System Reliability From Stress–Strength Relationship in Bivariate Pareto Distribution

Jang Sik Cho¹) · Kil Ho Cho²) · Young Joon Cha³)

Abstract

In this paper, we assume that strengths of two component system follow a bivariate pareto distribution. And these two components are subjected to a common stress which is independent of the strength of the components. We obtain maximum likelihood estimator (MLE) for the system reliability from stress–strength relationship. Also we derive asymptotic properties of the MLE and present a numerical study.

Key Words: Bivariate pareto distribution; Common stress; Maximum likelihood estimator; Stress–strength; System reliability.

1. Introduction

In recent years, bivariate pareto distributions have been proposed in the modelling of life times of two–component systems working in a changing environment. Lindley and Singpurwalla(1986) considered the distribution of life lengths measured in a laboratory environment as independent exponential distributions proved that, when they work in a different environment which may be harsher, the same or gentler than the original, the resulting density of life lengths has a bivariate pareto distribution. However, the assumption of independence is unrealistic as in many systems the component life lengths have a

1) Assistant Professor, Department of Statistical Information Science, Kyungsoong University, Pusan, 608–736, Korea
   E–mail : jscho@star.ks.ac.kr
2) Professor, Department of Statistics, Kyungpook National University, Daegu, 702–701, Korea
3) Professor, Department of Information Statistics, Andong National University, Andong, 760–749, Korea

In this paper, we consider two-component system. We assume that strengths of two components system follow a bivariate pareto distribution. And these two components are subjected to a common stress which is independent of the strength of the components. From stress-strength relationship, we obtain MLE for the system reliability and derive asymptotic properties of the MLE. Also we present a numerical example by giving a data set which is generated by computer.

2. Estimation of System Reliability

Let \( (X_1, X_2) \) be strengths of two components that follow a bivariate pareto(BVP) distribution. Then the joint survival function of \( (X_1, X_2) \) is given by

\[
\overline{F}(x_1, x_2) = P(X_1 > x_1 , X_2 > x_2)
\]

\[
= \left( \frac{x_1}{\beta} \right)^{-\lambda_1} \cdot \left( \frac{x_2}{\beta} \right)^{-\lambda_2} \cdot \max\left( \frac{x_1}{\beta}, \frac{x_2}{\beta} \right)^{-\lambda_3}, \quad \beta \leq \min(x_1, x_2) < \infty, \quad (2.1)
\]

where \( \lambda_1, \lambda_2, \lambda_3 > 0 \).

The above BVP model is not absolutely continuous with respect to Lebesgue measure on \( R^2 \). That is, in the present model there is provision for simultaneous failure of the components. The marginals of \( X_i, \quad i = 1, 2 \) are given by

\[
\overline{F}(x_i) = P(X_i > x_i) = \left( \frac{x_i}{\beta} \right)^{-(\lambda_i + \lambda_3)}, \quad i = 1, 2. \quad (2.2)
\]

which are the survival functions of pareto with parameters \( (\lambda_i + \lambda_3, \beta), \quad i = 1, 2 \).

From (2.1) and (2.2), we can see that the random variables \( X_1 \) and \( X_2 \) are independent if and only if \( \lambda_3 = 0 \). And \( X_1 \) and \( X_2 \) are identically distributed if and only if \( \lambda_1 = \lambda_2 \). The probability that \( X_1 \) and \( X_2 \) are equal to each other is

\[
P[X_1 = X_2] = \frac{\lambda_3}{\lambda}, \quad \lambda = \lambda_1 + \lambda_2 + \lambda_3.
\]
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We assume $\beta = 1$ in BVP model, the joint survival function of $(X_1, X_2)$ is given by

$$
\overline{F}(x_1, x_2) = x_1^{-\lambda_1} \cdot x_2^{-\lambda_2} \cdot (\max(x_1, x_2))^{-\lambda_3}.
$$

(2.3)

We call the survival function (2.3) as BVP type 2 and the survival function (2.1) as BVP type 1.

Let $n_1$ be the number of observations with $x_{1i} < x_{2i}$ in the sample, and let $n_2$ be the number of observations with $x_{2j} < x_{1j}$ in the sample, and let $n_3$ be the number of observations with $x_{1i} = x_{2i}$ in the sample. Then the likelihood function of the sample of size $n$ is

$$
L = \lambda_1^{n_1} \cdot \lambda_2^{n_2} \cdot \lambda_3^{n_3} \cdot (\lambda_1 + \lambda_2)^{n_2} \cdot (\lambda_2 + \lambda_3)^{n_1} \cdot \beta^{\sum_{i=1}^{n} x_{1i}} \cdot \left( \prod_{i=1}^{n} \max(x_{1i}, x_{2i}) \right)^{-\lambda_1} \cdot \left[ \prod_{i=1}^{n} x_{2i} \right]^{-\lambda_2} \cdot \left[ \prod_{i=1}^{n} x_{1i} \right]^{-\lambda_3}.
$$

(2.4)

Let $Y$ be the common random stress that follow pareto distribution with parameter $(\mu, \beta)$, that is, distribution function $G(y) = 1 - \left( \frac{y}{\mu} \right)^{-\beta}$ and that $Y$ is independent on $(X_1, X_2)$. Then the reliability of the system reliability from stress–strength relationship is given by

$$
R = P[Y < \max(X_1, X_2)].
$$

(2.5)

For $z > \beta$, the distribution of $Z = \max(X_1, X_2)$ is given by

$$
H(Z) = P[Z < z] = P[X_1 < z, X_2 < z] = 1 - \beta \cdot (\lambda_1 + \lambda_2) \cdot z^{-(\lambda_1 + \lambda_2)} - \beta \cdot (\lambda_2 + \lambda_3) \cdot z^{-(\lambda_2 + \lambda_3)} + \beta \lambda \cdot z^{-\lambda}.
$$

(2.6)

Hence the survival function of $Z$ is

$$
\overline{H}(Z) = P[Z > z] = \beta \cdot (\lambda_1 + \lambda_2) \cdot z^{-(\lambda_1 + \lambda_2)} + \beta \cdot (\lambda_2 + \lambda_3) \cdot z^{-(\lambda_2 + \lambda_3)} - \beta \lambda \cdot z^{-\lambda}.
$$

(2.7)

Now, the system reliability $(R)$ is

$$
R = P[Y < Z],
$$

$$
= \int_{\beta}^{\infty} \overline{H}(y) dG(y).
$$
\[
\hat{\mu} \left[ \frac{1}{\hat{\lambda}_1 + \hat{\lambda}_3} + \frac{1}{\hat{\lambda}_2 + \hat{\lambda}_3 + \hat{\mu}} - \frac{1}{\hat{\lambda} + \hat{\mu}} \right].
\] (2.8)

In this paper, we focus only on BVP type 2 model. From (2.4), the MLE’s of \((\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3)\) can be obtained as follows:

\[
\frac{n_1}{\hat{\lambda}_1} + \frac{n_2}{\hat{\lambda}_1 + \hat{\lambda}_3} - \sum^n \log(x_i) = 0. \tag{2.9}
\]

\[
\frac{n_2}{\hat{\lambda}_2} + \frac{n_1}{\hat{\lambda}_2 + \hat{\lambda}_3} - \sum^n \log(x_i) = 0. \tag{2.10}
\]

\[
\frac{n_3}{\hat{\lambda}_3} + \frac{n_2}{\hat{\lambda}_1 + \hat{\lambda}_3} + \frac{n_1}{\hat{\lambda}_2 + \hat{\lambda}_3} - \sum^n \log(\max(x_1, x_2)) = 0. \tag{2.11}
\]

By either Newton–Raphson procedure or Fisher’s method of scoring, we can easily obtain MLEs of \((\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3)\).

Let \((X_{i1}, X_{i2})\) and \(Y_i, \ i = 1, \ldots, n\) be i.i.d. random sample of size \(n\) and \(k_1\) be the number of observations with \(Y_i < \max(X_{1i}, X_{2i})\) in the sample. Then the distribution of \(k_1\) is binomial\((n, R)\). Hence, the natural estimate of \(R\) is

\[
\hat{R} = \frac{k_1}{n} \quad \text{which is asymptotic normal distribution with mean } R \text{ and variance } R(1-R)/n.
\]

Since the MLE of \(\mu\) is given by \(\hat{\mu} = n/\hat{\sum} \log(y_i)\), the estimate of system reliability \(R\) in (2.8) based on MLE’s of \((\lambda_1, \lambda_2, \lambda_3, \mu)\) is

\[
\hat{R}_{\text{MLE}} = \hat{\mu} \left[ \frac{1}{\hat{\lambda}_1 + \hat{\lambda}_3} + \frac{1}{\hat{\lambda}_2 + \hat{\lambda}_3 + \hat{\mu}} - \frac{1}{\hat{\lambda} + \hat{\mu}} \right], \quad \hat{\lambda} = \hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3. \tag{2.12}
\]

By consistency of MLE and delta method, we can see that the asymptotic distribution of \(\hat{R}_{\text{MLE}}\) is normal distribution with mean \(R\) and variance

\[
\Lambda \cdot \Lambda^{-1}((\lambda_1, \lambda_2, \lambda_3, \mu) \cdot \Lambda^{-1}/n, \text{ where } \Lambda = (\partial R/\partial \lambda_1, \partial R/\partial \lambda_2, \partial R/\partial \lambda_3, \partial R/\mu) \text{ (See Lehmann(1983), chapter 5).}
\]

Here, the elements of Fisher information matrix \(I(\lambda_1, \lambda_2, \lambda_3, \mu)\) are given by

\[
I_{11} = \frac{1}{\lambda_1^2} + \frac{\lambda_2}{\lambda(\lambda_1 + \lambda_3)^2}, \quad I_{13} = \frac{\lambda_2}{\lambda(\lambda_1 + \lambda_3)^2}, \quad I_{22} = \frac{1}{\lambda_2^2} + \frac{\lambda_1}{\lambda(\lambda_2 + \lambda_3)^2},
\]
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\[
I_{23} = \frac{\lambda_1}{\lambda(\lambda_2 + \lambda_3)^2}, \quad I_{33} = \frac{\lambda_1}{\lambda(\lambda_2 + \lambda_3)^2} + \frac{\lambda_2}{\lambda(\lambda_1 + \lambda_3)^2} + \frac{1}{\lambda \lambda_3}, \quad I_{44} = \frac{1}{\mu^2},
\]

\[
I_{12} = I_{14} = I_{31} = I_{34} = 0.
\]

Therefore, \(100(1 - \alpha)\%\) approximately confidence interval for system reliability \(R\) based on MLE is as follows:

\[
\left( R_{MLE} - z_{\alpha/2} \cdot \sqrt{\Lambda \cdot I(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\mu}) \cdot \Lambda / n}, \quad R_{MLE} + z_{\alpha/2} \cdot \sqrt{\Lambda \cdot I(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\mu}) \cdot \Lambda / n} \right).
\]

(2.13)

3. Numerical Example

In this section, we present a numerical example by giving a data set which is generated by computer. We generate a strength sample of size 30 from BVP with parameter \( (\lambda_1 = 1.0, \lambda_2 = 1.0, \lambda_3 = 0.5) \) and stress sample from pareto with parameter \( \mu = 3.0 \). The data is given Table 1 in the form of triplet \((x_1, x_2, y)\).

MLEs of the parameters in BVP model are \(\hat{\lambda}_1 = 0.9230, \quad \hat{\lambda}_2 = 0.9911, \quad \hat{\lambda}_3 = 0.4496\).

The estimate of variance–covariance matrix of \((\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3)\) is

\[
\begin{pmatrix}
0.0519 & 0.0568 & -0.0089 \\
0 & 0.0084 & 0.0273 \\
-0.0089 & 0.0084 & 0.0273
\end{pmatrix}
\]

The MLE of \(\mu\) is \(\hat{\mu} = 2.8557\) and its estimate of variance is \(\text{Var}(\hat{\mu}) = 0.2718\).

The estimate of system reliability is \(R_{MLE} = 0.7929\) and its estimate of variance is \(\text{Var}(R_{MLE}) = 0.0023\). Hence, 95% confidence interval for system reliability is \((0.6973, 0.8884)\).

In a similar way one can extend numerically the MLE of \(R\) when the common stress\((Y)\) is other distribution or the strengths\((X_1, X_2)\) are other distribution.
<Table 1> Generated samples \((x_1, x_2, y)\) from BVP distribution and univariate pareto distribution.

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<th>(x_{2i})</th>
<th>(y_i)</th>
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References


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