Categories of two types of uniform spaces

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Abstract

In a strictly two-sided, commutative biquantale, we study the relationships between the categories of Hutton \((L, \otimes)\)-uniform spaces and \((L, \odot)\)-uniform spaces. We investigate the properties of them.

Key words: Hutton \((L, \otimes)\)-uniform spaces, \((L, \odot)\)-uniform spaces

1. Introduction

Recently, Gutiérrez García and his colleagues [1] introduced \(L\)-valued Hutton uniformity where a quadruple \((L, \leq, \otimes, *)\) is defined by a GL-monoid \((L, *)\) dominated by \(\otimes\), a cl-quasi-monoid \((L, \leq, \odot)\). Kubiak and his colleagues [10] studied the relationships between the categories of \(I(L)\)-uniform spaces and \(L\)-uniform spaces. Kim and his colleagues [7], as a somewhat different aspect in [1], introduced the notion of Hutton \((L, \odot)\)-uniformities as a view point of the approach using uniform operators defined by Rodabaugh [13] and \((L, \odot)\)-uniformities in a sense Lowen [11] and Höhle [12] based on powersets of the form \(L^{X \times X}\).

In this paper, we show that the category \(\text{HUnif}\) of all Hutton \((L, \otimes)\)-uniform spaces and \(H\)-uniformly continuous maps and the category \(\text{Unif}\) of all \((L, \odot)\)-uniform spaces and uniformly continuous maps are isomorphic. Moreover, we define the subspaces of them.

2. Preliminaries

Definition 2.1. [3,4,7,8,12] A triple \((L, \leq, \odot)\) is called a strictly two-sided, commutative biquantale (stbs-biquantale, for short) iff it satisfies the following properties:

\((L1)\) \(L = (L, \leq, \lor, \land, \top, \bot)\) is a completely distributive lattice where \(\top\) is the universal upper bound and \(\bot\) is the universal lower bound;

\((L2)\) \((L, \odot)\) is a commutative semigroup;

\((L3)\) \(a = a \odot \top\), for each \(a \in L\);

\((L4)\) \(\odot\) is distributive over arbitrary joins, i.e.

\[\bigvee_{i \in \Gamma} (a_i \odot b) = \bigvee_{i \in \Gamma} (a_i \odot b), \quad \forall i \in \Gamma\]

\((L5)\) \(\odot\) is distributive over arbitrary meets, i.e.

\[\bigwedge_{i \in \Gamma} (a_i \odot b) = \bigwedge_{i \in \Gamma} (a_i \odot b), \quad \forall i \in \Gamma\]

Remark 2.2. [3,4,7,8,12] Let \((L, \leq, \odot)\) be a stbs-biquantale. For each \(x, y \in L\), we define

\[x \rightarrow y = \bigvee \{ z \in L \mid x \odot z \leq y \}.\]

Then it satisfies Galois correspondence, that is,

\[(x \odot y) \leq z \iff x \leq (y \rightarrow z).\]

In this paper, we always assume that \((L, \leq, \odot, *)\) is a stbs-biquantale with strong negation \(*\) where \(a^* = a \circ 0\) unless otherwise specified.

Let \(X\) be a nonempty set. All algebraic operations on \(L\) can be extended pointwisely to the set \(L^X\) as follows: for all \(x \in X\), \(f, g \in L^X\), \(\lambda \in L^X\) and \(\alpha \in L\),

\((L1)\) \(f \leq g\) iff \(f(x) \leq g(x)\);

\((L2)\) \(f \odot g(x) = f(x) \odot g(x)\);

\((L3)\) \(1_X(x) = \top\), \(\alpha \odot 1_X(x) = \alpha\) and \(1_0(x) = \bot\);

\((L4)\) \((\alpha \rightarrow \lambda)(x) = \alpha \rightarrow \lambda(x)\) and \((\lambda \rightarrow \alpha)(x) = \lambda(x) \rightarrow \alpha\);

\((L5)\) \((\alpha \circ \lambda)(x) = \alpha \circ \lambda(x)\).

Definition 2.3. [7] Let \(\Omega(X)\) be a subset of \((L^X)(L^X)\) such that

\((O1)\) \(\lambda \leq \phi(\lambda)\), for every \(\lambda \in L^X\),

\((O2)\) \(\phi(\bigvee_{i \in \Gamma} \lambda_i) = \bigvee_{i \in \Gamma} \phi(\lambda_i)\), for \(\{\lambda_i\}_{i \in \Gamma} \subset L^X\),

\((O3)\) \(\alpha \circ \phi(\lambda) = \phi(\alpha \circ \lambda)\), for \(\lambda \in L^X\).
Lemma 2.4. [7] For \( \phi, \phi_1, \phi_2 \in \Omega(X) \), we define, for all \( \lambda \in L^X \),
\[
\phi^{-1}(\lambda) = \bigwedge \{ \rho \in L^X | \phi(\rho^\ast) \leq \lambda^\ast \},
\]
\[
\phi_1 \circ \phi_2(\lambda) = \phi_1(\phi_2(\lambda)),
\]
\[
\phi_1 \circ \phi_2(\lambda) = \bigwedge \{ \phi_1(\lambda_1) \circ \phi_2(\lambda_2) | \lambda = \lambda_1 \circ \lambda_2 \}.
\]

For \( \phi_1, \phi_2, \phi_3 \in \Omega(X) \), the following properties hold: (1) If \( \phi(1_{x}) = \rho_x \) for all \( x \in X \), then \( \phi(\lambda) = \bigvee_{x \in X} \lambda(x) \circ \rho_x \).

(2) If \( \phi_1(1_{x}) = \phi_2(1_{x}) \) for all \( x \in X \), then \( \phi_1 = \phi_2 \).

(3) \( \phi^{-1} \in \Omega(X) \), \( (\phi^{-1})^{-1} = \phi \) and \( \phi_1 \circ \phi_2 \in \Omega(X) \).

(4) If \( \phi_1 \leq \phi_2 \), then \( \phi_1^{-1} \leq \phi_2^{-1} \).

(5) \( \phi_1 \circ \phi_2 \in \Omega(X) \).

(6) \( \phi_1 \circ \phi_2 \leq \phi_1 \) and \( \phi_1 \circ \phi_2 \leq \phi_2 \).

(7) \( \phi_1 \circ \phi_2 \in \Omega(X) \).

(8) \( \phi_1 \circ \phi_2 \circ \phi_3 = \phi_1 \circ (\phi_2 \circ \phi_3) \).

(9) Define \( \phi^\perp \in \Omega(X) \) as \( \phi^\perp(1_{x}) = 1_X \), \( \forall x \in X \).

Then \( \phi \leq \phi^\perp \) for all \( \phi \in \Omega(X) \).

Definition 2.5. [7] A nonempty subset \( U \) of \( \Omega(X) \) is called a Hutton \((L, \circ)\)-quasi-uniformity on \( X \) if it satisfies the following conditions:

(QU1) If \( \phi \leq \psi \) with \( \phi \in U \) and \( \psi \in \Omega(X) \), then \( \psi \in U \).

(QU2) For each \( \phi, \psi \in U \), \( \phi \circ \psi \in U \).

(QU3) For each \( \phi \in U \), there exists \( \psi \in U \) such that \( \psi \circ \phi \leq \phi \).

The pair \( (X, U) \) is said to be a Hutton \((L, \circ)\)-quasi-uniform space.

A Hutton \((L, \circ)\)-quasi-uniform space is said to be a Hutton \((L, \circ)\)-quasi-uniform space if it satisfies:

(U) For each \( \phi \in U \), there exists \( \phi^{-1} \in U \).

Definition 2.6. [7] Let \( E(X \times X) = \{ u \in L^{X \times X} | u(x, x) = 1 \} \) be a subset of \( L^{X \times X} \). A nonempty subset \( D \) of \( E(X \times X) \) is called an \((L, \circ)\)-quasi-uniformity on \( X \) if it satisfies the following conditions:

(QD1) If \( u \leq u \) with \( u \in D \) and \( v \in E(X \times X) \), then \( u \in D \).

(QD2) For each \( u, v \in D \), \( u \circ v \in D \).

(QD3) For each \( u \in D \), there exists \( v \in D \) such that \( u \circ v \leq u \) where

\[
v \circ u(z, y) = \bigvee_{z \in X} (v(z, z) \circ u(z, y)).
\]

The pair \((X, D)\) is said to be an \((L, \circ)\)-quasi-uniform space.

An \((L, \circ)\)-quasi-uniform space is said to be an \((L, \circ)\)-uniform space if it satisfies:

(D) For each \( u \in D \), there exists \( u^\ast \in U \) where \( u^\ast(x, y) = u(y, x) \).

Definition 2.7. [7] A function \( u : X \times X \rightarrow L \) is called an \( \circ \)-quasi-equivalence relation iff it satisfies the following properties:

(E1) \( u(x, x) = 1 \) for all \( x \in X \).

(E2) \( u(x, y) \circ u(y, z) \leq u(x, z) \).

An \( \circ \)-quasi-equivalence relation is called an \( \circ \)-equivalence relation on \( X \) if it satisfies

(E) \( u(x, y) = u(y, x) \).

We denote \( u^2 = u \circ u \) and \( u^{n+1} = u^n \circ u \) for each \( u \in L^{X \times X} \).

Theorem 2.8. [7] Let \( u : X \times X \rightarrow L \) be an \( \circ \)-equivalence relation. We define a mapping \( D_u \) as follows:

\[
D_u = \{ v \in E(X \times X) | \exists n \in N, u^n \leq v \}.
\]

Then \( D_u \) is an \((L, \circ)\)-uniformity on \( X \).

Theorem 2.9. [7] We define a mapping \( \Gamma : E(X \times X) \rightarrow \Omega(X) \) as follows:

\[
\Gamma(u)(\lambda)(y) = \bigvee_{x \in X} \lambda(x) \circ u(x, y).
\]

Then we have the following properties:

(1) For \( u \in E(X \times X) \), \( \Gamma(u) \in \Omega(X) \) and \( \Gamma(u) \) has a right adjoint mapping \( \Gamma(u)^{-1} \) defined by

\[
\Gamma(u)^{-1}(\lambda) = \bigvee_{\rho \in L^X} \{ \rho \in L^X | \Gamma(u)(\rho) \leq \lambda \}.
\]

(2) \( \Gamma \) is injective and join preserving.

(3) \( \Gamma \) has a right adjoint mapping \( \Lambda : \Omega(X) \rightarrow E(X \times X) \) as follows:

\[
\Lambda(\phi)(x, y) = \phi(1_{x})(y).
\]

(4) \( \Gamma \circ \Lambda = 1_{\Omega(X)} \) and \( \Lambda \circ \Gamma = E(X \times X) \).

Theorem 2.10. [7] Let \( u, u_1, u_2 \in E(X \times X) \). Then we have the following properties:

(1) If \( u_1 \leq u_2 \), \( \Gamma(u_1) \leq \Gamma(u_2) \).

(2) \( \Gamma(u_1 \circ u_2) \leq \Gamma(u_1) \circ \Gamma(u_2) \).

(3) \( \Gamma(1_L) = 1_L \).

(4) \( \Gamma(u)^{-1} = \Gamma(u^\ast) \).

(5) \( \Gamma(u)^{-1}(\lambda) \rightarrow \bot = \Gamma(u)(\lambda) \rightarrow \bot \), for all \( \lambda \in L^X \).

(6) \( \Gamma(u_1 \circ u_2) = \Gamma(u_2) \circ \Gamma(u_1) \).

(7) \( \Gamma(\alpha \circ u) = \alpha \circ \Gamma(u) \).

(8) If \( u \) is an \( \circ \)-equivalence relation on \( X \), then \( \Gamma(u)^{-1} = \Gamma(u^\ast) = \Gamma(u) = \Gamma(u) \).
Theorem 2.11. [7] Let \( u : X \times X \to L \) be an \( \circ \)-equivalence relation. We define a mapping \( U_u \) as follows:
\[
U_u = \{ \phi \in \Omega(X) \mid \exists n \in N, \Gamma(u^n) \leq \phi \}.
\]
Then \( U_u \) is a Hutton \( (L, \circ) \)-uniformity on \( X \).

Theorem 2.12. [7] Let \( \phi, \phi_1, \phi_2 \in \Omega(X) \). Then we have the following properties:

1. If \( \phi_1 \leq \phi_2 \), then \( \Lambda(\phi_1) \leq \Lambda(\phi_2) \).
2. \( \Lambda(\phi_1) \otimes \Lambda(\phi_2) = \Lambda(\phi_1 \circ \phi_2) \).
3. \( \Lambda(1_L) = 1_\Lambda \).
4. \( \Lambda(\phi^\circ) = \Lambda(\phi^{-1}) \).
5. \( \Lambda(\phi_1) \circ \Lambda(\phi_2) = \Lambda(\phi_2 \circ \phi_1) \).
6. \( \Lambda(\alpha \circ \phi) = \alpha \circ \Lambda(\phi) \).
7. If \( \phi \circ \phi = \phi \) and \( \phi \circ \phi = \phi^{-1} \), then \( \Lambda(\phi) \) is an \( \circ \)-equivalence relation.

Theorem 2.13. [7] Let \( D \) be an \( (L, \circ) \)-uniform space. We define a mapping \( U_D \subset \Omega(X) \) as follows:
\[
U_D = \{ \phi \in \Omega(X) \mid \exists u \in D, \Gamma(u) \leq \phi \}.
\]
Then \( U_D \) is a Hutton \( (L, \circ) \)-uniformity on \( X \).

Theorem 2.14. [7] Let \( U \) be a Hutton \( (L, \circ) \)-uniformity on \( X \). We define a mapping \( D_U \subset E(X \times X) \) as follows:
\[
D_U = \{ u \in E(X \times X) \mid \exists \phi \in U, \Lambda(\phi) \leq u \}.
\]
Then:

1. \( D_U \) is an \( (L, \circ) \)-uniformity on \( X \).
2. \( D_{U_D} = D \) and \( U_{D_U} = U \).

3. Properties of two types of uniform spaces

Let \( f : X \to Y \) be a function. We define the image and preimage operators
\[
\begin{align*}
\text{f}^\Rightarrow : (L^X)(L^X) &\to (L^Y)(L^Y), \\
\text{f}^\Leftarrow : (L^Y)(L^Y) &\to (L^X)(L^X)
\end{align*}
\]
such that for each \( \phi \in (L^X)(L^X) \) and \( \psi \in (L^Y)(L^Y) \) for all \( \mu, \mu_1, \mu_2 \in L^X \), \( \rho, \rho_1, \rho_2 \in L^Y \),
\[
\begin{align*}
\text{f}^\Rightarrow(\phi)(\rho) &= (f^- \circ \phi \circ f^-)(\rho) = f^- \circ (\phi \circ f^-)(\rho), \\
\text{f}^\Leftarrow(\psi)(\mu) &= (f^- \circ \psi \circ f^-)(\mu) = f^- \circ (\psi \circ f^-)(\mu).
\end{align*}
\]

Lemma 3.1. For each \( \psi, \psi_1, \psi_2 \in \Omega(Y) \) and \( \phi, \phi_2 \in \Omega(X) \), we have the following properties.

1. The pair \( (f^\Rightarrow, f^\Leftarrow) \) is a Galois connection; i.e.,
\[ f^\Rightarrow \circ f^\Leftarrow = f^\Leftarrow \circ f^\Rightarrow. \]

2. \( f^- \circ (\mu_1 \circ \mu_2) \leq f^- \circ (\mu_1) \circ f^- \circ (\mu_2) \) with equality if \( f \) is injective and \( f^- \circ (\rho_1 \circ \rho_2) = f^- \circ (\rho_1) \circ f^- \circ (\rho_2). \)

3. \( f^- \circ (\psi) \in \Omega(X) \).

4. If \( \psi_1 \leq \psi_2 \), then \( f^- \circ (\psi_1) \leq f^- \circ (\psi_2). \)

5. \( f^- \circ (\psi_1) \circ f^- \circ (\psi_2) \leq f^- \circ (\psi_1 \circ \psi_2) \) with equality if \( f \) is onto.

Proof. (1) We prove the following statements:
\[
\begin{align*}
f^- \circ (f^\Rightarrow(\psi))(\mu) &= f^- \circ (f^- \circ (\psi))(f^- \circ (\mu)) \\
&= f^- \circ (f^- \circ (\psi \circ f^- \circ (\mu)))).
\end{align*}
\]

Similarly, \( f^- \circ (f^\Leftarrow(\phi))(\mu) \leq f^- \circ (\psi)(\mu) \).

(2) \( f^- \circ f^\Rightarrow = f^- \circ f^\Leftarrow \).

(3) \( f^- \circ f^\Rightarrow = f^- \circ f^\Leftarrow \).

(4) \( f^- \circ f^\Rightarrow = f^- \circ f^\Leftarrow \).

(5) \( f^- \circ f^\Rightarrow = f^- \circ f^\Leftarrow \).

(6) \( f^- \circ f^\Rightarrow = f^- \circ f^\Leftarrow \).

By the definition of \( f^- \circ (\psi)(\mu) \), there exists \( \rho, \rho_1 \in L^Y \) with \( \psi(\rho) \leq f^- \circ (\mu) \) such that
\[
(\text{f}^\Rightarrow(\psi)^{-1}(\mu)) \leq f^- \circ (\rho). \]

On the other hand, since
\[
\psi(\mu) = \psi(\mu^*) \leq (f^- \circ (\mu^*)) \leq \mu^*,
\]
we have \( f^- \circ (\psi)(f^- \circ (\mu^*)) \leq f^- \circ (\mu^*) \leq \mu^* \). So,
\[
(\text{f}^\Rightarrow(\psi)^{-1}(\mu)) \leq f^- \circ (\mu^*).
\]

It is a contradiction. Thus, \( (\text{f}^\Rightarrow(\psi)^{-1}(\mu)) \leq f^- \circ (\mu^*). \)

Similarly, \( f^- \circ (\psi)(\mu^*) \leq f^- \circ (\mu^*) \) \( f^- \circ (\mu^*) \)

Thus, \( f^- \circ (\psi)(\mu^*) \leq f^- \circ (\mu^*) \) \( f^- \circ (\mu^*) \)

(7) Suppose there exist \( \mu \in L^X \) and \( x \in X \) such that
\[
\begin{align*}
\text{f}^\Rightarrow(\psi_1 \circ \psi_2)^{-1} &\leq f^- \circ (\psi_1 \circ \psi_2)(\mu)(x) \\
&= f^- \circ (\psi_1 \circ \psi_2)(\mu)(x).
\end{align*}
\]

Then there exist \( \nu \in L^Y \) such that
\[
\begin{align*}
\text{f}^\Rightarrow(\psi_1 \circ \psi_2)(\nu)(x) &\leq \text{f}^\Leftarrow(\psi_1) \circ \psi_2(x) \\
&= \text{f}^\Leftarrow(\psi_1) \circ \psi_2(x).
\end{align*}
\]
Since $\mu \leq f^-(f^-(\mu)) = f^-(\nu_1) \circ f^-(\nu_2)$ from (2), we have

\[
\begin{align*}
(f^\infty(\nu_1) \circ f^\infty(\nu_2))(\mu) & \leq (f^\infty(\nu_1) \circ f^\infty(\nu_2))(f^-(\nu_1) \circ f^-(\nu_2)) \\
& \leq f^\infty(\nu_1)(f^-(\nu_1)) \circ f^\infty(\nu_2)(f^-(\nu_2)) \\
& = f^-(\nu_1)(f^-(\nu_1)) \circ f^-(\nu_2)(f^-(\nu_2)) \\
& \leq f^-(\nu_1)(\nu_1) \circ f^-(\nu_2)(\nu_2).
\end{align*}
\]

Thus, $\left(f^\infty(\psi_1) \circ f^\infty(\psi_2)\right)(\mu)(x) \leq \psi_1(\nu_1)(f(x)) \circ \psi_2(\nu_2)(f(x))$. It is a contradiction. Hence $f^\infty(\psi_1) \circ f^\infty(\psi_2) \leq f^\infty(\psi_1 \circ \psi_2)$.

Suppose there exist $\rho \in L^X$ and $x \in X$ such that

\[
\begin{align*}
(f^\infty(\psi_1) \circ f^\infty(\psi_2))(\rho)(x) & \leq f^\infty(\psi_1 \circ \psi_2)(\rho)(x) \\
= (\psi_1 \circ \psi_2)(f^-(\rho))(f(x)).
\end{align*}
\]

Then there exist $p \in L^X$ with $\rho = p_1 \circ p_2$ such that

\[
\begin{align*}
(f^\infty(\psi_1)(p_1) \circ f^\infty(\psi_2)(p_2))(x) & \leq f^\infty(\psi_1 \circ \psi_2)(\rho)(x) \\
& \leq (\psi_1 \circ \psi_2)(f^-(p))(f(x)) \\
& \leq (\psi_1 \circ \psi_2)(f^-(p_1) \circ f^-(p_2))(f(x)) \\
& = (\psi_1 f^-(p_1) \circ \psi_2 f^-(p_2))(f(x)).
\end{align*}
\]

Since $f^-(\rho) \leq f^-(p_1) \circ f^-(p_2)$ from (2),

\[
\begin{align*}
f^\infty(\psi_1 \circ \psi_2)(\rho)(x) & \leq (\psi_1 \circ \psi_2)(f^-(p))(f(x)) \\
& \leq (\psi_1 \circ \psi_2)(f^-(p_1) \circ f^-(p_2))(f(x)) \\
& = (\psi_1 f^-(p_1) \circ \psi_2 f^-(p_2))(f(x)).
\end{align*}
\]

It is a contradiction. Thus

\[
f^\infty(\psi_1) \circ f^\infty(\psi_2) \geq f^\infty(\psi_1 \circ \psi_2).
\]

We will show $f^\infty(\phi_1) \circ f^\infty(\phi_2) \geq f^\infty(\phi_1 \circ \phi_2)$ from:

\[
\begin{align*}
f^\infty(\phi_1) \circ f^\infty(\phi_2)(\mu) & = \Lambda\{f^\infty(\phi_1)(\mu_1) \circ f^\infty(\phi_2)(\mu_2) \mid \mu_1 \circ \mu_2 \} \\
& \geq \Lambda\{f^-(\phi_1)(f^-(\mu_1)) \circ f^-(\phi_2)(f^-(\mu_2)) \mid f^-(\mu) \circ f^-(\mu_2) \} \\
& \geq f^-(\Lambda\{\phi_1(f^-(\mu_1)) \circ \phi_2(f^-(\mu_2)) \mid f^-(\mu) \circ f^-(\mu_2) \}) \\
& \geq f^-(\phi_1 \circ \phi_2)(f^-(\mu)).
\end{align*}
\]

(8) From (6), we have for all $\mu \in L^X$,

\[
\begin{align*}
f^-(f^\infty(\psi)(\mu)) & = f^-(f^-(\psi^{-1})(\mu)) \quad \text{(by (6))} \\
& \leq \psi^{-1}(f^-(\mu)).
\end{align*}
\]

\[\square\]

Example 3.2. Let $X = \{a, b, c\}$ and $Y = \{x, y\}$ be sets and $L = [0, 1]$ an unit interval. Define a binary operation $\circ$ (called Łukasiewicz conjunction) on $[0, 1]$ by

\[x \circ y = \max\{0, x + y - 1\}.\]

Then $([0, 1], \cdot, 0, 1)$ is a stsc-biquantale (ref.[2-4]). Let $\mu, \nu \in [0, 1]^X$ as follows:

\[\mu(a) = 0.7, \mu(b) = 0.5, \mu(c) = 0.8, \nu(a) = 0.6, \nu(b) = 0.9, \nu(c) = 0.7.\]

Then $(\mu \circ \nu)(a) = 0.3, (\mu \circ \nu)(b) = 0.4, (\mu \circ \nu)(c) = 0.5$.

Let $f : X \to Y$ be a function by $f(a) = f(b) = x, f(c) = y$. Then $f^-(\mu)(x) = 0.7, f^-(\mu)(y) = 0.8$ and $f^-(\nu)(x) = 0.9, f^-(\nu)(y) = 0.7$. Thus, $(f^-(\mu) \circ f^-(\nu))(x) = 0.6$ and $(f^-(\mu) \circ f^-(\nu))(y) = 0.5$.

But $f^-(\mu \circ \nu)(x) = 0.4, f^-(\mu \circ \nu)(y) = 0.5$. Hence $f^-(\mu \circ \nu) \neq f^-(\mu) \circ f^-(\nu)$ because $f$ is not injective.

Lemma 3.3. Let $f : X \to Y$ be a function. For each $v, v_1, v_2 \in E(Y \times Y), \phi \in \Omega(Y)$ and $\lambda \in L^X$, we have:

(1) $f^\infty(\Gamma(v)) = f^\infty \circ \Gamma(v) \circ f^-= \Gamma((f \times f)^-(v))$.

(2) $(f \times f)^-(\Lambda(v)) = (f \times f)^-(\Lambda(v))$.

(3) $(f \times f)^-(\psi^*) = (f \times f)^-(\psi^*)^{-1}$.

(4) $(f \times f)^-(v_1 \circ v_2) = (f \times f)^-(v_1) \circ (f \times f)^-(v_2)$.

(5) $(f \times f)^-(v) \circ (f \times f)^-(v) \leq (f \times f)^-(v \circ v)$.

(6) If $v$ is an $\circ$-equivalence relation on $Y$, then $(f \times f)^-(v)$ is an $\circ$-equivalence relation on $X$.

Proof. (1) It is proved from:

\[
\begin{align*}
f^\infty(\Gamma(v))(\lambda)(x) & = f^\infty \circ \Gamma(v) \circ f^-(\lambda)(x) \\
& = \Gamma(v)(f^-(\lambda))(f(x)) \\
& \geq \Lambda_{v \in Y}\{f^-(\lambda)(y) \circ v(y, f(x))\} \\
& \geq \Lambda_{v \in X}\{f^-(\lambda)(f(z)) \circ v(f(z), f(x))\} \\
& = \Lambda_{v \in X}\{f^-(\lambda)(z) \circ (f \times f)^-(v)(z, x)\} \\
& = \Gamma((f \times f)^-(v))(\lambda)(x).
\end{align*}
\]

(2)

\[
\begin{align*}
\Lambda(f^\infty(\phi))(x, y) & = f^\infty(\phi)(1_{x\times y})(y) \\
& = f^-(\phi)(1_{x\times y})(f(x)) \\
& \geq \phi(1_{y \times x})(f(y), f(x)) \\
& = (f \times f)^-(\Lambda(\phi))(x, y).
\end{align*}
\]

(3)

\[
\begin{align*}
\Gamma((f \times f)^-(v^*))(v^*)(\lambda)(x) & = ((f \times f)^-(v^*))(\lambda) \\
& = \Lambda_{v \in X}\{\lambda(y) \circ [(f \times f)^-(v^*)(y, x)]\} \\
& = \Lambda_{v \in X}\{\lambda(y) \circ v^*(f(y), f(x))\} \\
& \geq \Lambda_{v \in X}\{\lambda(y) \circ ((f \times f)^-(v^*)(y, x))\} \\
& = \Gamma(((f \times f)^-(v^*))^*)(\lambda)(x).
\end{align*}
\]
Furthermore, by Lemma 2.10(4), $\Gamma(((f \times f)^{-1}(v))^u) = (f \times f)^{-1}(v)^{-1}$.

(4) It is easily proved.

\begin{align*}
(f \times f)^{-1}(v) &= (f \times f)^{-1}(v)(x_1, x_2) \\
&= \bigvee_{x \in X} (f \times f)^{-1}(v)(x_1, x_2) \\
&= \bigvee_{x \in X} v(f(x_1), f(x_2)) \cup v(f(x_1), f(x_2)) \\
&\leq \bigvee_{y \in Y} v(f(x_1), y) \cup v(y, f(x_2)) \\
&= v \circ (f \times f)(x_1, x_2) \\
&= (f \times f)^{-1}(v)(x_1, x_2).
\end{align*}

(6) We have to check the axioms of Definition 2.7.

(E1) $(f \times f)^{-1}(v)(x, x) = v(f(x), f(x)) = 1$.

(E2) $(f \times f)^{-1}(v) \circ (f \times f)^{-1}(v) \leq (f \times f)^{-1}(v) = (f \times f)^{-1}(v)$.

(E) $(f \times f)^{-1}(v^u) = (((f \times f)^{-1}(v))^u)^u$.

\begin{definition}
Let $(X, \mathcal{U}_1)$ and $(Y, \mathcal{U}_2)$ be Hutton $(L, \otimes)$-uniform spaces. A function $f : (X, \mathcal{U}_1) \rightarrow (Y, \mathcal{U}_2)$ is $H$-uniformly continuous if $f^\infty(\psi) \in \mathcal{U}_1$, for every $\psi \in \mathcal{U}_2$.

(2) Let $(X, \mathcal{D}_1)$ and $(Y, \mathcal{D}_2)$ be $(L, \circ)$-uniform spaces. A function $f : (X, \mathcal{D}_1) \rightarrow (Y, \mathcal{D}_2)$ is uniformly continuous if $(f \times f)^{-1}(v) \in \mathcal{D}_1$, for every $v \in \mathcal{D}_2$.
\end{definition}

\begin{theorem}
Let $(X, \mathcal{U}_1)$, $(Y, \mathcal{U}_2)$ and $(Z, \mathcal{U}_3)$ be Hutton $(L, \otimes)$-uniform spaces. If $f : (X, \mathcal{U}_1) \rightarrow (Y, \mathcal{U}_2)$ and $g : (Y, \mathcal{U}_2) \rightarrow (Z, \mathcal{U}_3)$ are $H$-uniformly continuous, then $g \circ f : (X, \mathcal{U}_1) \rightarrow (Y, \mathcal{U}_3)$ is $H$-uniformly continuous.

(2) Let $(X, \mathcal{D}_1)$, $(Y, \mathcal{D}_2)$ and $(Z, \mathcal{D}_3)$ be $(L, \circ)$-uniform spaces. If $f : (X, \mathcal{D}_1) \rightarrow (Y, \mathcal{D}_2)$ and $g : (Y, \mathcal{D}_2) \rightarrow (Z, \mathcal{D}_3)$ are uniformly continuous, then $g \circ f : (X, \mathcal{D}_1) \rightarrow (Y, \mathcal{D}_3)$ is uniformly continuous.
\end{theorem}

\begin{proof}
(1) Since $f^\infty(g^\infty(\psi)) = (g \circ f)^\infty(\psi)$ for each $\psi \in \mathcal{U}_3$, it is easily proved.

(2) For each $v \in \mathcal{D}_3$, $(g \circ f) \times (g \circ f)^{-1}(v) = (f \times f)^{-1}((g \times g)^{-1}(v)) \in \mathcal{D}_1$.
\end{proof}

\begin{theorem}
Let $(X, \mathcal{D}_1)$ and $(Y, \mathcal{D}_2)$ be $(L, \circ)$-uniform spaces. If $f : (X, \mathcal{D}_1) \rightarrow (Y, \mathcal{D}_2)$ is uniformly continuous, then $f : (X, \mathcal{U}_{D_1}) \rightarrow (Y, \mathcal{U}_{D_2})$ is $H$-uniformly continuous.
\end{theorem}

\begin{proof}
For each $\psi \in \mathcal{U}_{D_2}$, there exists $v \in \mathcal{D}_2$ with $\Gamma(\psi) \leq \psi$. Since $f$ is uniformly continuous, for $v \in \mathcal{D}_2$, $(f \times f)^{-1}(v) \in \mathcal{D}_1$. By Lemma 3.3(1), since

\begin{align*}
\Gamma((f \times f)^{-1}(v)) = f^\infty(\Gamma(\psi)) \leq f^\infty(\psi)
\end{align*}

we have $f^\infty(\psi) \in \mathcal{U}_{D_1}$.
\end{proof}

\begin{theorem}
Let $(X, \mathcal{U}_1)$ and $(Y, \mathcal{U}_2)$ be Hutton $(L, \otimes)$-uniform spaces. A function $f : (X, \mathcal{U}_1) \rightarrow (Y, \mathcal{U}_2)$ is $H$-uniformly continuous if $f^\infty(\psi) \in \mathcal{U}_1$, for every $\psi \in \mathcal{U}_2$.

(2) Let $(X, \mathcal{D}_1)$ and $(Y, \mathcal{D}_2)$ be $(L, \circ)$-uniform spaces. A function $f : (X, \mathcal{D}_1) \rightarrow (Y, \mathcal{D}_2)$ is uniformly continuous if $(f \times f)^{-1}(v) \in \mathcal{D}_1$, for every $v \in \mathcal{D}_2$.
\end{theorem}

\begin{proof}
(1) Since $f^\infty(g^\infty(\psi)) = (g \circ f)^\infty(\psi)$ for each $\psi \in \mathcal{U}_3$, it is easily proved.

(2) For each $v \in \mathcal{D}_3$, $(g \circ f) \times (g \circ f)^{-1}(v) = (f \times f)^{-1}((g \times g)^{-1}(v)) \in \mathcal{D}_1$.
\end{proof}
Proof. (1) For each $u \in D_{U_2}$, there exists $\psi \in U_2$ with $\Lambda(\psi) \leq u$. Since $f$ is $H$-uniformly continuous, for $\psi \in U_2$, $f^\equiv(\psi) \in U_1$. By Lemma 3.2(2), since

$$(f \times f)^{-}(\Lambda(\phi)) = \Lambda(f^\equiv(\phi)) \leq (f \times f)^{-}(\nu)$$

we have $(f \times f)^{-}(\nu) \in D_{U_1}$. Conversely, since $(U_1)_{D_{U_1}} = U_i$ for $i = 1, 2$ from Theorem 2.14, it is easily proved. (2) Since $(D_i)_{U_{D_i}} = D_i$ for $i = 1, 2$ from Theorem 2.14, it is easily proved. \hfill \square

The class of all Hutton $(\Lambda, \otimes)$-uniform spaces and $H$-uniformly continuous maps forms a category, which is denoted by $\mathcal{HUnif}$. Moreover, the class of all $(\Lambda, \circ)$-uniform spaces and uniformly continuous maps forms a category, which is denoted by $\mathcal{Unif}$.

**Theorem 3.9.** Define maps $F : \mathcal{HUnif} \rightarrow \mathcal{Unif}$ and $G : \mathcal{Unif} \rightarrow \mathcal{HUnif}$ by $F(X, U) = (X, D_{U_1})$, $F(f) = f$ and $G(X, D) = (X, U_D)$, $G(g) = g$, respectively. Then $F$ and $G$ are functors and $\mathcal{HUnif}$ and $\mathcal{Unif}$ are isomorphic.

**Proof.** By Theorems 3.6-8, $F$ and $G$ are functors. From Theorem 2.14, $F \circ G(X, D) = (X, D)$ and $G \circ F(X, U) = (X, U)$. So, $\mathcal{HUnif}$ and $\mathcal{Unif}$ are isomorphic. \hfill \square

**Theorem 3.10.** Let $(Y, U)$ be a Hutton $(\Lambda, \otimes)$-uniform space, $X$ a set and $f : X \rightarrow Y$ a function. Define a subset $U''$ of $\Omega(X)$ as follows:

$$U'' = \{\phi \in \Omega(X) \mid \exists \psi \in U, \ f^\equiv(\psi) \leq \phi\}.$$  

Then we have the following properties.

1. The structure $U''$ is the coarsest Hutton $(\Lambda, \otimes)$-uniformity on $X$ for which each $f$ is $H$-uniformly continuous.
2. A map $g : (Z, U_1) \rightarrow (X, U'')$ is $H$-uniformly continuous if $f \circ g : (Z, U_1) \rightarrow (Y, U)$ is $H$-uniformly continuous.

**Proof.** (1) First, we will show that $U''$ is a Hutton $(\Lambda, \otimes)$-uniformity on $X$.

(Q1) Obvious. (Q2) If $\phi_1, \phi_2 \in U''$, there exists $\psi_i \in U$ with $f^\equiv(\psi_i) \leq \phi_i$ for $i = 1, 2$. Since $f^\equiv(\psi_1) \otimes f^\equiv(\psi_2) = f^\equiv(\psi_1 \otimes \psi_2) \leq \phi_1 \otimes \phi_2$ from Lemma 3.1(7), we have $\phi_1 \otimes \phi_2 \in U''$. (Q3) For each $\phi \in U''$, there exists $\psi \in U$ with $f^\equiv(\psi) \leq \phi$. For $\psi \in U$, since $(Y, U)$ is a Hutton $(\Lambda, \otimes)$-uniform space, by (Q3), there exists $\gamma \in U$ with $\Lambda(\gamma) \leq \psi$. By Lemma 3.1(5), since

$$f^\equiv(\gamma) \circ f^\equiv(\gamma) \leq f^\equiv(\gamma \circ \gamma) \leq f^\equiv(\psi) \leq \phi,$$

then $f^\equiv(\gamma) \in U''$.  

(U) For each $\phi \in U''$, there exists $\psi \in U$ with $f^\equiv(\psi) \leq \phi$. For $\psi \in U$, since $(Y, U)$ is a Hutton $(\Lambda, \otimes)$-uniform space, by (U), there exists $\psi^{-1} \in U$. By Lemma 3.1(6), we have

$$f^\equiv(\psi^{-1}) = (f^\equiv(\psi))^{-1} \leq \phi^{-1}.$$  

Thus, $\phi^{-1} \in U''$.

Second, by definition of $U''$, $f^\equiv(\psi) \in U''$, for all $\psi \in U$. Hence $f : (X, U) \rightarrow (Y, U)$ is $H$-uniformly continuous.

Finally, let $f : (X, U_1) \rightarrow (Y, U)$ be $H$-uniformly continuous. For each $\phi \in U''$, there exists $\psi \in U$ with $f^\equiv(\psi) \leq \phi$. Since $f^\equiv(\psi) \in U_1$, then $\phi \in U_1$. Hence $U'' \subseteq U_1$.

(2) Necessity of the composition condition is clear since the composition of $H$-uniformly continuous maps is $H$-uniformly continuous. If $\phi \in U''$, there exists $\psi \in U$ such that $f^\equiv(\psi) \leq \phi$. Since $f \circ g$ is $H$-uniformly continuous, for $\psi \in U$,

$$(f \circ g)^\equiv(\psi) = g^\equiv \circ f^\equiv(\psi) \in U_1.$$  

Since $g^\equiv(\phi) \geq g^\equiv \circ f^\equiv(\psi) \in U_1$, we have $g^\equiv(\phi) \in U_1$. \hfill \square

**Theorem 3.11.** Let $(Y, D)$ be an $(\Lambda, \otimes)$-uniform space, $X$ a set and $f : X \rightarrow Y$ a function. Define a subset $D'$ of $E(X \times X)$ as follows:

$$D' = \{u \in E(X \times X) \mid \exists v \in U, \ (f \times f)^{-}(v) \leq u\}.$$  

Then we have the following properties.

1. The structure $D'$ is the coarsest $(\Lambda, \otimes)$-uniformity on $X$ for which each $f$ is $H$-uniformly continuous.
2. A map $g : (Z, D_1) \rightarrow (X, D')$ is uniformly continuous iff $f \circ g : (Z, D_1) \rightarrow (Y, D)$ is uniformly continuous.
3. $(D_1)_{D_D} = D_D'$.
4. If $(Y, U)$ is a Hutton $(\Lambda, \otimes)$-uniform space, then $D_{U_U} = D_U'$.  

**Proof.** (1) and (2) are similarly proved as in Theorem 3.10. (3) If $\phi \in D_D'$, there exists $u \in D'$ with $\Gamma(u) \leq \phi$. Since $u \in D'$, there exists $v \in D$ such that $(f \times f)^{-}(v) \leq u$. So, $v \in D$ implies $\Gamma(v) \in U_D$. Since $f^\equiv(\Gamma(v)) = \Gamma((f \times f)^{-}(v)) \leq \Gamma(u) \leq \phi$ from Lemma 3.3(1), we have $\phi \in U_{D_D'}$. Hence $D_D' \subseteq U_{D_D'}$.

If $\phi \in U_{D_D'}$, there exists $\psi \in U_D$ with $f^\equiv(\psi) \leq \phi$. Since $\psi \in U_D$, there exists $v \in D$ such that $\Gamma(v) \leq \psi$ and $(f \times f)^{-}(v) \in D_F$. Since

$$\Gamma((f \times f)^{-}(v)) = f^\equiv(\Gamma(v)) \leq f^\equiv(\psi) \leq \phi,$$

then $f^\equiv(\psi) \in U''$. Then $\phi \in D_D'$. Thus, $D_D' = U_{D_D'}$.

(4) If $(Y, U)$ is a Hutton $(\Lambda, \otimes)$-uniform space, then $D_{U_U} = D_U'$.
we have $\phi \in U_D^f$. Hence $U_D^f \supseteq U_D^f$.

(4) If $u \in U_D^f$, there exists $v \in U_U$ with $(f \times f)^{-1}(v) \leq u$. Since $v \in U_U$, there exists $\psi \in U$ such that $\Lambda(\psi) \leq v$. It follows $f^\omega(\psi) \in U^f$. By Lemma 3.3(2), since $\Lambda(f^\omega(\psi)) = (f \times f)^{-1}(\Lambda(\psi) \leq (f \times f)^{-1}(u) \leq u$, we have $u \in D_U^f$. Hence $D_U^f \supseteq U_D^f$.

If $u \in D_U^f$, there exists $\phi \in U_U^f$ with $\Lambda(\phi) \leq u$. Since $\phi \in U_U^f$, there exists $\psi \in U$ such that $f^\omega(\psi) \leq \phi$. Since $\Lambda(\psi) \in U_U$, and

$$(f \times f)^{-1}(\Lambda(\psi)) = (f \times f)^{-1}(\Lambda(f^\omega(\psi))) \leq \Lambda(\phi) \leq u,$$

we have $u \in D_U^f$. Hence $D_U \supseteq U_D^f$.


\[\Box\]

**Theorem 3.12.** Let $w : X \to Y$ be a $\cap$-equivalence relation and $f : X \to Y$ a function. Then $U_{D_w}^f = U_{D_w}^f$ is defined as follows:

$$U_{D_w}^f = \{\psi \in \Omega(Y) \mid \exists n \in N, f^\omega(\Gamma(w^n)) \leq \phi\}.$$ 

**Proof.** From Theorems 2.8 and 2.11, we obtain

$$D_w = \{v \in E(X \times X) \mid \exists n \in N, \Gamma(v^n) \leq u\},$$

$$U_{D_w}^f = \{\psi \in \Omega(Y) \mid \exists n \in N, \Gamma(w^n) \leq \phi\}.$$ 

Since $(f \times f)^{-1}(u)$ is an $\cap$-equivalence relation on $X$ from Lemma 3.3(6), we obtain

$$D_w^f = \{u \in E(X \times X) \mid \exists n \in N, (f \times f)^{-1}(u^n) \leq u\}.$$ 

Since $\Gamma((f \times f)^{-1}(w, w)) = \Gamma((f \times f)^{-1}(w^n)) = f^\omega(\Gamma(w^n))$ from Lemma 3.3(1), we have

$$U_{D_w}^f = \{\phi \in \Omega(X) \mid \exists n \in N, f^\omega(\Gamma(w^n)) \leq \phi\}.$$ 

\[\Box\]

**Example 3.13.** Let $X, Y, f$ and $(L = [0, 1], \cap)$ be defined as in Example 3.4. Let $w \in E(X \times X)$ be an $\cap$-equivalence relation on $X$ as

$$w(x, x) = w(y, y) = w(x, y) = w(y, x) = 1, \quad w(x, z) = w(z, y) = 0.6, \quad w(x, z) = w(z, x) = 0.5.$$ 

Then

$$w^2(x, x) = w^3(y, y) = w^3(z, z) = w^3(x, y) = w^3(\lambda, y) = 1, \quad w^3(y, z) = w^3(z, u) = w^3(x, z) = w^3(z, x) = 0.$$ 

We obtain $D_w, D_{\omega D_w}, D_w^f$ and $U_{D_w}^f$ as follows:

$$D_w = \{v \in E(Y \times Y) \mid w^v \leq u\},$$

$$U_{D_w}^f = \{\psi \in \Omega(Y) \mid \Gamma(\psi) \leq \psi\},$$

$$D_w^f = \{u \in E(X \times X) \mid (f \times f)^{-1}(u) \leq u\},$$

$$U_{D_w}^f = \{\phi \in \Omega(X) \mid f^\omega(\Gamma(w^n)) \leq \phi\}.$$ 

From Theorems 3.9 and 3.10, we can define subspaces in the obvious way.

**Definition 3.14.** Let $A$ be a subset of $X$ and $i : A \to X$ an inclusion function.

(1) Let $(X, U)$ be a Hutton $(L, \cap)$-uniform space. The pair $(A, U_A)$ where $U_A = \{\psi \in \Omega(A) \mid \exists \psi \in U, \quad \Lambda(\psi) \leq \phi\}$ is said to be a subspace of $(X, U)$.

(2) Let $(X, D)$ be an $(L, \cap)$-uniform space. The pair $(A, D_A)$ where $D_A = \{u \in E(A \times A) \mid \exists \psi \in D, \quad (i \times i)^{-1}(u) \leq u\}$ is said to be a subspace of $(X, D)$.

**Example 3.15.** Let $X, Y, f$ and $(L = [0, 1], \cap)$ be defined as in Example 3.4.

(1) Define $\phi \in \Omega(Y)$ as follows:

$$\phi(1_{x_1}) = \phi(1_{x_2}) = 1_{x_1, y} \quad \phi(1_{x_1}) = \phi(1_{x_2})$$

Since

$$\phi \circ \phi(1_{x_1}) = \phi \circ \phi(1_{x_2}) = 1_{x_1, y}, \phi \circ \phi(1_{x_2}) = 1_{x_2, y},$$

by Lemma 2.4(2), $\phi \circ \phi = \phi$. We have $\phi \circ \phi = \phi$ because

$$\phi \circ \phi(1_{x_1}) = \phi \circ \phi(1_{x_2}) = 1_{x_1, y}, \phi \circ \phi(1_{x_2}) = 1_{x_2, y}.$$ 

Since

$$\phi^{-1}(1_{x_1}) = \phi^{-1}(1_{x_2}) = 1_{x_1, y}, \phi^{-1}(1_{x_2}) = 1_{x_2, y},$$

Hence $\phi^{-1} = \phi$. Define $U = \{\psi \in \Omega(X) \mid \phi \leq \psi\}$.

Then $U$ is a Hutton $(L, \cap)$-uniformity on $X$. We obtain $D_U = \{u \in E(X \times X) \mid \Lambda(\phi) \leq u\}$. Since $\phi \circ \phi = \phi$ and $\phi^{-1} = \phi$, by Theorem 2.12(7), $\Lambda(\phi)$ is an $\cap$-equivalence relation such that

$$\Lambda(\phi)(x, y) = \phi(1_{x_1}) = 1_{x_1, y} = 1,$$

$$\Lambda(\phi)(x, z) = 1, \Lambda(\phi)(y, y) = 1, \Lambda(\phi)(y, z) = 0,$$

$$\Lambda(\phi)(x, z) = 0, \Lambda(\phi)(z, y) = 0, \Lambda(\phi)(z, z) = 1.$$ 

Furthermore, $\Lambda(\phi) \circ \Lambda(\phi) = \Lambda(\phi), \Lambda(\phi)^{-1} = \Lambda(\phi)^* = \Lambda(\phi)$ and $\Lambda(\phi) \circ \Lambda(\phi) = \Lambda(\phi) \circ \phi = \Lambda(\phi)$. Hence $D_U$ is an $(L, \cap)$-uniformity on $X$ and $U_{D_U} = U$.

(2) We obtain $f^\omega(\phi) \in \Omega(X)$ as follows:

$$f^\omega(1_{x_1}) = f^\omega(1_{x_2}) = f^\omega(1_{x_1}) = 1_{x_1, c}, \quad f^\omega(1_{x_2}) = 1_{x_2, c}.$$ 

Then

$$U^f = \{\psi \in \Omega(X) \mid f^\omega(\phi) \leq \psi\}$$

is a Hutton $(L, \cap)$-uniformity on $X$. We obtain $D_U^f = \{u \in E(X \times X) \mid \Lambda(f^\omega(\phi)) \leq u\}$ where

$$\Lambda(f^\omega(\phi))(x, y) = \begin{cases} 1 & x \in \{a, b, c\}, \quad y \in \{a, b, c\}, \\ 1 & x = d, \quad y = d, \\ 0 & \text{otherwise}. \end{cases}$$

We obtain $D_U^f = \{u \in E(X \times X) \mid \exists \psi \in D_U, (f \times f)^{-1}(u) \leq u\}$. Since $(f \times f)^{-1}(\Lambda(\phi)) = \Lambda(f^\omega(\phi))$, we have $D_U = D_U^f$. 

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