Inference on $P(Y<X)$ in an Exponential Distribution

Joongdae Kim$^1$ · Yeung-Gil Moon$^2$ · Junho Kang$^3$

Abstract

Inference for probability $P(Y<X)$ in two parameter exponential distribution will be considered when the scale parameters are known or not: point and interval estimations, and test for a null hypothesis.

Keywords: exponential distribution, point and interval estimation, test for hypothesis

1. Introduction

A two-parameter exponential distribution is given by

$$f(x; \mu, \sigma) = \frac{1}{\sigma} e^{-(x-\mu)/\sigma}, \quad x > \mu, \quad \text{where } \sigma > 0, \mu \in \mathbb{R},$$  
(1.1)

it will be denoted $X \sim \text{EXP}(\mu, \sigma)$.

It is very important for us to consider an exponential distribution in parametric inferences. Here we shall consider inference for $P(Y<X)$ in two parameter exponential distribution.

The probability that a Weibull random variable $Y$ is less than another independent Weibull random variable $X$ was considered (McCull(1991)). Many other authors have considered the probability $P(Y<X)$, where $X$ and $Y$ are independent random variables.

The problem of estimating and of drawing inferences about, the probability that

\begin{itemize}
  \item 1) Associated Professor, Department of Computer Information, Andong Junior College, Andong, 760-300, Korea. e-mail : jdkim@andong.ac.kr
  \item 2) Assistant Professor, Department of Quality Management, Kangwon Tourism College, Taebaek, 235-711, Korea.
  \item 3) Associated professor, School of Computer Engineering, Kaya University, Korung, 717-800, Korea.
\end{itemize}
a random variable $Y$ is less than an independent random variable $X$, arises in a reliability.

When $Y$ represents the random value of a stress that a device will be subjected to in service and $X$ represents the strength that varies from item to item in the population of devices, then the reliability $R$, i.e. the probability that a randomly selected device functions successfully, is equal to $P(\{Y < X\})$. The same problem also arises in the context of statistical tolerance where $\delta$ represents, say, $Y$ the diameter of a draft and $X$ the diameter of a bearing that is to be mounted on the shaft. The probability that the bearing fits without interference is then $P(\{Y < X\})$.

In biometrics, $Y$ represents a patient’s remaining years of life if treated with drug A and $X$ represents the patient’s remaining years when treated with drug B. If the choice of drug is left to the patient, person’s deliberations will center on whether $P(\{Y < X\})$ is less than or greater than $1/2$.

Here, we shall consider inferences on $P(\{Y < X\})$ in two parameter exponential distribution when the scale parameters are known or not : point and interval estimations, and test for a null hypothesis.

2. Inference on $P(X < Y)$

Let $X$ and $Y$ be independent two parameter exponential random variables, $X \sim \text{EXP}(\mu_x, \sigma_x)$ and $Y \sim \text{EXP}(\mu_y, \sigma_y)$, respectively.

Then,

$$P(X < Y) = \int \int_{\mu_y < x} f_Y(y; \mu_y, \sigma_y) \cdot f_X(x; \mu_x, \sigma_x) dx$$

$$= 1 - \frac{e^{-\delta/\sigma_y}}{1 + \sigma_x/\sigma_y}, \quad \text{where } \delta = \mu_y - \mu_x. \quad (2.1)$$

where $f_X(x)$ and $f_Y(y)$ are the density functions of $X$ and $Y$, respectively.

To consider inferences on $P(X < Y)$, assume $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ be two independent random samples from $X \sim \text{EXP}(\mu_x, \sigma_x)$ and $Y \sim \text{EXP}(\mu_y, \sigma_y)$, respectively.

Then the MLE $\widehat{\delta}$ of $\delta$ is

$$\widehat{\delta} = \widehat{\mu}_y - \widehat{\mu}_x = Y_{(1)} - X_{(1)}, \quad \text{where } X_{(1)} \text{ and } Y_{(1)} \text{ are the first order statistics of } X_i's \text{ and } Y_i's, \text{ respectively.}$$

(2.2)
By the result of Johnson, etal.(1995),

**Fact 1.** (a) $X_{(1)}$ follows an exponential distribution with a location parameter $\mu_s$ and a scale parameter $\sigma_s/m$.

(b) If $X_1, X_2, \ldots, X_m$ are iid exponential distributions with a scale parameter $\sigma$ and a location parameter $\mu$, then $\sum_{i=1}^{m} (X_i - X_{(1)})$ follows a gamma distribution with a shape parameter $m-1$ and a scale parameter $\sigma$.

(c) If a random variable $X$ follows a gamma distribution with a shape parameter $\alpha$ and a scale parameter $\beta$, then $E\left(\frac{1}{X^k}\right) = \frac{\Gamma(a-k)}{\Gamma(a)\beta^a}$, for $a > k$.

From Fact 1(a), we can obtain the expectation and variance of $\tilde{\delta}$:

$$E(\tilde{\delta}) = \delta + \frac{\sigma_s}{n} - \frac{\sigma_s}{m} \quad \text{and} \quad \text{Var}(\tilde{\delta}) = \frac{\sigma_s^2}{m^2} + \frac{\sigma_s^2}{n^2}. \quad (2.3)$$

Let $D = Y_{(1)} - X_{(1)}$. Then we can obtain the density function of $D$:

$$f_D(d) = \begin{cases} 
\frac{\Gamma(nh)}{n\sigma_x + n\sigma_y} e^{-\frac{d}{\sigma_x} - \frac{d}{\sigma_y}} e^{-\frac{\mu_x}{\sigma_x} (d - \delta)} & \text{if } d \geq \delta, \\
\frac{\Gamma(nh)}{n\sigma_x + m\sigma_y} e^{-\frac{d}{\sigma_x} + \frac{\mu_x}{\sigma_x}} e^{-\frac{\mu_y}{\sigma_y} (d - \delta)} & \text{if } d < \delta. 
\end{cases} \quad (2.4)$$

2-A. When the scale parameters $\sigma_x = \sigma_y = \sigma_0$ is known

From the result (2.1),

$$R = P(X < Y) = 1 - \frac{1}{2} e^{\delta \sigma}, \quad \delta = \mu_y - \mu_x.$$ 

Then the probability depends on $\delta$ only. Because $R$ is a monotone function in $\delta$, inference on $\delta$ is equivalent to inference on $R$. We hereafter confine attention to the parameter $\tilde{\delta}$ (see McCool(1991)).

When the scale parameters $\sigma_x = \sigma_y = \sigma_0$ is known, let $T = D - \delta$. Then from the pdf (2.4) of $D$, we have the pdf of $T$:
\[ f_{\mathcal{T}}(t) = \begin{cases} 
\frac{m}{m+n} \cdot \frac{n}{\sigma_0} e^{-\frac{n}{\sigma_0} t}, & \text{if } t \geq 0 \\
\frac{n}{m+n} \cdot \frac{m}{\sigma_0} e^{-\frac{m}{\sigma_0} t}, & \text{if } t < 0.
\end{cases} \tag{2.5} \]

Based on a pivotal quantity \( \mathcal{T} \), we shall consider an \((1 - p_1 - p_2)\)100% confidence interval of \( \delta \). For a given \( 0 < p_1 < 1 \), there exists an \( b_1 \) such that

\[ p_1 = \int_{-\infty}^{b_1} \frac{n}{m+n} \cdot \frac{m}{\sigma_0} e^{-\frac{m}{\sigma_0} t} \, dt, \]

and hence,

\[ b_1 = -\frac{\sigma_0}{2m} \cdot \chi^2_{\alpha/2, \frac{m+n}{m} p_1}. \tag{2.6} \]

where \( \alpha = \int_{\chi^2_{\alpha/2}}^{\infty} \, dt \). \( \chi^2_{2}(t) \) is the pdf of chi-square distribution of df 2.

For another given \( 0 < p_2 < 1 \), there exists an \( b_2 \) such that

\[ p_2 = \int_{b_1}^{\infty} \frac{n}{m+n} \cdot \frac{m}{\sigma_0} e^{-\frac{m}{\sigma_0} t} \, dt, \]

and hence,

\[ b_2 = -\frac{\sigma_0}{2m} \cdot \chi^2_{\alpha/2, \frac{m+n}{m} p_2}. \tag{2.7} \]

Therefore, \((Y_{(1)} - X_{(1)}) - b_2, Y_{(1)} - X_{(1)} - b_1)\) is an \((1 - p_1 - p_2)\)100% confidence interval of \( \delta \) in two parameter exponential distribution.

Next We wish to test the null hypothesis \( H_0 : \mu_x = \mu_y \) against \( H_1 : \mu_x \neq \mu_y \).
Let \( \Theta = \{ (\mu_x, \mu_y) \mid \mu_x \in R, \mu_y \in R \} \) and \( \theta = (\mu_x, \mu_y) \).

Then the joint pdf of \((X_1, \ldots, X_m, Y_1, \ldots, Y_n)\) is

\[ L(\theta) = f_\theta(x, y) = \prod_{1}^{m} \frac{1}{\sigma_0} e^{-\frac{1}{\sigma_0}(x_i-\mu_x)} \cdot \prod_{1}^{n} \frac{1}{\sigma_0} e^{-\frac{1}{\sigma_0}(y_i-\mu_y)}, \text{ for all } x_i > \mu_x, y_i > \mu_y. \]

From the likelihood function, we can obtain the MLE's of \( \mu_x \) and \( \mu_y \)

\[ \hat{\mu}_x = X_{(1)} \quad \text{and} \quad \hat{\mu}_y = Y_{(1)}. \]

If \( \mu_x = \mu_y = \mu \), then the MLE of \( \mu \) is

\[ \hat{\mu} = \min (X_{(1)}, Y_{(1)}) = \left( Y_{(1)} + X_{(1)} - | Y_{(1)} - X_{(1)} | \right)/2. \]
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From definition of a likelihood ratio test (Rohatgi(1976)), the likelihood ratio test function can be obtained:

\[
\lambda(x, y) = \exp\left(-\frac{1}{2}\left| D \right| \left(\frac{m}{2\sigma_0} + \frac{n}{2\sigma_0}\right) + D\left(\frac{m}{2\sigma_0} - \frac{n}{2\sigma_0}\right)\right),
\]

where, \( D = Y_{(1)} - X_{(1)} \).

Therefore, \( \lambda(x, y) < c \) is equivalent to \( D < b_1 \) or \( D > b_2 \). \( \quad (2.8) \)

Under \( H_0 : \mu_x = \mu_y \), i.e. \( \delta = 0 \), we hold \( T = D - \delta = D \), and hence, for given \( 0 < \alpha < 1 \) we can find \( b_1 \) and \( b_2 \) of (2.8), through the results (2.6) and (2.7) if \( p_1 = p_2 = \alpha/2 \).

2-B. When the scale parameters \( \sigma_x = \sigma_y = \sigma \) is unknown

First we wish to know whether two scale parameters are equal or not:

To test the null hypothesis \( H_0 : \sigma_x = \sigma_y = \sigma \) against \( H_1 : \sigma_x \neq \sigma_y \),

\( \mu_x \in \mathbb{R}, \quad \mu_y \in \mathbb{R} \)

Let \( \Theta = \{ (\sigma_x, \sigma_y, \mu_x, \mu_y) : \sigma_x, \sigma_y > 0, \mu_x, \mu_y \in \mathbb{R}, \mu_x = \mu_y \} \) and \( \theta = (\sigma_x, \sigma_y, \mu_x, \mu_y) \).

Then the joint pdf of \( (X_1, \ldots, X_m, Y_1, \ldots, Y_n) \) is

\[
L(\theta) = f_\theta(x, y) = \prod_{r=1}^{m} \frac{1}{\sigma_x} e^{-\frac{1}{\sigma_x}(x_i - \mu_x)} \cdot \prod_{r=1}^{n} \frac{1}{\sigma_y} e^{-\frac{1}{\sigma_y}(y_j - \mu_y)}, \text{ for all } x_i > \mu_x, \quad y_j > \mu_y.
\]

Differentiating with respect to \( \sigma_x \) and \( \sigma_y \), we can obtain the MLE’s

\[
\hat{\sigma}_x = \frac{1}{m} \sum_{i=1}^{m} x_i, \quad \hat{\sigma}_y = \frac{1}{n} \sum_{j=1}^{n} y_j, \quad \text{and} \quad \hat{\mu}_x = X_{(1)} \quad \text{and} \quad \hat{\mu}_y = Y_{(1)}.
\]

If \( \sigma_x = \sigma_y = \sigma \), then the MLE of \( \sigma \) is

\[
\hat{\sigma} = \frac{1}{n + m} \left( \sum_{i=1}^{m} (X_i - \hat{\mu}_x) + \sum_{j=1}^{n} (Y_j - \hat{\mu}_y) \right). \quad (2.9)
\]

From definition of a likelihood ratio test (Rohatgi(1976)), the likelihood ratio test function can be obtained:

\[
\lambda(x, y) = \left(\frac{\hat{\sigma}_x}{\sigma}\right)^m \cdot \left(\frac{\hat{\sigma}_y}{\sigma}\right)^n = \left(\frac{m + n}{m}\right)^m \cdot \left(\frac{m + n}{n}\right)^n \cdot \left(\frac{1}{1 + 1/U}\right)^m \cdot \left(\frac{1}{1 + U}\right)^n.
\]
where \[ U = \frac{\sum_{i=1}^{m}(X_i - X_{(1)})}{\sum_{i=1}^{n}(Y_i - Y_{(1)})}. \]

Therefore, \( \lambda(x, y) < c \) is equivalent to \( U < u_1 \) or \( U > u_2 \). \hspace{1cm} (2.10)

From Fact 1(b) and the results of Rohatgi(1976), we have the followings:

**Fact 2.** (a) \[ Z = \frac{2 \sum_{i=1}^{m}(X_i - X_{(1)})}{\sigma_x} \quad \text{and} \quad W = \frac{2 \sum_{i=1}^{m}(Y_i - Y_{(1)})}{\sigma_y} \]

follows chi-square distribution with df’s 2(m-1) and 2(n-1), respectively.

(b) The random variables Z and W are independent.

Under \( H_0 : \sigma_x = \sigma_y = \sigma \), from Fact 2, \( U = \frac{\sum_{i=1}^{m}(X_i - X_{(1)})}{\sum_{i=1}^{n}(Y_i - Y_{(1)})} \) follows a F-distribution with df’s 2(m-1) and 2(n-1). And hence, for a given \( 0 < \alpha < 1 \), 

\[ u_2 = F_{\alpha/2}(2(m - 1), 2(n - 1)) \quad \text{and} \quad u_1 = 1/F_{1-\alpha/2}(2(n - 1), 2(m - 1)) \]

from (2.10).

If \( \sigma_x = \sigma_y = \sigma \), then from the result (2.1),

\[ R = P(X < Y) = 1 - \frac{1}{2} e^{\delta/\sigma} \quad \text{where} \quad \delta = \mu_y - \mu_x. \]

Let \( \beta = \delta/\sigma \). Then, an estimator of \( \beta \) is defined as:

\[ \tilde{\beta} = \frac{(m + n)(Y_{(1)} - X_{(1)})}{\sum_{i=1}^{m}(X_i - X_{(1)}) + \sum_{i=1}^{n}(Y_i - Y_{(1)})}, \]

from results (2.2) and (2.9).

From the results (2.3) and Fact 1(c), we can obtain the followings:

\[ E(\tilde{\beta}) = \beta + \frac{3}{m + n - 3} \beta + \frac{m^2 - n^2}{mn(m + n - 3)} \]

and

\[ Var(\tilde{\beta}) = \frac{(m + n)^2(m^2 + n^2)}{m^2n^2(m + n - 3)^2(m + n - 4)}. \]
References


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