Goodness-of-fit tests for the inverse Weibull or extreme value distribution based on multiply type-II censored samples

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Abstract

The inverse Weibull distribution has been proposed as a model in the analysis of life testing data. Also, inverse Weibull distribution has been recently derived as a suitable model to describe degradation phenomena of mechanical components such as the dynamic components (pistons, crankshaft, etc.) of diesel engines. In this paper, we derive the approximate maximum likelihood estimators of the scale parameter and the shape parameter in the inverse Weibull distribution under multiply type-II censoring. We also develop four modified empirical distribution function (EDF) type tests for the inverse Weibull or extreme value distribution based on multiply type-II censored samples. We also propose modified normalized sample Lorenz curve plot and new test statistic.

Keywords: Approximate maximum likelihood estimator, goodness-of-fit test, inverse Weibull distribution, modified normalized sample Lorenz curve, multiply type-II censored sample.

1. Introduction

The probability density function (PDF) and the cumulative distribution function (CDF) of the two-parameter inverse Weibull distribution are given by

\[ g(x; \sigma, \lambda) = \lambda \sigma^{-\lambda} x^{-\lambda-1} \exp \left[ - (x \sigma)^{-\lambda} \right], \quad x > 0, \sigma > 0, \lambda > 0 \] \hspace{1cm} (1.1)

and

\[ G(x; \sigma, \lambda) = \exp \left[ - (x \sigma)^{-\lambda} \right], \quad x > 0, \sigma > 0, \lambda > 0, \] \hspace{1cm} (1.2)

where \( \sigma \) and \( \lambda \) are scale and shape parameters respectively.

This distribution has been recently proposed as a model in the analysis of life testing data. Calabria and Pulcini (1990) derived the maximum likelihood estimators and the least-square
estimators of the parameters in the inverse Weibull distribution. Calabria and Pulcini (1994) have studied Bayes 2-sample prediction for inverse Weibull distribution. Maswadah (2003) derived the conditional confidence intervals for the parameters based on the generalized order statistics in inverse Weibull distribution. Mahmoud et al. (2003) derived exact expression for the single moments of order statistics and calculated the double moments of order statistics from the inverse Weibull distribution. Also, they used these moments to obtain the best linear unbiased estimates (BLUEs) for the location and scale parameters.

The most common censoring schemes are type-I and type-II censoring, but the conventional type-I and type-II censoring schemes do not have flexibility. Multiply type-II censoring is a generalization of type-II censoring. Multiply type-II censored sampling arises in a life-testing experiment whenever the experimenter does not observe the failure times of some units placed on a life-test. Another situation where multiply censored samples arise naturally is when some units failed between two points of observation with the exact times of failure of these units unobserved.

The approximated maximum likelihood estimating method was first developed by Balakrishnan (1989) for the purpose of providing explicit estimators of the scale parameter in the Rayleigh distribution. Fei et al. (1995) studied the estimation for the two-parameter Weibull distribution and extreme-value distribution under multiply type-II censoring. They compared the mean squared errors of the maximum likelihood estimators, approximate maximum likelihood estimators (AMLEs), and BLUEs of the parameters in the extreme value distribution. Balakrishnan et al. (2004) discussed point and interval estimation for the extreme value distribution under progressively type-II censoring. Kang (2007) proposed some explicit estimators of scale parameters in the half-triangle distribution under multiply type-II censoring by the approximate maximum likelihood methods. Han and Kang (2008) derived the AMLEs of the scale parameter and the location parameter in a double Rayleigh distribution based on multiply type-II censoring samples. Shin and Lee (2012) suggested an estimation method of the parameter in an exponential distribution based on progressively type-II interval censored sample with semi-missing observation. Kang et al. (2013) obtained Bayes estimators and corresponding credible intervals with the highest posterior density and Bayes predictive intervals for unknown parameters based on progressively type-II censored data from an exponentiated half logistic distribution.


In this paper, we derive the AMLEs of the scale parameter $\sigma$ and the shape parameter $\lambda$ under multiply type-II censored sample. We also propose three modified EDF type tests, including the modified Kolmogorov-Smirnov test, and the modified Anderson-Darling test, and the modified Cramer-von Mises test for the inverse Weibull distribution with unknown parameters based on multiply type-II censored samples using the AMLEs. For each test, Monte Carlo techniques are used to generate the critical values. The powers of these tests are also investigated under normal, lognormal, gamma, Weibull distributions.

We consider the modified EDF type tests using AMLEs and the modified normalized
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sample Lorenz curve (NSLC) plot to test for the half-logistic distribution based on multiply type-II censored samples. We also propose new test statistics based on the NSLC for the inverse Weibull distribution under multiply type-II censored samples.

2. Approximate maximum likelihood estimators

We assume that \( n \) items are put on a life test, but only \( a_1 \), \( a_2 \), ..., \( a_s \) failures are observed, the rest are unobserved or missing, where \( a_1, a_2, ..., a_s \) are considered to be fixed. If this censoring arises, the scheme is known as multiply type-II censoring scheme.

If \( X \) is a inverse Weibull random variable, then \( Y = \log X \) has extreme-value distribution with location \( \mu = \log(1/\lambda) \) and scale parameter \( \theta = 1/\lambda \) with PDF and CDF given respectively as:

\[
f(y; \mu, \theta) = \frac{1}{\theta} \exp \left( \frac{-y - \mu}{\theta} \right) \exp \left( -\frac{y - \mu}{\theta} \right)
\]

and

\[
F(y; \mu, \theta) = \exp \left( -\frac{y - \mu}{\theta} \right).
\]

Let us assume that the following multiply type-II censored sample from a sample of size \( n \) is

\[
Y_{a_1:n} \leq Y_{a_2:n} \leq \cdots \leq Y_{a_s:n},
\]

where \( 1 \leq a_1 < a_2 < \cdots < a_s \leq n \), \( a_0 = 0 \), \( a_{s+1} = n + 1 \), \( F(y_{a_0:n}) = 0 \), and \( F(y_{a_{s+1}:n}) = 1 \).

The likelihood function based on the multiply type-II censored sample \( (2.3) \) is given by

\[
L = \frac{n!}{\prod_{j=1}^{s+1} (a_j - a_{j-1} - 1)!} \prod_{j=1}^{s+1} [F(z_{a_j:n}) - F(z_{a_{j-1}:n})]^{a_j - a_{j-1} - 1} \frac{1}{\sigma^s} \prod_{j=1}^{s} f(z_{a_j:n})
\]

\[
= \frac{1}{\sigma^s} \frac{n!}{\prod_{j=1}^{s+1} (a_j - a_{j-1} - 1)!} [F(z_{a_1:n})]^{a_s - 1} \prod_{j=2}^{s} [F(z_{a_j:n}) - F(z_{a_{j-1}:n})]^{a_j - a_{j-1} - 1},
\]

where \( z_{i:n} = (y_{i:n} - \mu)/\theta \), and \( f(z) \) and \( F(z) \) are the pdf and the cdf of the standard Extreme-value distribution, respectively.

Since \( f'(z)/f(z) = e^{-z^2} - 1 \), we can obtain the likelihood equations as follows:

\[
\frac{\partial \ln L}{\partial \theta} = -\frac{1}{\theta} \sum_{j=1}^{s} (a_j - 1)e^{-z_{a_j:n}^2}z_{a_j:n} - (n - a_s) f(z_{a_s:n}) z_{a_s:n} + \sum_{j=1}^{s} e^{-z_{a_j:n}^2} z_{a_j:n}
\]

\[
- \sum_{j=1}^{s} z_{a_j:n} + \sum_{j=2}^{s} (a_j - a_{j-1} - 1) \frac{f(z_{a_j:n}) z_{a_j:n} - f(z_{a_{j-1}:n}) z_{a_{j-1}:n}}{f(z_{a_j:n}) - F(z_{a_{j-1}:n})}
\]

\[
= 0,
\]
and

\[ \frac{\partial \ln L}{\partial \mu} = -1 \left[ (a_1 - 1)e^{-\xi_{a,1}^n} - (n - a_s) \frac{f(z_{a,n})}{1 - F(z_{a,n})} + \sum_{j=1}^{s} e^{-\xi_{a,j}^n} - s \right. \\
+ \left. \sum_{j=2}^{s} (a_j - a_{j-1} - 1) \frac{f(z_{a_j,n}) - f(z_{a_{j-1},n})}{F(z_{a_{j-1}} - F(z_{a_{j},n}))} \right] \
\]

(2.6)

Since the likelihood equations are very complicated, the equations (2.5) and (2.6) do not admit explicit solutions for \( \theta \) and \( \mu \), respectively.

Let \( \xi_i = F^{-1}(p_i) = -\ln[-\ln(p_i)] \) where \( p_i = i/(n + 1), q_i = 1 - p_i \). First, we can approximate the following function by

\[ \frac{f(z_{a,n})}{1 - F(z_{a,n})} z_{a,n} = \kappa_1 + \delta_1 z_{a,n}, \tag{2.7} \]

and

\[ \frac{f(z_{a,j,n}) z_{a,j,n} - f(z_{a_{j-1},n}) z_{a_{j-1},n}}{F(z_{a,j,n}) - F(z_{a_{j-1},n})} \approx \alpha_{1j} + \beta_{1j} z_{a,j,n} + \gamma_{1j} z_{a_{j-1},n}, \tag{2.9} \]

where

\[ \kappa_1 = - \frac{\xi_{a,s}^2}{q_a} \left[ f'(\xi_{a,s}) + \frac{f^2(\xi_{a,s})}{q_a} \right], \quad \delta_1 = \frac{1}{q_a} \left[ f(\xi_{a,s}) + \xi_{a,s} f'(\xi_{a,s}) + \frac{f^2(\xi_{a,s})}{q_a} \xi_{a,s} \right], \]

\[ \alpha_{1j} = K^2 - \frac{\xi_{a,j}^2 f'(\xi_{a,j}) - \xi_{a,j-1}^2 f'(\xi_{a,j-1})}{p_{a,j} - p_{a,j-1}}, \quad \beta_{1j} = \frac{(1 - K) f(\xi_{a,j}) + \xi_{a,j} f'(\xi_{a,j})}{p_{a,j} - p_{a,j-1}}, \]

\[ \gamma_{1j} = \frac{(1 - K) f(\xi_{a,j-1}) + \xi_{a,j-1} f'(\xi_{a,j-1})}{p_{a,j} - p_{a,j-1}}, \quad K = \frac{f(\xi_{a,j}) - f(\xi_{a,j-1})}{p_{a,j} - p_{a,j-1}}. \]

By substituting the equation (2.7), (2.8), and (2.9) into the equation (2.5), we can derive an estimator of \( \theta \) as follows:

\[ \hat{\theta} = -B_1 + C_1 \hat{\mu}, \tag{2.10} \]

where

\[ A_1 = s + (a_1 - 1)e^{-\xi_{a,1}^n} - (n - a_s)\kappa_1 + \sum_{j=1}^{s} e^{-\xi_{a,j}^n} + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)\alpha_{1j}, \]

\[ B_1 = (a_1 - 1)e^{-\xi_{a,1}^n} - (n - a_s)\delta_1 Y_{a,n} + \sum_{j=1}^{s} e^{-\xi_{a,j}^n} - (n - a_s)\delta_1 Y_{a,n} + \sum_{j=1}^{s} e^{-\xi_{a,j}^n} - (n - a_s)\delta_1 Y_{a,n} \]

and

\[ + \sum_{j=1}^{s} Y_{a,j,n} + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)(\beta_{1j} Y_{a,j,n} + \gamma_{1j} Y_{a_{j-1},n}). \]
\[ C_1 = (a_1 - 1)e^{-\xi_{a_1}} (1 - \xi_{a_1}) - (n - a_s)\delta_1 + \sum_{j=1}^{s} e^{-\xi_{a_j}} (1 - \xi_{a_j}) - s + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)(\beta_{1j} + \gamma_{1j}). \]

Second, we can approximate the following functions by

\[ \frac{f(z_{a_{j:n}})}{1 - F(z_{a_{j:n}})} \approx \kappa_2 + \delta_2 z_{a_{j:n}}, \tag{2.11} \]

\[ e^{-\xi_{a_{j:n}}} \approx e^{-\xi_{a_j}} (1 + \xi_{a_j}^2) - e^{-\xi_{a_{j-1}}} z_{a_{j-1:n}}, \tag{2.12} \]

\[ \frac{f(z_{a_{j-1:n}})}{F(z_{a_{j-1:n}}) - F(z_{a_{j:n}})} \approx \alpha_{2j} + \beta_{2j} z_{a_{j:n}} + \gamma_{2j} z_{a_{j-1:n}}, \tag{2.13} \]

and

\[ \frac{f(z_{a_{j-1:n}})}{F(z_{a_{j-1:n}}) - F(z_{a_{j:n}})} \approx \alpha_{3j} + \beta_{3j} z_{a_{j:n}} + \gamma_{3j} z_{a_{j-1:n}}, \tag{2.14} \]

where

\[ \kappa_2 = \frac{1}{q_{a_1}} \left[ f(\xi_{a_1}) - \xi_{a_1} f'(\xi_{a_1}) - \frac{f^2(\xi_{a_1})}{q_{a_1}} \xi_{a_1} \right], \quad \delta_2 = \frac{1}{q_{a_1}} \left[ f'(\xi_{a_1}) + \frac{f^2(\xi_{a_1})}{q_{a_1}} \right], \]

\[ \alpha_{2j} = \frac{(1 + K) f(\xi_{a_{j-1}}) - \xi_{a_j} f'(\xi_{a_j})}{p_{a_j} - p_{a_{j-1}}}, \quad \beta_{2j} = \frac{f'(\xi_{a_1})}{p_{a_j} - p_{a_{j-1}}} - \left( \frac{f(\xi_{a_j})}{p_{a_j} - p_{a_{j-1}}} \right)^2, \]

\[ \gamma_{2j} = \frac{f(\xi_{a_j}) f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}} + \left( \frac{f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}} \right)^2, \]

\[ \alpha_{3j} = \frac{(1 + K) f(\xi_{a_{j-1}}) - \xi_{a_{j-1}} f'(\xi_{a_{j-1}})}{p_{a_{j-1}} - p_{a_{j-1}}}, \quad \gamma_{3j} = \frac{f'(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}} + \left( \frac{f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}} \right)^2. \]

By substituting the equations (2.11)~(2.14) into the equation (2.5), we can derive an estimator of \( \theta \) as follows:

\[ \hat{\theta}_2 = \frac{-B_2 + \sqrt{B_2^2 - 4sC_2}}{2s}, \tag{2.15} \]

where

\[ B_2 = (a_1 - 1)e^{-\xi_{a_1}} (1 + \xi_{a_1}^2) Y_{a_1:n} - (n - a_s)\kappa_2 Y_{a_{s:n}} + \sum_{j=1}^{s} e^{-\xi_{a_j}} (1 + \xi_{a_j}^2) Y_{a_{j:n}} - \sum_{j=1}^{s} Y_{a_{j:n}} \]

\[ + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)(\alpha_{2j} Y_{a_{j:n}} - \gamma_{3j} Y_{a_{j-1:n}}) - (a_1 - 1)\alpha_2 - (n - a_s)\kappa_2 \]

\[ + \sum_{j=1}^{s} e^{-\xi_{a_j}} (1 + \xi_{a_j}^2) - s + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)(\alpha_{2j} - \gamma_{3j}) \tilde{\mu}, \]
\[
C_2 = -(a_1 - 1)e^{-\xi_1}(Y_{1:n} - \bar{\mu})^2 - (n - a_s)\delta_2(Y_{a_s:n} - \bar{\mu})^2 - \sum_{j=1}^s e^{-\xi_j}(Y_{a_j:n} - \bar{\mu})^2 + \sum_{j=2}^s (a_j - a_{j-1} - 1)\left[\beta_2j(Y_{a_j:n} - \bar{\mu})^2 + 2\gamma_2j(Y_{a_j:n} - \bar{\mu})(Y_{a_{j-1}:n} - \bar{\mu}) - \gamma_3j(Y_{a_{j-1}:n} - \bar{\mu})^2\right].
\]

Next, equation (2.6) does not admit an explicit solution for \( \mu \). But we can expand the following function as follows;
\[
\frac{f(z_{a_j:n}) - f(z_{a_{j-1}:n})}{F(z_{a_j:n}) - F(z_{a_{j-1}:n})} \approx \alpha_{4j} + \beta_{4j}z_{a_j:n} + \gamma_{4j}z_{a_{j-1}:n}
\]
where \( \alpha_{4j} = \alpha_{2j} - \alpha_{3j}, \beta_{4j} = \beta_{2j} - \beta_{3j}, \) and \( \gamma_{4j} = \gamma_{2j} - \gamma_{3j} \).

By substituting the equations (2.11), (2.12), and (2.16) into the equation (2.6), we can derive an estimator of \( \mu \) as follows;
\[
\hat{\mu} = \frac{E}{D},
\]
where
\[
D = A_{\mu}C_1 - A_1C_{\mu}, \quad E = A_{\mu}B_1 - A_1B_{\mu},
\]
\[
A_{\mu} = (a_1 - 1)e^{-\xi_1}(1 + \xi_{a_1}^2) - (n - a_s)\alpha_2 + \sum_{j=1}^s e^{-\xi_j}(1 + \xi_{a_j}^2) - s + \sum_{j=2}^s (a_j - a_{j-1} - 1)\alpha_4j;
\]
\[
B_{\mu} = -(a_1 - 1)e^{-\xi_1}Y_{1:n} - (n - a_s)\delta_2Y_{a_s:n} - \sum_{j=1}^s e^{-\xi_j}Y_{a_j:n} + \sum_{j=2}^s (a_j - a_{j-1} - 1)(\beta_{4j} + \gamma_{4j}Y_{a_{j-1}:n}),
\]
\[
C_{\mu} = -(a_1 - 1)e^{-\xi_1} - (n - a_s)\delta_2 - \sum_{j=1}^s e^{-\xi_j} + \sum_{j=2}^s (a_j - a_{j-1} - 1)(\beta_{4j} + \gamma_{4j}Y_{a_{j-1}:n}).
\]

Since \( \theta = 1/\lambda \) and \( \mu = log(1/\sigma) \), we can obtain the AMLEs of the shape parameter \( \lambda \) and the scale parameter \( \sigma \) as follows;
\[
\lambda_i = \frac{1}{\delta_i}, \quad i = 1, 2
\]
and
\[
\sigma = \frac{1}{e^{\lambda_i}}.
\]

3. EDF and graphical methods in the goodness-of-fit tests

In the inference of distribution, the EDF and graphical methods generally have used. In this section, we consider some goodness-of-fit tests of the inverse Weibull distribution based on multiply type-II censored samples. Also we propose the graphical method and new test statistic based on modified NSLC.
3.1. Modified empirical distribution function type tests

A well known EDF $F_n(x)$ is

$$F_n(x) = \frac{\text{the number of } X's \leq x}{n}. \quad (3.1)$$

For complete samples under a simple hypothesis, the Kolmogorov-Smirnov ($D$), the Cramer-von Mises ($W^2$), and the Anderson-Darling ($A^2$) are defined as

$$D^+ = \sup_x [F_n(x) - F_0(x)],$$
$$D^- = \sup_x [F_0(x) - F_n(x)],$$
$$D = \max [D^+, D^-], \quad (3.2)$$

$$W^2 = n \int_{-\infty}^{\infty} [F_n(x) - F_0(x)]^2 dF_0(x), \quad (3.3)$$

and

$$A^2 = n \int_{-\infty}^{\infty} \frac{[F_n(x) - F_0(x)]^2}{F_0(x)[1 - F_0(x)]} dF_0(x). \quad (3.4)$$

In the form given above, test statistics can only be used with complete samples, i.e. no censoring. Modification of the test statistics for censored samples and for composite hypothesis $H_0$ with unspecified parameters has been studied by Pettitt and Stephens (1976). Kang and Lee (2006) developed three EDF type modified Kolmogorov-Smirnov test, modified Anderson-Darling test, and modified Cramer-von Mises test for the two-parameter exponential distribution based on multiply type-II censored samples.

Now, we use three EDF type modified Kolmogorov-Smirnov test, modified Anderson-Darling test, and modified Cramer-von Mises test for multiply type-II censored samples from the inverse Weibull distribution using the proposed estimators $\hat{\sigma}$ and $\hat{\lambda}_k$, $k = 1, 2$ as follows;

$$D^+_k = \max_{1 \leq a_j \leq s} \left[ \frac{a_j}{s} - F(x_{a_j}; \hat{\sigma}, \hat{\lambda}_k) \right],$$
$$D^-_k = \max_{1 \leq a_j \leq s} \left[ F(x_{a_j}; \hat{\sigma}, \hat{\lambda}_k) - \frac{a_j}{s} \right],$$
$$D_k = \max_{1 \leq a_j \leq s} \left[ D^+_k, D^-_k \right], \quad (3.5)$$

$$W^2_k = \frac{1}{12s} + \sum_{j=1}^{s} \left[ F(x_{a_j}; \hat{\sigma}, \hat{\lambda}_k) - \frac{2a_j - 1}{2s} \right]^2, \quad (3.6)$$

and

$$A^2_k = -s - \frac{1}{s} + \sum_{j=1}^{s} (2a_j - 1) \left[ \ln F(x_{a_j}; \hat{\sigma}, \hat{\lambda}_k) + \ln \{1 - F(x_{a_j+1}; \hat{\sigma}, \hat{\lambda}_k)\} \right]. \quad (3.7)$$
3.2. Modified normalized sample Lorenz curve

The Lorenz curve is extensively used in the study of income distribution and used to be a powerful tool for the analysis of a variety of scientific problems. Cho et al. (1999) proposed the transformed Lorenz curve that can be used in the study of symmetric distribution. The transformed Lorenz curve is defined by

\[ TL(r_i) = \frac{\sum_{j=1}^{i} X_{j:n}}{\sum_{j=1}^{n} X_{j:n}}, \quad r_i = \frac{i}{n}, \quad i = 1, 2, \ldots, n. \]  

(3.8)

Kang and Cho (2001) proposed the NSLC for the complete sample as follows;

\[ NSLC(r_i) = \frac{TSL(r_i)}{TSL_F(r_i)}, \quad r_i = \frac{i}{n}, \quad i = 1, 2, \ldots, n, \]  

(3.9)

where

\[ TSL(r_i) = \frac{\sum_{j=1}^{i} (X_{j:n} - X_{1:n})}{\sum_{j=1}^{n} (X_{j:n} - X_{1:n})} - r_i + 1, \]

\[ TSL_F(r_i) = \frac{\sum_{j=1}^{i} \left[ F^{-1}(p_j) - F^{-1}(p_1) \right]}{\sum_{j=1}^{n} \left[ F^{-1}(p_j) - F^{-1}(p_1) \right]} - r_i + 1. \]

Now, we propose modified NSLC based on multiply type-II censored samples. The modified NSLC based on multiply type-II censored samples is given by

\[ MNSLC(r_i) = \frac{MTSL(r_i)}{TSL_F(r_i)}, \quad r_i = \frac{a_i}{n}, \quad i = 1, 2, \ldots, s, \]  

(3.10)

where

\[ MTSL(r_i) = \frac{\sum_{j=1}^{i} (X_{a_j:n} - X_{a_1:n})}{\sum_{j=1}^{a} (X_{a_j:n} - X_{a_1:n})} - r_i + 1. \]

Also, we propose the modified NSLC plot for multiply type-II censored samples using \((X,Y)=(1-r_i, 1-MNSLC_{i,k})\). If data come from the inverse Weibull distribution, the modified NSLC plot is \(y \approx 0\) (see, Figure 3.1 and Figure 3.2). The value of \(1-MNSLC_{i,k}\) increases and then decreases as \(1-r_i\) increases when the alternative is Pareto and Weibul distributions. But the value of \(1-MNSLC_{i,k}\) decreases and then increases as \(1-r_i\) increases when the alternative is beta, lognormal and normal distributions.

![Figure 3.1 Modified NSLC plot: Complete data (n=30, m=0)](image-url)
Goodness-of-fit test for the inverse Weibull distribution

We also propose new test statistics based on the modified NSLC for the inverse Weibull distribution under multiply type-II censored samples. The new test statistic is defined by

$$T_{S_k} = \frac{1}{2s} \sum_{i=1}^{s-1} (|1 - MNSLC_{i,k}| + |1 - MNSLC_{i+1,k}|)(a_{i+1} - a_i), \quad k = 1, 2.$$  (3.11)

4. Simulation study

In order to evaluate the performance of the proposed estimators, the mean squared errors of all proposed estimators were simulated by a Monte Carlo method for sample size $n = 20, 40$ and various choices of censoring ($m = n - s$ is the number of unobserved or missing data). The simulation procedure was repeated 10,000 times. The 95th percentile were found and these critical values ($\alpha = 0.05$) are given in Table 4.1.

The probabilities of type I error for four tests under the inverse Weibull distribution are given in Table 4.2. As expected, when underlying distribution is inverse Weibull, the probabilities of type I error for four tests are close to the nominal significance level 0.05.

Table 4.1 Critical values ($\alpha = 0.05$)

<table>
<thead>
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<th>n</th>
<th>m</th>
<th>$a_j$</th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$A_1^2$</th>
<th>$A_2^2$</th>
<th>$W_1^2$</th>
<th>$W_2^2$</th>
<th>$T_{S_1}$</th>
<th>$T_{S_2}$</th>
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<tr>
<td>0</td>
<td>1~20</td>
<td>0.183</td>
<td>0.183</td>
<td>0.798</td>
<td>0.741</td>
<td>0.127</td>
<td>0.123</td>
<td>0.165</td>
<td>0.163</td>
<td>0.163</td>
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<tr>
<td>20</td>
<td>1~17</td>
<td>0.323</td>
<td>0.330</td>
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<td>4.643</td>
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<td>0.430</td>
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Table 4.2 The probabilities of type I error for four tests under the inverse Weibull distribution

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<th>( D_1 )</th>
<th>( D_2 )</th>
<th>( A_1^1 )</th>
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<th>( W_1^1 )</th>
<th>( W_2^1 )</th>
<th>( T S_1 )</th>
<th>( T S_2 )</th>
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<td>1 2 6<del>9 12</del>15 17~20</td>
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<td>6<del>10 16</del>25 31~40</td>
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<td>0.056</td>
<td>0.056</td>
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Table 4.3 The powers of the tests for the normal alternative distribution (N(5,1))

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<th>( A_1^2 )</th>
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<th>( W_1^2 )</th>
<th>( W_2^2 )</th>
<th>( T S_1 )</th>
<th>( T S_2 )</th>
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</thead>
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<tr>
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</tr>
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Table 4.4 The powers of the tests for the gamma alternative distribution (G(3))

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<th>( A_1^3 )</th>
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<th>( W_1^3 )</th>
<th>( W_2^3 )</th>
<th>( T S_1 )</th>
<th>( T S_2 )</th>
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<td>1.000</td>
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</tr>
<tr>
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<td>0.056</td>
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<td>1.000</td>
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<td>1.000</td>
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<td>0.056</td>
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Suk-Bok Kang · Jun-Tae Han · Yeon-Ju Seo · Jina Jeong
Goodness-of-fit test for the inverse Weibull distribution

Table 4.5 The powers of the tests for the lognormal alternative distribution (LN(0,1))

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<th>(A_1^2)</th>
<th>(A_2^2)</th>
<th>(W_1^2)</th>
<th>(W_2^2)</th>
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<td>0.900</td>
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<td>0.940</td>
<td>0.939</td>
<td>0.877</td>
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<td>0.638</td>
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<td>0.802</td>
<td>0.836</td>
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<td>0.813</td>
<td>0.776</td>
<td>0.526</td>
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Table 4.6 The powers of the tests for the Weibull alternative distribution (Wei(2,2))

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<th>(A_1^2)</th>
<th>(A_2^2)</th>
<th>(W_1^2)</th>
<th>(W_2^2)</th>
<th>(TS_1)</th>
<th>(TS_2)</th>
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<td>0.999</td>
<td>0.999</td>
<td>1.000</td>
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<td>0.860</td>
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<td>0.982</td>
<td>0.960</td>
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<td>4−17</td>
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<td>0.985</td>
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<td>0.938</td>
<td>0.928</td>
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<td>0.985</td>
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</table>

The powers of four tests with significance level 0.05 for the inverse Weibull distribution based on multiply type-II censored samples are investigated under 4 alternative distributions. These values are given in Table 4.3−4.6. For the normal alternative distribution, all the tests showed good performance. For the gamma alternative distribution, the modified Anderson-Darling statistic is more powerful than other test statistics. For the lognormal alternative distribution, the modified EDF type tests are more powerful than \(TS_k\). But the \(TS_1\) and \(TS_2\) are more powerful than other test statistics when the Weibull alternative distribution. The tests that use estimator \(\hat{\lambda}_1\) are generally more powerful than the tests that use the estimator \(\hat{\lambda}_2\) for the \(D_k, W_k^2, A_k^2\) and \(TS_k\) statistics. All computation are programmed in Microsoft Visual C++ 6.0 and random numbers for simulations are generated by IMSL subroutines.
5. Conclusions

In most cases of censored and truncated samples, the maximum likelihood method does not provide explicit estimators. So we discuss another method for the purpose of providing the explicit estimators. We use three modified EDF types test for the inverse Weibull distribution based on multiply type-II censored samples using AMLEs. We propose the modified NSLC plot to test for the inverse Weibull distribution based on multiply type-II censored samples. We also propose new test statistics based on the NSLC for the inverse Weibull distribution under multiply type-II censored samples. In addition we believe that the modified NSLC plot can be extended to cases with various censoring.

References