Interfacial Boundary Estimation in Stratified Flow of Two Immiscible Liquids Using Hybrid-type Fourier Series

Bong Seok Kim*, Bong-Yeol Choi**, Kyung Youn Kim***

Abstract

In stratified flows of two immiscible liquids, due to the vibration in a pipe, the shape of the interface is not always periodic and it causes the different end points of the interfacial boundary. In this case the performance is not good. To solve this, in this paper, the hybrid-type Fourier series is proposed, which consists of both the polynomial and the trigonometric terms. Under the stationary interfacial boundary during acquiring a full set of voltage data, the performance of the proposed method is evaluated through the numerical experiments. The results show that the proposed method performs better than the conventional Fourier series in estimating the interfacial boundary.

Key words: electrical resistance tomography, interfacial boundary estimation, Fourier series, stratified flow, two-phase flow

1. Introduction

Flows of two immiscible liquids in pipelines are of interest in many engineering applications. As an example, liquid hydrocarbons such as crude oil and gasoline transported in pipelines over long distances often contain free water[1]. Electrical resistance tomography (ERT) is an imaging modality for determining the electrical properties inside an object. Currents are applied through the electrodes attached around the object and the voltages are measured. Based on the currents and voltages, a cross-sectional image of resistivity distribution can be reconstructed.

Although the temporal resolution of ERT is excellent, the spatial resolution is usually poor. In some cases such as two-phase flows, the resistivity values of the liquid and gas/air phases are known beforehand, whereas the interfacial boundaries are unknowns. Therefore, many researchers have recently focused on boundary estimation problems instead of the image reconstruction problems using ERT. The boundary estimation problems can be categorized into two classes: One is closed boundary problem. In the case that the anomalies are enclosed by a background substance in the domain those closed boundaries are estimated[2-4]. The other is open boundary problem, in which the domain is divided into two disjoint regions which are separated by an open boundary[5-8]. This paper considers the open boundary problem.

Butler and Bonnecaze[5] considered the free surface of a homogeneous conducting fluid flowing through an open channel. The free surface was parameterized with Chebyshev polynomials, whose coefficients were the unknowns to be estimated. Tossavainen et al.[6] introduced a free surface estimation method for air-liquid interfaces. The unknown free surface was represented using two
different parameterizations: the boundary nodes and control points of a Bézier curve. The end points of the open boundary were fixed, which is not desirable. After that, Tossavainen et al.[7] proposed an improved method for the free surface. They used a fixed computational domain and introduced a constraint to enforce the open boundary to be confined within the domain, therefore, this constraint allowed of unfixed end points. Kim et al.[8] considered dynamic interfacial boundary estimation in stratified flow of two immiscible liquids. The interfacial boundary was parameterized with discrete front points located on the interface and their unknown positions were estimated.

This paper considers the estimation problem of the interfacial boundary between two immiscible liquids using ERT. It is assumed that the conductivities of two liquids are known beforehand. The interfacial boundary can be expressed as coefficients of the Fourier series. In the case of closed regions, the boundaries are expressed using the $xy$-coordinates. However, in the case of the free surface, the boundary can be only expressed by the $y$-coordinate. Therefore, we consider the coefficients of the Fourier series only using the $y$-coordinate for the interfacial boundary estimation[5].

Generally, the Fourier series comprises of trigonometric terms and can be effectively used to estimate the periodic boundaries[3]. However, in the case of the open boundary problem, the shape of the interfacial surface is not always periodic due to the vibration in the pipe. Thus, in this case the representation performance is not good, provided the Fourier series is only used. To solve this problem, in this paper, the hybrid-type Fourier series is proposed, which consists of both the trigonometric and the polynomial terms to represent the boundary shape. Moreover, the fixed computational domain is used. Furthermore, it is shown that the interfacial boundary is stationary during acquiring a full set of measurement data. The estimation performance of the proposed method is evaluated through the numerical experiments. The results demonstrate that the proposed method performs better than the conventional method in estimating the interfacial boundary in flows of two immiscible liquids.

II. Representation of the Interfacial Boundary

Let us suppose a stratified flow of two immiscible liquids as shown in figure 1. If the interfacial boundary $f(s)$ is sufficiently smooth, it can be approximated in the form

$$f(s) \approx \tilde{f}(\phi) = \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} = \sum_{n=1}^{N} \left[ \gamma_n^x \phi_n^x(s) \right] \left[ \gamma_n^y \phi_n^y(s) \right]$$

(1)

where $\gamma_n^x$ are Fourier coefficients, $\phi_n(s)$ are periodic and smooth basis functions and $N$ is the number of the Fourier coefficients. In this paper, the interesting region is only inside the circular domain and this computational domain is fixed. Therefore, we express the $x$ coordinate of the curve as the linear function with respect to the curve parameter $s$

$$x(s) = \gamma_n^x \phi_n^x(s) = Rs, \quad y_n^x = R, \quad \phi_n^x(s) = s$$

(2)

where $R$ is the radius of the domain. And the $y$ coordinate of the curve can be expressed as a Fourier series or a polynomial function with respect to the curve parameter $s$, that is, we use the basis functions of the form

$$\begin{cases} 
\phi_n^s = 1, & n = 1 \\
\phi_n^s = \sin((n/2)s), & n = 2, 4, 6, ... \\
\phi_n^s = \cos((n/2)s), & n = 3, 5, 7, ...
\end{cases}$$

$$\leftrightarrow \phi = \phi^s = \begin{bmatrix} 1, \sin(s), \cos(s), \sin(2s), \cos(2s), \ldots \end{bmatrix}^T$$

(3)

or

$$\phi = \begin{bmatrix} 1, s, s^2, \ldots \end{bmatrix}^T$$

(4)

where $s \in [-1, 1]$. Furthermore, using the expansion (1), the open boundary is identified with the vector
γ of the shape coefficients in the y coordinate alone
\[ γ = [γ_1, γ_2, ..., γ_n, ..., γ_N] \]  \hspace{1cm} (5)
where \( γ_n = γ_n^o \).

### 2.1. Hybrid-type Fourier series

In the open boundary problem, the shape of the interfacial surface is not always periodic, i.e. the end points of the interface sometimes are different because of the oscillation phenomenon in the pipe. In this case the interface cannot be sufficiently expressed by using only the terms (3) and (5) of the Fourier series. To solve this problem, this paper proposes the hybrid-type Fourier series that consists of both the polynomial and the trigonometric terms for the representation of the interfacial boundary. Therefore, we use the basis functions of the form
\[
\begin{align*}
\phi_n &= 1, \quad n = 1 \\
\phi_n &= s, \quad n = 2 \\
\phi_n &= \sin((n-1/2)πs), \quad n = 3, 5, 7,... \\
\phi_n &= \cos((n-1/2)πs), \quad n = 4, 6, 8,... \\
\end{align*}
\]
\[ \Rightarrow \phi = [1, s, \sin(πs), \cos(πs), \sin(2πs), \cos(2πs), ...,]^T \]  \hspace{1cm} (6)
where \( s \in [-1, 1] \).

Applying the Gram Schmidt process[10] to the above basis functions (6) on the interval \([-1, 1]\), we can find the coefficients of the hybrid-type Fourier series. To obtain the coefficients, we find the orthogonal sequences \( p_n \) on the basis of the Gram Schmidt orthonormalization. And then the normalized vectors \( ϕ_n \) can be obtained as follows:
\[ \varphi_n = \frac{p_n}{\|p_n\|} \]  \hspace{1cm} (7)
where the vectors \( ϕ_n \) form an orthonormal set.

Finally, we have the following coefficients
\[ γ_n = \langle f, ϕ_n \rangle \]  \hspace{1cm} (8)
where \( \langle \cdot, \cdot \rangle \) denotes the inner product and thus the interfacial boundary can be expressed as
\[ \hat{f}(s) = \sum_{n=1}^{N} γ_n ϕ_n(s) \]  \hspace{1cm} (9)

In case of \( N=8 \), for example, we have the following basis function
\[ \phi = [1, s, \sin(πs), \cos(πs), \sin(2πs), \cos(2πs),...,]^T \]  \hspace{1cm} (10)
and then the normalized vectors \( ϕ_n \) can be obtained from the Gram Schmidt orthonormalization:
\[ \varphi = [φ_1, φ_2, φ_3, φ_4, φ_5, φ_6, φ_7, φ_8]^T \]  \hspace{1cm} (11)
where
\[ \varphi_1 = \frac{\sqrt{2}}{2} \]  \hspace{1cm} (12a)
\[ \varphi_2 = \frac{\sqrt{6}}{2} s \]  \hspace{1cm} (12b)
\[ \varphi_3 = \frac{π}{\sqrt{π^2 - 6}} \left[ \sin(πs) - \frac{3}{π} \right] \]  \hspace{1cm} (12c)
\[ \varphi_4 = \cos(πs) \]  \hspace{1cm} (12d)
\[ \varphi_5 = \frac{\sqrt{2π^2 - 12}}{2π^2 - 15} \times \left[ \sin(2πs) + \frac{3}{2π} s \right] \]  \hspace{1cm} (12e)
\[ \varphi_6 = \cos(2πs) \]  \hspace{1cm} (12f)
\[ \varphi_7 = \frac{\sqrt{6π^2 - 45}}{6π^2 - 49} \times \left[ \sin(3πs) - \frac{1}{π} \right] \]  \hspace{1cm} (12g)
\[ + \frac{2}{π^2 - 6} \left( \sin(πs) - \frac{3}{π} \right) \]
\[ - \frac{2}{π^2 - 15} \left( \sin(2πs) + \frac{3}{2π} s \right) \]  \hspace{1cm} (12h)
\[ \varphi_8 = \cos(3πs) \]  \hspace{1cm} (12i)

### 2.2. Comparison with Fourier series

To examine the Fourier series characteristic according to each basis function mentioned in the previous section, we choose a simply test function as the object function:
\[ f(x) = \begin{cases} 0, & -1 \leq x \leq 0 \\ 1, & 0 \leq x \leq 1 \end{cases} \]  \hspace{1cm} (13)
Now we can estimate the representation of the Fourier series based on the basis function and the Gram Schmidt process.

(i) Case 1: The basis function is comprised of a hybrid type form, 
\[ \phi = [1, s, \sin(πs), \cos(πs), \sin(2πs), \cos(2πs),]^T \]
Then the representation can be estimated as follows:
\[ f = \gamma_1 \varphi_1 + \gamma_2 \varphi_2 + \gamma_3 \varphi_3 + \gamma_4 \varphi_4 + \gamma_5 \varphi_5 + \gamma_6 \varphi_6 + \gamma_7 \varphi_7 + \gamma_8 \varphi_8 \\
= \frac{\sqrt{2}}{2} \cdot \varphi_1 + \frac{\sqrt{6}}{4} \cdot \varphi_2 + \frac{1}{2\sqrt{\pi^2 - 6}} \cdot \varphi_3 + 0 \cdot \varphi_4 \\
+ \frac{3\sqrt{2}(\pi^2 - 8)(\pi^2 - 6)\sqrt{\pi^2 - 6}}{2\pi\sqrt{\pi^2 - 15}} \cdot \varphi_5 + 0 \cdot \varphi_6 \\
+ \frac{3\sqrt{3}(\pi^2 - 6)\sqrt{6\pi^2 - 49}}{3\pi(2\pi^2 - 15)^{\frac{3}{2}}} \cdot \varphi_7 + 0 \cdot \varphi_8 \]

\[(14)\]

(ii) Case 2: The basis function is only comprised of the trigonometric form,
\[ \phi = \left[ 1, \sin(\pi \theta), \cos(\pi \theta), \sin(2\pi \theta), \cos(2\pi \theta) \right]^T \]

Then the estimated representation is as follows:
\[ \hat{f} = \gamma_1 \varphi_1 + \gamma_2 \varphi_2 + \gamma_3 \varphi_3 + \gamma_4 \varphi_4 + \gamma_5 \varphi_5 + \gamma_6 \varphi_6 + \gamma_7 \varphi_7 \]
\[ = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{4} \cdot \frac{\sqrt{6}}{2} + 0 \cdot \varphi_3 + 0 \cdot \varphi_4 \]
\[ + 0 \cdot \varphi_3 + \frac{2}{3\pi} \cdot \sin(3\pi \theta) + 0 \cdot \varphi_7 \]

(15)

(iii) Case 3: The basis function is only comprised of a polynomial form, \[ \phi = [1, s, s^2, s^3]^T \]
\[ \hat{f} = \gamma_1 \varphi_1 + \gamma_2 \varphi_2 + \gamma_3 \varphi_3 + \gamma_4 \varphi_4 \]
\[ = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{4} \cdot \frac{\sqrt{6}}{2} + 0 \cdot \varphi_3 \]
\[ - \frac{\sqrt{14}}{16} \cdot \frac{5\sqrt{14}}{5} \left( s^3 - \frac{3}{5} s \right) \]

(16)

To evaluate the representation performance of the Fourier series according to each basis function, we can use the error function as follows:
\[ \epsilon = \int_{-1}^{1} (f - \hat{f})^2 ds \]

(17)

where \( f \) denotes the true object function and \( \hat{f} \) is the approximated representation of the Fourier series. The results of three cases are plotted in figure 2 and the errors are shown in table 1. In the comparison with the resultant graphs and the errors, it is shown that the Fourier series using hybrid-type form is better than that using either the trigonometric or the polynomial in the representation performances.

III. Interfacial Boundary Estimation Using Hybrid-type Fourier Series

The estimation of the interfacial boundary is considered in a stratified flow of two immiscible

![Fig. 2. The representation of the Fourier series according to three different basis functions. (a) Hybrid type form, (b) trigonometric form and (c) polynomial form.](image)

<table>
<thead>
<tr>
<th>Case</th>
<th>( \epsilon )</th>
</tr>
</thead>
<tbody>
<tr>
<td>case 1</td>
<td>( 2.82 \times 10^{-2} )</td>
</tr>
<tr>
<td>case 2</td>
<td>( 4.97 \times 10^{-2} )</td>
</tr>
<tr>
<td>case 3</td>
<td>( 7.03 \times 10^{-2} )</td>
</tr>
</tbody>
</table>

Table 1. The errors between the true and estimated representations.
liquids. In this paper, we are only interested in identifying the interfacial boundary. So we only consider the fluid interface using the $y$-coordinate alone. Then this approach significantly reduces the computational complexity of the estimation problem.

In this paper, it is assumed that the conductivities of two immiscible liquids are known in advance and the fixed computational domain is used. Moreover, it is also assumed that the interfacial boundary is stationary during acquiring a full set of measured data. Thus, the open boundary can be expressed as a sum of the products of the Fourier coefficients (5) and the hybrid-type basis function (6). The coefficients are estimated by using the iterative Gauss-Newton (GN) algorithm

$$\gamma_{i+1} = \gamma_i + \left[ J_i^T J_i + \alpha I \right]^{-1} J_i^T (V - U_i)$$

where $\alpha$ denotes the regularization parameter, $V$ is measured voltages, $U_i \equiv U(\gamma_i)$ is the calculated voltages, and $J_i \equiv J(\gamma_i)$ is the Jacobian matrix with respect to the $i$-th coefficient vector $\gamma_i$, defined as

$$J_i \equiv \frac{\partial U(\gamma_i)}{\partial \gamma_i}$$

For the easiness of the computation of the Jacobian matrix, it can be rewritten as

$$J_i = \frac{\partial U_i}{\partial \sigma_e} \times \frac{\partial \sigma_e}{\partial \gamma_i}$$

where $\sigma_e$ is the conductivity of the $e$-th element in the finite element mesh. The first term of the Jacobian (20) can be calculated as [11]

$$\frac{\partial U_i}{\partial \sigma_e} = -\Phi \frac{\partial B}{\partial \sigma_e} \psi$$

$$\frac{\partial B_{r,k}}{\partial \sigma_e} = \int_{\Omega_e} \nabla \psi_{r} \cdot \nabla \psi_s \, d\Omega, \quad r,k = 1,2,\ldots,N_n$$

where $\Phi$ and $\Psi$ denote voltages corresponding to the measurement field and the injected current patterns, respectively, $\psi$ is a linear tent-like function and $N_n$ is the number of nodes of the domain. The second term of the Jacobian (20) can be computed as [3]

$$\frac{\partial \sigma_e}{\partial \gamma_n} = \frac{\sigma_e - \sigma_r}{S_e} \int_{s_1}^{s_2} x(s) \phi_n(s) \, ds$$

$$= \frac{\sigma_e - \sigma_r}{S_e} \int_{s_1}^{s_2} \phi_n(s) \, ds$$

Fig. 3. A schematic representation of the $e$-th element intercepted by the interfacial boundary $\partial \Omega$.

where $\sigma_i$ and $\sigma_e$ are the conductivity values in the regions $A_i$ and $A_e$, respectively, shown in figure 3, $S_i$ is the area of the $e$-th element, $R$ is the radius of the domain. The limits $s_1$ and $s_2$ are the values of the curve parameter in the intersection points of the element edge and the interface $f(s)$, and $\phi_n(s)$ is the basis functions in (6).

IV. Results

In this study, it is assumed that two-phase immiscible liquids flow separately through a pipe, i.e. stratified flow. The performance of the proposed method is evaluated using numerical data, and the results are compared with the conventional method.

A circular domain is employed, which could be considered as the cross-section of a process pipe. The adjacent currents with an amplitude of 10 mA are injected into the domain of 14 cm in radius through 16 electrodes.

Suppose that the interfacial boundary is stationary during injecting the currents and acquiring a full set of measurement data. It is also assumed that the conductivities of two immiscible liquids are known a priori. The conductivity values are set to 3.3 m$\Omega$/cm for the upper region and 1.7 m$\Omega$/cm for the lower region. We add zero mean Gaussian noises to the calculated voltages to generate noisy measurements and the noise level is set so that 1% of the maximum voltage range. The regularization parameter is set to be $\alpha = 1 \times 10^{-5}$.

Numerical experiments are carried out as follows: 1) The true interfacial boundaries with the same and the different end points are expressed by the
Interfacial Boundary Estimation in Stratified Flow of Two Immiscible Liquids
Using Hybrid-type Fourier Series

Table 2. True and estimated Fourier coefficients for the numerical results.

<table>
<thead>
<tr>
<th>Case of the same end points</th>
<th>true</th>
<th>estimated (trigonometric)</th>
<th>estimated (hybrid-type)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[4.00, 0.00, 1.50, 1.00, 0.50, 0.30, -0.20, 0.10]</td>
<td>[-3.94, 0.00, 1.73, 0.86, 0.77, 0.15, 0.00, 0.00]</td>
<td>[-4.00, -0.53, 1.87, 1.05, 0.03, 0.41, 0.00, 0.00]</td>
</tr>
<tr>
<td>Case of the different end points</td>
<td>true</td>
<td>estimated (trigonometric)</td>
<td>estimated (hybrid-type)</td>
</tr>
<tr>
<td></td>
<td>[-1.00, -5.50, 1.50, -1.00, 0.50, 0.20, -0.10, 0.10]</td>
<td>[-1.24, 0.00, 3.00, 1.15, 8.15, 2.59, 0.00, 0.00]</td>
<td>[-1.07, -4.95, 1.63, -0.84, 1.29, 0.55, 0.00, 0.00]</td>
</tr>
</tbody>
</table>

Fig. 4. Numerical results in the cases of (a) the same end points and (b) the different end points. True interface (bold solid line), initial guess (dotted line), trigonometric form (dashed line) and hybrid type form (weak solid line).

Fig. 5. RMSEs for the Fourier coefficients in the cases of (a) the same end points and (b) the different end points. Trigonometric form (dashed line) and hybrid-type form (solid line).

3rd order hybrid-type Fourier series to simulate a two-phase stratified flow, respectively.

2) The iterative GN method is used to estimate the
Fourier coefficients based on the 2nd-order trigonometric basis functions of the Fourier series. The Fourier coefficients of the interface are estimated using the 2nd-order hybrid-type basis functions. In order to analyze the performance, the root-mean-square errors (RMSE) are computed

$$RMSE = \frac{||\gamma_{true} - \gamma_{est}||}{||\gamma_{true}||}$$ (23)

The estimated results are plotted in figure 4 and the RMSE values according to the iteration number are shown in figure 5. Figure 4(a) shows the estimated interfaces with the same end points using the 2nd-order trigonometric and hybrid-type Fourier series, respectively. The estimation performances of two methods look similar to each other in figure 4(a), but the trigonometric form gives a better performance than the hybrid-type form, as shown in figure 5(a). Figure 4(b) shows the estimated interfacial boundaries in the case of the different end points using the 2nd-order trigonometric and hybrid-type Fourier series, respectively. From figures 4(b) and 5(b), it can be seen clearly that the performance of the proposed method is much improved as compared to the trigonometric form. In Table 2, true and estimated Fourier coefficients are listed using the 2nd-order trigonometric and hybrid-type Fourier series, respectively.

V. Conclusions

This paper has considered the interfacial boundary problems in stratified flows of two immiscible liquids in the pipeline using the ERT data. To solve periodic and non-periodic problems, the hybrid-type Fourier series with the polynomial and the trigonometric terms is proposed to estimate the shape of the interfacial boundary. Moreover, it is assumed that the interface is stationary during acquiring a full set of measurement data. The performance of the proposed method is evaluated through the noisy numerical data. The results show that the performance of the proposed method is a little less than that of the conventional Fourier series method in the cases of the detection of the interface with the same end points, whereas the proposed method is much improved in the case of the different end points.

References

Interfacial Boundary Estimation in Stratified Flow of Two Immiscible Liquids
Using Hybrid-type Fourier Series


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