Cusum of squares test for discretely observed sample from diffusion processes

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Abstract

In this paper, we consider the change point problem in diffusion processes based on discretely observed sample. Particularly, we consider the change point test for the dispersion parameter when the drift has unknown parameters. In performing a test, we employ the cusum of squares test based on the residuals. It is shown that the test has a limiting distribution of the sup of a Brownian bridge. A simulation result as to the Ornstein-Uhlenbeck process is provided for illustration. It demonstrates the validity of our test.

Keywords: Diffusion process, discretely observed sample, residual based cusum test.

1. Introduction

The diffusion process has long been popular in analyzing random phenomena occurring in various fields such as finance, engineering, physical and medical sciences. As a representative text, we can refer to Karatzas and Shreve (1991) and Shirayev (1999). Since the application of diffusion processes to real world is various, much attraction has been drawn to statistical inference for diffusion processes and many sophisticated methods have been established by researchers and practitioners. For a general review, we refer to Prakasa Rao (1999) and Kutoyants (2004). According to past experience, time series models are not well fitted to financial time series in many occasions due to structural changes governed by the change of financial policies and critical events. This phenomenon is observed in most financial time series data with high volatility. See, for instance, Lee et al. (2004), who empirically verified by using the cusum test that most stock prices of Nikei 225 suffer from serious parameter

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changes when the underlying model of the data is assumed to be a GARCH(1,1) model. With
regard to the parameter change test for time series models, we refer to Lee et al. (2003),
also Lee et al. (2006) and the articles cited in these papers for continuous time stochastic
processes. Recently, Gregorio and Iacus (2008) considered the change point test for the
dispersion parameter in diffusion processes based on discretely observed sample. However,
they only handled the case that the drift is completely known with no unknown parameters.
Thus, in this paper, we consider the change point test for diffusion processes when the drift
has unknown parameters.

In Section 2, it is shown that under regularity conditions, the cusum of squares test based
on residuals has a limiting distribution of the sup of a Brownian bridge. In Section 3, we
carry out a simulation study.

2. Main result

Let us consider the stochastic differential equation:

\[ dX_t = a(X_t; \theta)dt + \sigma dW_t, \quad X_0 = x_0, \quad t \geq 0, \]

where \( \theta \) is a \( p \)-dimensional unknown parameter, \( a \) is a known real valued function, and
\( W = \{ W_t; t \geq 0 \} \) is a standard Brownian motion. Suppose that \( X_{t_i}, t_i = ih_n, i = 1, \ldots, n, \)
are observed, where \( \{ h_n \} \) is a sequence of positive real numbers such that \( h_n \to 0 \)
and \( nh_n \to \infty \), and one wishes to test the following hypotheses:

\[ H_0 : \sigma \text{ is constant over } i = 1, \ldots, n \text{ vs. not } H_0. \]

To task this, we assume that

(A1) There exist constants \( C, m > 0 \) such that for any \( \theta \) and \( x, y, \)

\[ |a(x; \theta) - a(y; \theta)| \leq C|x - y|, \]

\[ \sup_{\theta' \in N_\theta} ||\dot{a}(x; \theta')|| \leq C(1 + |x|^m), \]

where \( \dot{a} = \partial a/\partial \theta \), and \( N_\theta \) is an open neighborhood of \( \theta \).

(A2) Under \( H_0 \), \( \sup_t E|X_t|^\gamma < \infty \) for all \( \gamma > 0 \).

(A3) Under \( H_0 \), there exists an estimator \( \hat{\theta}_n \) of \( \theta \), such that \( (nh_n)^{1/2}(\hat{\theta}_n - \theta) = O_P(1) \).

(A4) \( nh_n^4 \to 0 \) as \( n \to \infty \).

Sufficient conditions for (A3) can be found in Kessler (1997).

By using the Euler approximation, we can express

\[ X_{t_i} - X_{t_{i-1}} \asymp h_n a(X_{t_{i-1}}; \hat{\theta}_n) + \sigma(W_{t_i} - W_{t_{i-1}}). \]

In view of this, we define the residuals as

\[ \hat{\eta}_i = \{ X_{t_i} - X_{t_{i-1}} - h_n a(X_{t_{i-1}}; \hat{\theta}_n) \}/h_n^{1/2}. \]

Then, as in Lee et al. (2004), we consider the cusum of squares test based on the residuals
and obtain the following result.
Theorem 2.1 Assume that (A1)-(A4) hold. Let

\[ T_n = \frac{1}{\sqrt{nT_n}} \max_{1 \leq k \leq n} \left| \frac{k}{n} \sum_{i=1}^{k} \hat{\eta}_i^2 - \left( \frac{k}{n} \right) \sum_{i=1}^{n} \hat{\eta}_i^2 \right|, \]

where \( \hat{\tau}^2_n = \frac{1}{n} \sum_{i=1}^{n} \hat{\eta}_i^4 - \left( \frac{1}{n} \sum_{i=1}^{n} \hat{\eta}_i^2 \right)^2 \). Then, under \( H_0 \),

\[ T_n \overset{d}{\to} \sup_{0 \leq u \leq 1} |W^0(u)|, \quad n \to \infty, \]

where \( W^0 \) is a Brownian bridge.

Proof. Put \( \eta_i = \sigma(W_{t_i} - W_{t_{i-1}})h_n^{-1/2}, \Delta_i = \int_{t_{i-1}}^{t_i} \{a(X_s; \theta) - a(X_{t_{i-1}}; \theta)\} ds \), and \( d_i = a(X_{t_{i-1}}; \theta) - a(X_{t_{i-1}}; \hat{\theta}_n) \). Then, we can express \( \hat{\eta}_i = \eta_i + \Delta_i h_n^{-1/2} + d_i h_n^{1/2} \).

Note that

\[ 1 \sqrt{n} \sum_{i=1}^{n} \Delta_i^2 h_n^{-1} = o_P(1), \quad (2.3) \]

where we have used (A4) and the fact that \( E\Delta_i^2 \leq Ch_n^3 \) for some \( C > 0 \) (cf. Kessler, 1997). Moreover, by (A1)-(A3), for all large \( n \),

\[ 1 \sqrt{n} \sum_{i=1}^{n} d_i^2 h_n \leq C^2 \sum_{i=1}^{n} (1 + |X_{t_{i-1}}|^m)^2 ||\hat{\theta}_n - \theta||^2 h_n = o_P(1). \quad (2.4) \]

By using (2.3) and (2.4), we can eventually get

\[ \max_{1 \leq k \leq n} \frac{1}{\sqrt{n}} \sum_{i=1}^{k} |\hat{\eta}_i^2 - \eta_i^2| = o_P(1). \quad (2.5) \]

Since \( \eta_1, \ldots, \eta_n \) are iid \( N(0, \sigma^2) \), we have

\[ 1 \sqrt{nT_n} \max_{1 \leq k \leq n} \left| \frac{k}{n} \sum_{i=1}^{k} \hat{\eta}_i^2 - \left( \frac{k}{n} \right) \sum_{i=1}^{n} \hat{\eta}_i^2 \right| \overset{d}{\to} \sup_{0 \leq u \leq 1} |W^0(u)|, \]

where \( \tau^2 \) is the variance of \( \eta_i^2 \). Consequently, (2.5) implies that \( (\hat{\tau}_n^2/\tau^2)T_n \) converges to \( \sup_{0 \leq u \leq 1} |W^0(u)| \) in distribution. Since \( \hat{\tau}_n^2 \) converges to \( \tau^2 \) in probability, we establish the theorem. \( \square \)

Remark. The test does not detect the change in the drift parameter as will be seen in our simulation study. Thus, the test is free from any drift changes.

3. Simulation study

In this section, we evaluate the performance of the cusum of squares test through a simulation study. In this study, we consider the Ornstein-Uhlenbeck (O-U) process:

\[ dX_t = (\alpha - \mu X_t)dt + \sigma dW_t, \quad X_0 = 0, \quad t \geq 0. \quad (3.1) \]
Table 3.1 Empirical sizes of $T_n$ at nominal levels 0.1, 0.05 and 0.01

<table>
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<th>Level</th>
<th>n</th>
<th>0.10</th>
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<tr>
<td></td>
<td>500</td>
<td>0.087</td>
<td>0.043</td>
<td>0.008</td>
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<tr>
<td></td>
<td>1000</td>
<td>0.098</td>
<td>0.046</td>
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<td></td>
<td>2000</td>
<td>0.107</td>
<td>0.054</td>
<td>0.009</td>
</tr>
</tbody>
</table>

Table 3.2 Empirical powers of $T_n$ at nominal level 0.05 when $(\alpha, \mu, \sigma)$ changes from $(0.5, 1.0, 1.0)$ to $(\alpha', \mu', \sigma')$ at $[0.5n]$ for $n=500, 1000, 2000$

<table>
<thead>
<tr>
<th>$(\alpha', \mu', \sigma')$</th>
<th>$(\alpha, \mu, \sigma)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0.5, 1.4)$</td>
<td>$(0.5, 1.0, 2.0)$</td>
</tr>
<tr>
<td>500</td>
<td>0.997</td>
</tr>
<tr>
<td>1000</td>
<td>1.000</td>
</tr>
<tr>
<td>2000</td>
<td>1.000</td>
</tr>
</tbody>
</table>

The empirical sizes and powers are calculated as the rejection number of the null hypothesis $H_0$: No changes occur in $\sigma$ out of 1000 repetitions. The empirical quantile value for $\sup_{0 \leq u \leq 1} |W(u)|$ can be found in Table 1 of Lee et al. (2003), page 784. In each simulation, we generate the sample with $n=500, 1000$ and $2000$, and employ the sampling time length $h = n^{-1/3}$. For the empirical size, we consider the O-U process with $\alpha=0.5$, $\mu=1$ and $\sigma=1$. The empirical sizes are calculated at the nominal level 0.01, 0.05 and 0.1, respectively. In order to examine the power, we consider the O-U process with $\alpha=0.5$ and $\mu=1$, and change $\sigma$ from 1 to 1.4 and 2 at $[0.5n]$. We also consider the O-U process with $\sigma=1$ and change $\alpha$ and $\mu$ at $[0.5n]$. Table 3.1 shows that the cusum of squares test has no size distortions. Table 3.2 indicates that the test produces good powers. It is noteworthy that the test does not detect the change in the drift as anticipated. The simulation study enables us to conclude that the cusum of squares test is a proper tool to detect the change of the dispersion parameter in diffusion processes.

References


