m_1 \oplus_M m_2, n_1), (m_1 \oplus_M m_2, n_2)\}$ and for any $(m, n) \in M \times N$, the unary operation ”$\ast$” by, $(m, n)^\ast = * (m, n) = * (m^\ast M, n^\ast N) = (m^\ast M, n^\ast N)$ and constant $0 = (0_M, 0_N)$. First, we show that the hyperoperation ”$\oplus$” is well defined. Let $(m_1, n_1) = (m_1', n_1')$ and $(m_2, n_2) = (m_2', n_2')$. Then,

\[(2) \quad (m_1, n_1) \oplus (m_2, n_2) = \{(m_1 \oplus_M m_2, n_1), (m_1 \oplus_M m_2, n_2)\} = \{(m_1' \oplus_M m_2', n_1'), (m_1' \oplus_M m_2', n_2')\} = (m_1', n_1') \oplus (m_2', n_2')\]

Moreover, since $(m, n) = (m', n')$ implies that $* (m, n) = * (m', n')$ then ”$\ast$” is well-defined. Now, by some modifications we can show that $(M \times N, \oplus, \ast, 0)$ is a hyper $MV$-algebra. □

Theorem 4.2. Let $(M, \oplus_M, \ast_M, 0_M, \leq_M)$ and $(N, \oplus_N, \ast_N, 0_N, \leq_N)$ be two $MV$-chains. Then there exist a binary hyperoperation ”$\oplus$”, a unary operation ”$\ast$” and constant ”$0$” on $M \times N$, such that $(M \times N, \oplus, \ast, 0)$ is a strong hyper $MV$-algebra.
Proof. Let \((M, \oplus_M, \ast_M, 0_M)\) be an MV-algebra and \((N, \oplus_N, \ast_N, 0_N)\) be an MV-chain. Now, for any \((m_1, n_1), (m_2, n_2) \in M \times N\), we define the binary hyperoperation ”\(\oplus\)” on \(M \times N\) by, \((m_1, n_1) \oplus (m_2, n_2) = \{(m_1 \oplus_M m_2, n_1), (m_1 \oplus_M m_2, n_2)\}\) and for any \((m, n) \in M \times N\), the unary operation ”\(*\)” by, \((m, n)^* = *(m, n) = \ast_M(m, \ast_N(n)) = (m^\ast_M, n^\ast_N)\) and we let constant \(0 = (0_M, 0_N)\). By Theorem 4.1, \((M \times N, \oplus, \ast, 0)\) is a hyper MV-algebra. Now, we define a binary relation ”\(\ll\)” on \(M \times N\) by, \((x, y) \ll (z, w)\) if and only if \((0_M, 0_N)^* \in (x, y)^* \oplus (z, w)\). We show that for any \((x, y), (z, w) \in M \times N\), if \((x, y) \ll (z, w)\) then \(x \lesssim z\) and \(y \lesssim w\). For this, let \((x, y) \ll (z, w)\). Then by the hypothesis,

\[
(0_M, 0_N)^* = (0_M^*, 0_N^*) \in (x, y)^* \oplus (z, w) = (x^\ast_M, y^\ast_N) \oplus (z, w)
\]

and so \((0_M^*, 0_N^*) = (x^\ast_M \oplus_M z, y^\ast_N)\) or \((0_M^*, 0_N^*) = (x^\ast_M \oplus_M z, w)\). If \((0_M^*, 0_N^*) = (x^\ast_M \oplus_M z, y^\ast_N)\), then \(y = 0_N, x^\ast_M \oplus_M z = 0_M^*\). Now since \((M, \oplus_M, \ast_M, 0_M)\) is an MV-chain, \(x \lesssim z\) and \(y \lesssim 0_N\). If \((0_M^*, 0_N^*) = (x^\ast_M \oplus_M z, w)\), then \(w = 0_N^*, x^\ast_M \oplus_M z = 0_M^*\). Now, since \((N, \oplus_N, \ast_N, 0_N)\) is an MV-chain, \(x \lesssim z\) and \(y \lesssim 0_N\). Hence, in any cases, we have, \(x \lesssim z\) and \(y \lesssim w\). Therefore, \((M \times N, \oplus, \ast, 0)\) is a strong hyper MV-algebra.

**Lemma 4.3.** Let \(X\) and \(Y\) be two sets such that \(|X| = |Y|\). If \((X, \oplus_X, \ast_X, 0_X)\) is a (strong) hyper MV-algebra, then there exist a binary hyperoperation ”\(\oplus_Y\)” , a unary operation ”\(*_Y\)” and constant ”\(0_Y\)” on \(Y\), such that \((Y, \oplus_Y, \ast_Y, 0_Y)\) is a strong hyper MV-algebra and \((X, \oplus_X, \ast_X, 0_X) \simeq (Y, \oplus_Y, \ast_Y, 0_Y)\).

**Proof.** The proof is similar to the proof of Lemma 3.1, by some modifications.

**Corollary 4.4.** Let \((M, \oplus_M, \ast_M, 0_M, \lesssim)\) be an MV-chain. Then for any set \(\{a, b\}\):

(i) there exist a binary hyperoperation ”\(\oplus\)” , a unary operation ”\(*\)” and constant ”\(0\)” on \(M \times \{a, b\}\), such that \((M \times \{a, b\}, \oplus, \ast, 0)\) is a strong hyper MV-algebra.

(ii) If \(M\) is infinite, then there exist a binary hyperoperation ”\(\odot\)” , a unary operation ”\(\ast\)” and constant ”\(0\)” on \(M\), such that \((M, \odot, \ast, 0)\) is a strong hyper MV-algebra. and \((M \times \{a, b\}, \odot, \ast, 0) \cong (M, \odot, \ast, 0)\).

**Proof.** (i) First, we define the partial relation ”\(\leq\)” on set \(\{a, b\}\) by \(\leq := \{(a, a), (b, b), (a, b)\}\). Hence \((\{a, b\}, \leq)\) is a totally ordered set. Now we consider the following binary and unary operations:
Then clearly \((\{a, b\}, a, \oplus, \ast)\) is a the smallest nontrivial MV-chain. Now, we define the binary hyperoperation "\(\oplus\)" on \(M \times \{a, b\}\) as follows:
\[
(m_1, t) \oplus (m_2, s) = \{(m_1 \oplus_M m_2, t), (m_1 \oplus_M m_2, s)\}
\]
Similar to proof of Theorem 4.2, \((M \times \{a, b\}, \oplus, \ast, 0)\) is a strong hyper MV-algebra.

(ii) Since \(M\) is infinite set, then by Lemmas 2.5 and 4.3, there exist a binary hyperoperation "\(\circ\)", a unary operation "\(\ast\)" and constant "0" on \(M\), such that \((\{a, b\}, \circ, 0)\) is a strong hyper MV-algebra and
\[
(M \times \{a, b\}, \oplus, \ast, 0) \cong (M, \circ, 0).
\]

**Theorem 4.5.** Let \((X, \leq, x_0, y_0)\) be a totally ordered set with smallest element \(x_0\) and greatest element \(y_0\). Then, there exist a binary hyperoperation "\(\circ\)" and a unary operation "\(\ast\)" on \(X\), such that \((X, \circ, \ast, x_0)\) is a (strong) hyper MV-algebra.

**Proof.** Firstly, if \(X = \{x_0, y_0\}\), then by the following tables:
\[
\begin{array}{c|c|c}
\circ & x_0 & y_0 \\
\hline
x_0 & \{x_0, y_0\} & \{x_0, y_0\} \\
y_0 & \{x_0, y_0\} & \{x_0, y_0\}
\end{array}
\]
\[
\begin{array}{c|c|c}
\ast & x_0 & y_0 \\
\hline
x_0 & x_0 & y_0 \\
y_0 & y_0 & x_0
\end{array}
\]

\((X, \circ, \ast, x_0)\) is a (strong) hyper MV-algebra. Now, let \(|X| \geq 3\). For any \(x, y \in X\), we define a binary hyperoperation "\(\circ\)" and unary operation "\(\ast\)" as follows:

\[
x \circ y = \begin{cases} 
\{x_0, x, y\} & \text{if } x \neq y \\
\{x_0, y_0, x\} & \text{if } x = y
\end{cases}
\]
\[
x^{\ast} = \begin{cases} 
y_0 & \text{if } x = x_0 \\
x_0 & \text{if } x = y_0 \\
x & \text{otherwise}
\end{cases}
\]

First, we show "\(\circ\)" is well-defined. Let \(x = x'\) and \(y = y'\). If \(x \neq y\), then, \(x \circ y = \{x_0, x, y\} = \{x_0, x', y'\} = x' \circ y'\). Now, let \(x = y\). Then, \(x \circ y = \{x_0, x, y_0\} = \{x_0, x', y_0\} = x' \circ y'\). Hence "\(\circ\)" is well-defined. Clearly the unary operation "\(\ast\)" is well-defined, too. Now we show that \((X, \circ, \ast, x_0)\) is a hyper MV-algebra. Let \(x, y, z \in X\). Then,

\((HMV1)\): Case 1: If \(x = y = z\), then, \((x \circ y) \circ z = x \circ (y \circ z)\).

Case 2: If \(x = y \neq z\), then, \((x \circ y) \circ z = \{x_0, x, z, y_0\} = x \circ (y \circ z)\).

Case 3: If \(x \neq y = z\), then, \((x \circ y) \circ z = \{x_0, x, y, y_0\} = x \circ (y \circ z)\).
Case 4: If $x = z \neq y$, then, $(x \circ y) \circ z = \{x_0, x, y, y_0\} = x \circ (y \circ z)$.

Case 5: If $x \neq z \neq y$, then, $(x \circ y) \circ z = \{x_0, x, z, y\} = x \circ (y \circ z)$.

(HMV2): If $x \neq y$, then, $(x \circ y) = \{x_0, x, y\} = \{x_0, y, x\} = (y \circ x)$.

Now let $x = y$. Then, $(x \circ y) = \{x_0, x, y_0\} = (y \circ x)$.

(HMV3): By hypothesis $(x^*)^* = (x^*) = x$.

(HMV4): Case 1: If $x = x_0$ and $y = y_0$, then,

\[(x^* \circ y)^* \circ y = \{y_0, x_0\} = (y_0 \circ x) \circ x = (y^* \circ x)^* \circ x\]

Case 2: If $x = x_0$ and $y \neq y_0$, then, $(x^* \circ y)^* \circ y = \{y_0, x_0, y\} = (y_0 \circ x) \circ x = (y^* \circ x)^* \circ x$.

Case 3: If $x \neq x_0$ and $y = y_0$, then, $(x^* \circ y)^* \circ y = \{y_0, x_0, x\} = (y_0 \circ x) \circ x = (y^* \circ x)^* \circ x$.

Case 4: If $x \neq x_0$, $y \neq y_0$ and $x \neq y$, then, $(x^* \circ y)^* \circ y = \{y_0, x_0, x, y\} = (y_0 \circ x) \circ x = (y^* \circ x)^* \circ x$.

(HMV5): By hypothesis $x \circ x_0 = \{x, x_0\}$, then $x \in x \circ x_0$.

(HMV6): By hypothesis $x \circ x_0^* = \{x, x_0^*, x_0\}$ then $x_0^* \in x \circ x_0^*$. Therefore, $(X, \circ, *, x_0)$ is a hyper MV-algebra.

(HMV7): If $x \ll y$ and $y \ll x$, then $y_0 \in x^* \circ y$ and $y_0 \in y^* \circ x$.

Since $\{x, y\} \notin \{x_0, y_0\}$, then $x^* = x$ and $y^* = y$. This implies that $y_0 \in x \circ y = y \circ x$ and by hypothesis $x = y$.

Therefore, $(X, \circ, *, x_0)$ is a strong hyper MV-algebra. \hfill \Box

**Open Problem 4.6.** We proved that any bonded totally ordered set can be a strong hyper MV-algebra. Let $X$ be an infinite non bounded totally ordered set. Is there a binary hyperoperation "$\oplus$", a unary operation "$*$" and constant "0", such that $(X, \oplus, *, 0)$ is a (strong) hyper MV-algebra?

## 5. Fundamental MV-algebras

In this section, by using the notion of fundamental relation, we define the concept of fundamental MV-algebra and we prove that any MV-algebra is a fundamental MV-algebra. Let $(M, \oplus, *, 0)$ be a hyper MV-algebra and $A$ be a subset of $M$. Then with Now, in the following, the well-known idea of $\beta^*$ relation on hyperstructure [4, 16, 13] is transferred and applied to hyper MV-algebras.

Let $(M, \oplus, *, 0)$ be a hyper MV-algebra and $\mathcal{L}(A)$ denote the set of all finite combinations of elements $A$ with $\oplus$ and $*$. For example, $\mathcal{L}\{x_1, x_2\} = \{x_1 \oplus x_2, x_1 \oplus x_2, x_1 \oplus x_2, (x_1 \oplus x_2)^*, (x_1 \oplus x_2)^* \oplus x_1, \ldots\}$.
Then we set $\beta_1 = \{(x, x) \mid x \in M\}$ and for every integer $n \geq 1$, $\beta_n$ is the relation defined as follows:

$$x_1 \beta_n y \iff \exists (a_1, a_2, \ldots, a_n) \in X^n, \exists u \in L(a_1, a_2, \ldots, a_n) \text{ s.t. } \{x, y\} \subseteq u$$

Obviously, for every $n \geq 1$, the relations $\beta_n$ are symmetric, and the relation $\beta = \bigcup_{n \geq 1} \beta_n$ is reflexive and symmetric. Now, let $\beta^*$ be the transitive closure of $\beta$. Then $\beta^*$ is the smallest strongly regular equivalence relation on $M$, such that $M/\beta^*$ is an MV-algebra. (See [13]).

**Theorem 5.1.** [14] Let $(M_i, \oplus_i, *, 0_i)$ be a hyper MV-algebra and $\beta^*_i$ be a fundamental relation on $M_i$ for any $i = 1, 2, \ldots, n$. Then,

$$\frac{M_1 \times M_2 \times \ldots \times M_n}{\beta^*_1 \times \beta^*_2 \times \ldots \times \beta^*_n} \cong \frac{M_1}{\beta^*_1} \times \frac{M_2}{\beta^*_2} \times \ldots \times \frac{M_n}{\beta^*_n}$$

**Lemma 5.2.** Let $(M, \oplus, *, 0)$ be a hyper MV-algebra. Then for the fundamental relation $\beta^*$ and for any $m \in M$, we have $\beta^*(m^*) = (\beta^*(m))^*$.

**Proof.** Let $m \in M$. For any $t \in M$, if $t \in \beta^*(m^*)$, then there exist $n \geq 1, (a_1, a_2, \ldots, a_n) \in M^n$ and $u \in L(a_1, a_2, \ldots, a_n)$ such that $\{m, t\} \subseteq u$. Now, since $\{m, t\} = \{(m)^*, (t)^*\} \subseteq \beta^*(m)$, then $t^* \in \beta^*(m)$ and so $\beta^*(m^*) \subseteq (\beta^*(m))^*$. Let $t \in (\beta^*(m))^*$. Then $t^* \in \beta^*(m)$ and there exist $n \geq 1, (a_1, a_2, \ldots, a_n) \in M^n$ and $u \in L(a_1, a_2, \ldots, a_n)$ such that $\{m, t\} \subseteq u$. Now, since $\{m, t\} = \{(m^*), (t^*\} \subseteq \beta^*(m)$ and so $(\beta^*(m))^* \subseteq \beta^*(m)$.

**Lemma 5.3.** Let $(X, \oplus_X, *, X_0)$ and $(Y, \oplus_Y, *, Y_0)$ be two hyper MV-algebras and $f : (X, \oplus_X, *, X_0) \rightarrow (Y, \oplus_Y, *, Y_0)$ be a homomorphism. Then for any $x, y \in X$, $x \beta_1^* y$ implies that $f(x) \beta_2^* f(y)$.

**Proof.** Let $(X, \oplus_X, *, X_0)$ and $(Y, \oplus_Y, *, Y_0)$ be two hyper MV-algebras and $x, y \in X$. Since $x \beta_1^* y$, then there exist $u \in L(X)$, such that $\{x, y\} \subseteq u$. Now, for homomorphism $f : (X, \oplus_X, *, X_0) \rightarrow (Y, \oplus_Y, *, Y_0)$ we have $\{f(x), f(y)\} = f\{x, y\} \subseteq f(u) \in L(Y)$. Therefore, $f(x) \beta_2^* f(y)$.

**Example 5.4.** Let $(M_1, \oplus_1, *, 0_1)$ and $(M_2, \oplus_2, *, 0_2)$ be two hyper MV-algebras by the following tables:

<table>
<thead>
<tr>
<th>$\oplus_1$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0, 1}</td>
<td>{0, 1}</td>
</tr>
<tr>
<td>1</td>
<td>{0, 1}</td>
<td>{0, 1}</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\ast_1$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0, 1}</td>
<td>{0, 1}</td>
</tr>
<tr>
<td>1</td>
<td>{0, 1}</td>
<td>{0, 1}</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\oplus_2$</th>
<th>0</th>
<th>b</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{b}</td>
<td>{b, 1}</td>
</tr>
<tr>
<td>b</td>
<td>{b}</td>
<td>{b, 1}</td>
<td>{b, 1}</td>
</tr>
<tr>
<td>1</td>
<td>{b, 1}</td>
<td>{b, 1}</td>
<td>{b, 1}</td>
</tr>
</tbody>
</table>

and
Now, we define the map \( f : (M_2, \oplus_2, *, 0) \longrightarrow (M_1, \oplus_1, *, 0) \) by \( f(0) = 0 \) and \( f(1) = f(b) = 1 \). Moreover, \( \{ (M_1, \oplus_1, *, 0) \}_0 = \{ \beta^*(0) = \{ 0, 1 \}, \beta^*(0) = \{ 0, 1 \} \} \) and \( \{ (M_2, \oplus_2, *, 0) \}_0 = \{ \beta^*(0) = \{ 0 \}, \beta^*(b) = \{ b, 1 \}, \beta^*(b) = \{ b, 1 \} \} \).

Clearly \( f \) is a homomorphism which is not injective and \( f(b) \in \beta^*(f(1)) \), but \( b \not\in \beta^*(1) \).

**Lemma 5.5.** Let \((X, \oplus_X, *, 0_X)\) and \((Y, \oplus_Y, *, 0_Y)\) be hyper MV-algebras and \( f : (X, \oplus_X, *, 0_X) \rightarrow (Y, \oplus_Y, *, 0_Y) \) be a monomorphism. Then for any \( x, y \in X \), \( f(x) \beta^* f(y) \) implies that \( x \beta^*_X y \).

**Proof.** For any \( x, y \in X \), since \( f(x) \beta^*_X f(y) \), there exists \( v \in L(Y) \), such that \( \{ f(x), f(y) \} \subseteq v \). Now, for a monomorphism \( f : X \rightarrow Y \) we have \( \{ x, y \} = \{ f^{-1}(f(x)), f^{-1}(f(y)) \} = f^{-1}(f(x), f(y)) \subseteq f^{-1}(v) \subseteq U \). Therefore, \( x \beta^*_X y \). □

**Lemma 5.6.** Let \((X, \oplus_X, *, 0_X)\) and \((Y, \oplus_Y, *, 0_Y)\) be two hyper MV-algebras and \( f : (X, \oplus_X, *, 0_X) \rightarrow (Y, \oplus_Y, *, 0_Y) \) be an isomorphism. Then for any \( x, y \in X \), \( x \beta^*_X y \) if and only if \( f(x) \beta^*_Y f(y) \).

**Proof.** By Lemmas 5.3 and 5.5, the proof is straightforward. □

**Theorem 5.7.** Let \( X \) and \( Y \) be two nonempty sets and \( |X| = |Y| \).
If \((X, \oplus_X, *, 0_X)\) is a (strong) hyper MV-algebra, then there exist a binary hyperoperation \( " \oplus_X " \), a unary operation \( " \beta^*_X " \) and constant \( " 0_X " \) on \( Y \), such that \( (X, \oplus_X, *, 0_X) \cong (Y, \oplus_Y, \beta^*_Y, 0_Y) \).

**Proof.** Since \( |X| = |Y| \), then by Lemma 4.3, there exist a binary hyperoperation \( " \oplus_Y " \), a unary operation \( " \beta^*_Y " \) and constant \( 0_Y \) on \( Y \) such that \((Y, \oplus_Y, *, 0_Y)\) is a (strong) hyper MV-algebra. Moreover, there exists an isomorphism \( f : (X, \oplus_X, *, 0_X) \longrightarrow (Y, \oplus_Y, *, 0_Y) \), such that \( f(0_X) = 0_Y \). Now, we define the map \( \varphi : (X, \oplus_X, *, 0_X) \longrightarrow (Y, \oplus_Y, *, 0_Y) \) by \( \varphi(\beta^* x) = \beta^*(f(x)) \). First, we show that for any \( x, x_2 \in X \), \( \varphi(\beta^* (x_1) \oplus \beta^* (x_2)) = \varphi(\beta^* (x_1)) \oplus \varphi(\beta^* (x_2)) \). By Lemma 5.2, for any \( x \in X \),

\[
\varphi(\beta^* (x_1) \oplus \beta^* (x_2)) = \varphi(\beta^* (x_1) \oplus X x_2)) = \beta^*(f(x_1) \oplus X x_2)) = \beta^*(f(x_1) \oplus \beta^* (f(x_2))) = \varphi(\beta^* (x_1)) \oplus \varphi(\beta^* (x_2))
\]

Since \( f \) is bijection, then \( \varphi \) is a bijection. Now, we show that \( \varphi \) is well-defined. Let \( y_1, y_2 \in Y \). Then there exist the unique elements \( x_1, x_2 \in X \)
such that \( y_1 = f(x_1) \) and \( y_2 = f(x_2) \). Now, by Equation (3) and Lemma 5.6, \( \varphi (\beta^* (x_1)) = \varphi (\beta^* (x_2)) \) if and only if \( \beta^* (f(x_1)) = \beta^* (f(x_2)) \) if and only if \( \beta^* (x_1) = \beta^* (x_2) \). Therefore, \( \varphi \) is well-defined and one to one. In follow, we show that \( \beta \) is a homomorphism. Hence \( \varphi \) is an isomorphism. Therefore, \( \left( \frac{X, \odot, \oplus, 0_X}{\beta^*}, \varphi \right) \cong \left( \frac{Y, \odot, \oplus, 0_Y}{\varphi} \right) \).

**Definition 5.8.** An MV-algebra \((M, \oplus_M, *, 0_M)\), is called a fundamental MV-algebra, if there exists a nontrivial hyper MV-algebra \((N, \oplus_N, *, 0_N)\), such that \( \left( \frac{N, \oplus_N, *, 0_N}{\beta^*}, \varphi \right) \cong \left( \frac{M, \oplus_M, *, 0_M}{\beta^*} \right) \).

**Theorem 5.9.** Every MV-algebra can be a fundamental MV-algebra.

**Proof.** Let \((M, \oplus_M, *, 0_M)\) be an MV-algebra. Then by Theorem 4.1, for any MV-algebra \((N, \oplus_N, *, 0_N)\), \((M \times N, \oplus, *, (0_M, 0_N))\) is a hyper MV-algebra. First, we show that for any \((a, b) \in M \times N\), \( \beta^*(a, b) = \{(a, x) \mid x \in N\} \). For this let, \( u = \bigoplus_{i=1}^n (m_i, n_i) \in \ell(M \times N) \), where \((m_i, n_i) \in M \times N\). We have

\[
u = \bigoplus_{i=1}^n (m_i, n_i) = \{ \bigoplus_{i=1}^n (m_i, x) \mid m_i \in M, x \in N \}
\]

Now, if affect the unary operation * on element \( u \), then we obtain the type \( u = \{(a, x) \mid a \in M \text{ is fixed and } x \in N\} \). Hence, for any \((a, b), (c, d) \in M \times N\), \((a, b)\beta^*(c, d)\) if and only if \( a = c \). Now, we define the map \( \varphi : \left( \frac{M \times N, \oplus, *, (0_M, 0_N)}{\beta^*}, \varphi \right) \rightarrow \left( M, \oplus_M, *, 0_M \right) \) by \( \varphi(\beta^*(m, n)) = m \). It is clear that \( \beta^*(m, n) = \beta^*(m', n') \) if and only if \( m = m' \) if and only if \( \varphi(\beta^*(m, n)) = \varphi(\beta^*(m', n')) \). Then, \( \varphi \) is well defined and one to one. In follow, we show that \( \varphi \) is a homomorphism. For this we have

\[
\varphi(\beta^*(m, n) \odot \beta^*(m', n')) = \varphi(\beta^*(m \oplus_M m', n)) = m \oplus_M m' = \varphi(\beta^*(m, n)) \odot \varphi(\beta^*(m', n')).
\]

Moreover, by Lemma 5.2, for any \( m \in M \), \( \varphi((\beta^*(m, n))^*) = \varphi(\beta^*(m, n^*)) = m^* = (\varphi(\beta^*(m, n))^* \) and \( \varphi(\beta^*(0_M, 0_N)) = 0_M \). Clearly, \( \varphi \) is onto. Therefore, \( \varphi \) is an isomorphism. \( \square \)

**Corollary 5.10.** From every infinite countable set we can construct a fundamental MV-algebra.

**Proof.** By Corollary 3.6, there exists a binary operation "\( \oplus \)" a unary operation "\( \ast \)" and constant "\( 0 \)" such that \((M, \oplus, \ast, 0)\) is an MV-algebra. Now by Theorem 5.9, \((M, \oplus, \ast, 0)\) is a fundamental MV-algebra. \( \square \)
Theorem 5.11. Let \((M, \oplus, *, 0)\) be any finite MV-algebra. Then for any binary hyperoperation "\(\oplus\)”, unary operation "\(*\)” and constant "\(0\)” on \(M\), such that \((M, \oplus, *, 0)\) is a hyper MV-algebra, there is not any isomorphic between \((M, \oplus, *, 0)\) and \(\left(\frac{(M, \oplus, *, 0)}{\beta^*}, \oplus^*\right)\), that is \((M, \oplus, *, 0) \not\cong \left(\frac{(M, \oplus, *, 0)}{\beta^*}, \oplus^*\right)\).

Proof. Let \((M, \oplus, *, 0)\) be a finite MV-algebra, \(|M| = n\) and "\(\oplus\)” be a hyperoperation, "\(*\)” unary operation and "\(0\)” constant on \(M\), such that \((M, \oplus, *, 0)\) is a hyper MV-algebra. Then there exist \(x, y \in M\) such that \(|x \oplus y| \geq 2\). Hence, there are \(m, n \in x \oplus y\) such that \(\beta^*(m) = \beta^*(n)\). Since \(\frac{M}{\beta^*} = \{\beta^*(x) \mid x \in M\}\), then, \(|\frac{M}{\beta^*}| < n = |M|\). Therefore, \(\left(\frac{(M, \oplus, *, 0)}{\beta^*}, \oplus^*\right) \not\cong (M, \oplus, *, 0)\).

Now, in the follow we try to show that for any infinite countable set \(M\), there exist an operation "\(\oplus\)” an unary operation * and constant 0 and a hyperoperation "\(\oplus\)” on \(M\), such that \((M, \oplus, *, 0)\) is an MV-algebra and \((M, \oplus, *, 0)\) is a hyper MV-algebra. Moreover, \(\left(\frac{(M, \oplus, *, 0)}{\beta^*}, \oplus^*\right) \cong (M, \oplus, *, 0)\).

Theorem 5.12. Let \(M\) be an infinite countable set. Then there exist an operation "\(\oplus\)” a unary operation "\(*\)” and constant "\(0\)” and a binary hyperoperation "\(\circ\)” on \(M\) such that \(\left(\frac{(M, \oplus, *, 0)}{\beta^*}, \oplus^*\right) \cong (M, \oplus, *, 0)\). That is, \(M\) is a fundamental MV-algebra of itself.

Proof. Let \(M\) be an infinite countable set. Then by Corollary 5.10, there exist a binary operation "\(\oplus_M\)” a unary operation "\(*\)” and constant "\(0_M\)” such that \((M, \oplus_M, *, 0_M)\) is an MV-algebra. Moreover, by Corollary 4.4, there exist a binary hyperoperation "\(\oplus\)” a unary operation "\(*\)” and constant "\((0_M, a)\)” such that \((M \times \{a, b\}, \oplus, *, (0_M, a))\) is a strong hyper MV-algebra and by Theorem 5.7, there exist a binary hyperoperation "\(\circ\)” a unary operation "\(*\)” and constant "\(0\)” such that \((M, \circ, *, 0)\) is a strong hyper MV-algebra and

\[
(4) \quad \frac{(M \times \{a, b\}, \oplus, *, (0_M, a))}{\beta^*} \cong \frac{(M, \circ, *, 0)}{\beta^*}
\]

First, we show that for any \((m, t) \in M \times \{a, b\}, \beta^*(m, t) = \{(m, a), (m, b)\}\).

For this let \(u = \bigoplus_{i=1}^{n} (m_i, n_i) \in \ell(M \times \{a, b\})\), where \((m_i, n_i) \in M \times \{a, b\}\).

We have

\[
u = \bigoplus_{i=1}^{n} (m_i, n_i) = \{\bigoplus_{i=1}^{n} m_i, a, \bigoplus_{i=1}^{n} m_i, b\}\]
Now, if affect the unary operation $\ast$ on element $u$. Then we obtain the type $u = \{(m, a), (m, b) \mid m \in M \text{ is fixed}\}$ too. Hence, for any $(m, t), (n, s) \in M \times \{a, b\}$, $(m, t)\beta^*(n, s)$ if and only if $m = n$.

Now, we define the map $\varphi : \left(\frac{M \times \{a, b\}, \oplus, \ast, (0_M, a)}{\beta^*}\right) \to (M, \oplus_M, \ast_M, 0_M)$ by $\varphi(\beta^*(m, t)) = m$. It is clear that $\beta^*(m, t) = \beta^*(m', s)$ if and only if $m = m'$ if and only if $\varphi(\beta^*(m, t)) = \varphi(\beta^*(m', s))$. Then, $\varphi$ is well defined and one to one. Now, we show that $\varphi$ is a homomorphism. For this we have,

$$\varphi(\beta^*(m, t) \oplus \beta^*(m', s)) = \varphi(\beta^*(m \oplus_M m', t)) = m \oplus_M m'$$

$$= \varphi(\beta^*(m, t)) \oplus_M \varphi(\beta^*(m', s)).$$

Moreover, by Lemma 5.2, for any $m \in M$, $\varphi((\beta^*(m, t))^*) = \varphi(\beta^*(m^*, t^*)) = m^* = (\varphi(\beta^*(m, t))^*)^*$ and $\varphi(\beta^*(0_M, a)) = 0_M$. Clearly, $\varphi$ is onto. Hence, $\varphi$ is an isomorphism and so

(5) \[ \left(\frac{M \times \{a, b\}, \oplus, \ast, (0_M, a)}{\beta^*}\right) \cong (M, \oplus_M, \ast_M, 0_M) \]

Therefore, by (4) and (5), we have

\[ (M, \oplus_M, \ast_M, 0_M) \cong \left(\frac{M \times \{a, b\}, \oplus, \ast, (0_M, a)}{\beta^*}\right) \cong (M, \ominus, \ast, 0) \]

**Open Problem 5.13.** If $(M, \oplus, \ast, 0)$ is an infinite non-countable MV-algebra, then is it $(M, \oplus, \ast, 0)$ as a fundamental MV-algebra of itself?

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**References**


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