Estimating exponentiated parameter and distribution of quotient and ratio in an exponentiated Pareto

Yeung-Gil Moon¹ · Chang Soo Lee² · Junho Kang³

¹Department of Tourism Quality Management, Kangwon Tourism College
²Department of Mobile Engineering, Kyungwoon University
³Department of Special Physical Education, Kaya University
Received 10 July 2010, revised 15 September 2010, accepted 20 September 2010

Abstract

We shall consider estimations of an exponentiated parameter of the exponentiated Pareto distribution with known scale and threshold parameters. A quotient distribution of two independent exponentiated Pareto random variables is obtained. We also derive the distribution of the ratio of two independent exponentiated Pareto random variables.

Keywords: Approximate maximum likelihood estimator, exponentiated Pareto distribution, generalized hypergeometric function, quotient, ratio.

1. Introduction

Let \( F(x) \) be the density function of a continuous random variable \( X \). Then \( G(x) \equiv F^\alpha(x) \) is also a density function of a continuous random variable where \( \alpha \) is a positive real number. Hence, the distribution \( G(x) \) is called an exponentiated distribution of a given density function \( F(x) \) (Gupta, 2001). Gupta (2001) considered an exponentiated exponential family. Ali et al. (2006) considered the exponentiated Weibull. Lee and Won (2006) considered inferences on the reliability in an exponentiated uniform distribution. Ali et al. (2007) introduced some exponentiated distributions. Ali and Woo (2010) considered estimations of the tail probability and the reliability in exponentiated Pareto case. Ali et al. (2005) studied the ratio \( X/(X+Y) \) for the power function distribution. Woo (2008) considered estimations for the reliability and the distribution of ratio in two independent different variates. Moon et al. (2009) considered the reliability and the ratio in two exponentiated complementary power function distributions. Lee and Lee (2010) studied the reliability and the ratio in a right truncated Rayleigh distribution.

In this paper, we consider estimations of the exponentiated parameter in the exponentiated Pareto distribution with known scale and shape parameters. We then consider distributions of the quotient \( X/Y \) and the ratio \( X/(X+Y) \) in two independent exponentiated Pareto random variables \( X \) and \( Y \).

¹ Associated professor, Department of Tourism Quality Management, Kangwon Tourism College, Tae- baek 235-711, Korea.
² Corresponding author; Associated professor, Department of Mobile Engineering, Kyungwoon University, Gumi 730-850, Korea. E-mail: cslee@ikw.ac.kr
³ Associated professor, Department of Special Physical Education, Kaya University, Gimhae 621-070, Korea.
2. Estimation of the exponentiated parameter

From the density function of the Pareto distribution an exponentiated Pareto distribution is as given in Ali et al. (2007, 2010):

\[ G(x) = [1 - (\beta/x)^c]^a, \quad x \geq \beta, \quad \alpha > 0, \quad c > 0, \quad (2.1) \]

which the Pareto distribution is as given in Johnson et al. (1994), where parameters \( \alpha \) and \( \beta \) are called the exponentiated parameter and the threshold parameter, respectively.

When \( \alpha \) and \( c \) in the density function (2.1) are known, estimation problem of threshold parameter \( \beta \) was considered by Ali and Woo (2010). Here, we consider estimations of an exponentiated parameter in the density function (2.1) with known parameters \( c \) and \( \beta \).

Let us first consider a power function random variable with the density (2.2) in Ali et al. (2005) given by:

\[ p(x) = \alpha x^{\alpha - 1}, \quad 0 < x < 1, \quad \alpha \geq 1. \quad (2.2) \]

In considering estimation of the parameter \( \alpha \), we introduce the following well-known Lemma:

Lemma 1. Let \( Z_1, Z_2, ..., Z_n \) be a random sample from the density (2.2). Then

(a) The maximum likelihood estimation (MLE) \( \hat{\alpha} \) of \( \alpha \) is \( \hat{\alpha} = n / (-\sum_{i=1}^{n} \ln Z_i) \), whose the mean and the variance are \( n\alpha/(n-1) \) and \( n^2\alpha^2/((n-1)^2(n-2)), n > 2 \), respectively.

(b) An unbiased estimator \( \tilde{\alpha} \) of \( \alpha \) is \( \tilde{\alpha} = (n-1)/(-\sum_{i=1}^{n} \ln Z_i) \), whose the variance is \( \alpha^2/(n-1), n > 1 \).

(c) The unbiased estimator performs better in the sense of mean squared errors (MSE) than the MLE.

From the method of obtaining the approximate maximum likelihood estimator (AML) as in Balakrishnan and Cohen (1991), we obtain AML of \( \alpha \) in the density (2.2) as follows:

\[ \hat{\alpha}(d) = 2d - d^2(-\sum_{i=1}^{n} \ln Z_i/n), \quad \quad (2.3) \]

where \( d \) is any real number.

Especially from Lemma 1 (a) and (2.3), it’s clear that \( \hat{\alpha}(d) = \hat{\alpha} \), when \( d \) is the MLE \( \hat{\alpha} \) of \( \alpha \). If \( d \) is the unbiased estimator \( \tilde{\alpha} \) in Lemma 1 (b), then from (2.3) we introduce an AML of \( \alpha \) as follows:

\[ \hat{\alpha}(\tilde{\alpha}) = (n-1)(n^2 + 1)/(-n \sum_{i=1}^{n} \ln Z_i), \quad \quad (2.4) \]

whose the mean and the variance are given by:

\[ E(\hat{\alpha}(\tilde{\alpha})) = (n+1)\alpha/n \text{ and } \text{Var}(\hat{\alpha}(\tilde{\alpha})) = (n+1)^2\alpha^2/(n^2(n-2)), n > 2. \quad (2.5) \]

From Lemma 1 (a) and (b) and (2.5), we obtain the following:
Fact 1. The AML $\hat{\alpha}(\tilde{\alpha})$ performs better in a sense of MSE than the MLE $\hat{\alpha}$, but the AML $\tilde{\alpha}(\hat{\alpha})$ performs worse than the unbiased estimator $\tilde{\alpha}$.

Assume $X_1, X_2, \ldots, X_n$ be a random sample from the density function (2.1) with known $c$ and $\beta$.

Then using a transformation, we obtain the following:

Fact 2. $Z_1 \equiv 1 - (\beta/X_1)^c$ follows a power function distribution as in (2.2).

From Lemma 1 (b) and Facts 1 and 2, among three introduced estimators above, the first favorable estimator of an exponentiated parameter $\alpha$ in the density function (2.1) is given as:

$$\tilde{\alpha} = \frac{(n - 1)}{[-\sum_{i=1}^{n} \ln(1 - (\beta/X_i)^c)]},$$

whose the mean and the variance are as given in Lemma 1 (b).

From (2.4), the second favorable estimator of $\alpha$ is given by

$$\hat{\alpha}(\tilde{\alpha}) = \frac{(n - 1)(n^2 + 1)}{[-n \sum_{i=1}^{n} \ln(1 - (\beta/X_i)^c)]},$$

whose the mean and the variance are as given in (2.5).

3. Distribution of the quotient $X/Y$

Let $X$ and $Y$ be independent exponentiated Pareto random variables having the density function (2.1) each with parameters $(\alpha_1, \beta_1, c)$ and $(\alpha_2, \beta_2, c)$, respectively.

Then using formula 3.197(3) in Gradshteyn and Ryzhik (1965), the distribution of the quotient $Q = X/Y$ is obtained as follows:

$$F_Q(z) = F(-\alpha_2, 1; \alpha_1 + 1; (\frac{\beta_2}{\beta_1} \cdot 1)^c), \quad \text{if } z \geq \frac{\beta_2}{\beta_1}, \quad (3.1)$$

where $F(a, b; c; z)$ is the hypergeometric function.

From (3.1) and formula 15.2.1 in Abramowitz and Stegun (1970), we obtain the density of the quotient $Q$ as follows:

$$f_Q(z) = \frac{c\alpha_2}{1+\alpha_1} (\frac{\beta_2}{\beta_1})^c \cdot z^{-c-1} \cdot F(-\alpha_2, 1; \alpha_1 + 1; (\frac{\beta_2}{\beta_1} \cdot 1)^c), \quad \text{if } z \geq \frac{\beta_2}{\beta_1}. \quad (3.2)$$

From the density (3.2) and 7.512(5) in Gradshteyn and Ryzhik (1965), the $k$-th moment of the quotient $Q$ is obtained as follows:

$$E(Q^k) = (\frac{\alpha_2}{1+\alpha_1})(\frac{\beta_2}{\beta_1})^k (1 - \frac{k}{c})^{-1} F(1 - \alpha_2, 2, 1 - \frac{k}{c}; 2 + \alpha_1, 2 - \frac{k}{c}; 1), \quad \text{if } k < c, \quad (3.3)$$

where $F(a, b; c; p, q; 1)$ is the generalized hypergeometric function in Gradshteyn and Ryzhik (1965).
From (3.3) and formula 9.14(1) in Gradshteyn and Ryzhik (1965), Table 1 provides the asymptotic means, variances, and coefficients of skewness of the density (3.2) for \( c = 5 \). From Table 1 in the Appendix, we observe the following:

**Fact 3.** For the density (3.2) with \( c = 5 \),
(a) the density is skewed to the right when \((\alpha_1, \alpha_2)\) are as given in Table 1.
(b) the density with \( \alpha_1 > \alpha_2 \) has smaller mean and variance than the density with \( \alpha_1 < \alpha_2 \) has.

Let \( z = 1 \) in the density function \( F_Q(z) \) in (3.1). Then we obtain the following reliability:

\[
R = P(Y < X) = \begin{cases} 
F(-\alpha_2, 1; \alpha_1 + 1; (\frac{\beta_2}{\beta_1})^c), & \text{if } \beta_1 \geq \beta_2 \\
1 - F(-\alpha_1, 1; \alpha_2 + 1; (\frac{\beta_1}{\beta_2})^c), & \text{if } \beta_2 > \beta_1,
\end{cases}
\]

which is an extension of reliability in Ali and Woo (2010).

### 4. Distribution of the Ratio \( X/(X + Y) \)

In this section, we consider the distribution of the ratio \( R = X/(X + Y) \), where \( X \) and \( Y \) are independent exponentiated Pareto random variables having the density function (2.1) each with parameters \((\alpha_1, \beta_1, c)\) and \((\alpha_2, \beta_2, c)\), respectively.

In a similar manner as in Section 3, using formula 3.197(3) in Gradshteyn and Ryzhik (1965) we obtain the cdf of the ratio \( R \) as follows:

\[
F_R(r) = 1 - F(-\alpha_2, 1; \alpha_1 + 1; (\frac{\beta_2}{\beta_1})^c), \quad 0 < t < \frac{\beta_1}{\beta_1} + \beta_2.
\]

(4.1)

From (4.1) and formula 15.2.1 in Abramowitz and Stegun (1970), we obtain the density of the ratio \( R \) as follows:

\[
f_R(t) = (\frac{c\alpha_2}{\alpha_1 + 1}) (\frac{\beta_2}{\beta_1})^c \cdot \frac{t^{c-1}}{(1-t)^{c+1}}
\]

\[
F(1 - \alpha_2, 2; \alpha_1 + 2; (\frac{\beta_2}{\beta_1})^c), \quad 0 < t < \frac{\beta_1}{\beta_1} + \beta_2.
\]

(4.2)

From (4.2), formulas 1.110 and 7.512(5) in Gradshteyn and Ryzhik (1965), the k-th moment of the ratio \( R \) is obtained as follows:

\[
E(R^k) = \frac{c\alpha_2}{\alpha_1 + 1} \sum_{i=0}^{\infty} \frac{(-k)^i P_i}{i! (k + i + c)} (\frac{\beta_1}{\beta_2})^{k+i},
\]

\[
F(1 - \alpha_2, 2, (k + c + i)/c; 2 + \alpha_1, (k + 2c + i)/c; 1), \quad \text{if } \beta_2 > \beta_1.
\]

(4.3)
where \( a_i P_i = a(a-1)(a-2) \cdots (a-i+1) \) and \( a P_0 \equiv 1 \).

From (4.3) and formula 9.14(1) in Gradshteyn and Ryzhik (1965), Table 2 in the Appendix provides the asymptotic means, variances, and coefficients of skewness of the density (4.2) for \( c = 5 \) and \( \beta_1 = 1 \) and \( \beta_2 = 2 \). From Table 2, we observe the following:

Fact 4. For the density (4.2) with \( c = 5 \) and \( \beta_1 = 1 \) and \( \beta_2 = 2 \),

(a) the density (4.2) is skewed to the right when \( \alpha_1 > \alpha_2 \), otherwise it’s skewed to the left.

(b) its mean and variance are slightly increasing as \( \alpha_1 \) increases when \( \alpha_1 > \alpha_2 \), but are slightly decreasing as \( \alpha_1 \) increases when \( \alpha_1 < \alpha_2 \).

Appendix

### Appendix 1.1
Asymptotic mean, variance, and coefficient of skewness of the density (3.2) with \( c = 5 \) and units of mean and variance are \( \beta_2/\beta_1 \) and \( (\beta_2/\beta_1)^2 \) respectively.

<table>
<thead>
<tr>
<th>((\alpha_1, \alpha_2))</th>
<th>mean</th>
<th>variance</th>
<th>skewness</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1.5, 3.5)</td>
<td>1.20386</td>
<td>0.18101</td>
<td>2.7398</td>
</tr>
<tr>
<td>(3.5, 1.5)</td>
<td>0.01806</td>
<td>0.10180</td>
<td>2.66768</td>
</tr>
<tr>
<td>(3.5, 5.5)</td>
<td>1.14940</td>
<td>0.18179</td>
<td>2.59641</td>
</tr>
<tr>
<td>(5.5, 3.5)</td>
<td>0.97778</td>
<td>0.13092</td>
<td>2.58397</td>
</tr>
<tr>
<td>(5.5, 7.5)</td>
<td>1.12560</td>
<td>0.18122</td>
<td>2.55614</td>
</tr>
<tr>
<td>(7.5, 5.5)</td>
<td>1.00342</td>
<td>0.14379</td>
<td>2.55186</td>
</tr>
<tr>
<td>(7.5, 9.5)</td>
<td>1.11269</td>
<td>0.18079</td>
<td>2.53732</td>
</tr>
<tr>
<td>(9.5, 7.5)</td>
<td>1.01776</td>
<td>0.15115</td>
<td>2.53535</td>
</tr>
</tbody>
</table>

### Appendix 1.2
Asymptotic mean, variance, and coefficient of skewness of the density (4.2)

<table>
<thead>
<tr>
<th>((\alpha_1, \alpha_2))</th>
<th>mean</th>
<th>variance</th>
<th>skewness</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1.5, 3.5)</td>
<td>0.19267</td>
<td>0.01719</td>
<td>-0.66426</td>
</tr>
<tr>
<td>(3.5, 1.5)</td>
<td>0.08694</td>
<td>0.01809</td>
<td>0.94283</td>
</tr>
<tr>
<td>(3.5, 5.5)</td>
<td>0.16872</td>
<td>0.01926</td>
<td>-0.30083</td>
</tr>
<tr>
<td>(5.5, 3.5)</td>
<td>0.11085</td>
<td>0.01996</td>
<td>0.54016</td>
</tr>
<tr>
<td>(5.5, 7.5)</td>
<td>0.15947</td>
<td>0.02035</td>
<td>-0.40410</td>
</tr>
<tr>
<td>(7.5, 5.5)</td>
<td>0.11968</td>
<td>0.02026</td>
<td>-0.10117</td>
</tr>
<tr>
<td>(7.5, 9.5)</td>
<td>0.15456</td>
<td>0.02033</td>
<td>0.33500</td>
</tr>
</tbody>
</table>

### References


