Some Properties and Theorems on Intuitionistic Fuzzy Metric Space

Jong Seo Park*

*Department of Mathematic Education, Chinju National University of Education, Jinju 660-756, South Korea

Abstract

In this paper, we introduce and formulate the definitions of Banach operator type $k$ and Banach operator pair on intuitionistic fuzzy metric space. Thereafter we prove some properties and theorems on intuitionistic fuzzy metric space.

Key words: Intuitionistic fuzzy metric space, f-contraction, Banach operator type $k$, Banach operator pair, fixed point.

1. Introduction

Park et.al.[4] defined the intuitionistic fuzzy metric space, and we studied many contents on intuitionistic fuzzy metric space. Also, many authors([1],[3],[4] etc) studied some properties and theories on intuitionistic fuzzy metric space.

In this paper, we first introduce and formulate the definitions of Banach operator type $k$ and Banach operator pair on intuitionistic fuzzy metric space. Thereafter we prove some properties on intuitionistic fuzzy metric space. These results partially improve and generalize [6].

2. Preliminaries and Properties

Throughout this paper, $N$ denote the set of all positive integers. Now, we begin with some definitions, properties in intuitionistic fuzzy metric space as following:

Let us recall(see [5]) that a continuous $t$--norm is an operation $*$ : $[0,1] \times [0,1] \rightarrow [0,1]$ which satisfies the following conditions: (a)$*$ is commutative and associative, (b)$*$ is continuous, (c)$a*1 = a$ for all $a \in [0,1]$, (d)$a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$, $a,b,c,d \in [0,1]$). Also, a continuous $t$--conorm is an operation $\circ : [0,1] \times [0,1] \rightarrow [0,1]$ which satisfies the following conditions: (a)$\circ$ is commutative and associative, (b)$\circ$ is continuous, (c)$a*0 = a$ for all $a \in [0,1]$, (d)$a*c \geq b*d$ whenever $a \leq c$ and $b \leq d$, $a,b,c,d \in [0,1]$. Also, let us recall (see [2]) that the following conditions are satisfied: (a)For any $r_1, r_2 \in (0,1)$ with $r_1 > r_2$, there exist $r_3, r_4 \in (0,1)$ such that $r_1*r_3 \geq r_2$ and $r_4* r_2 \leq r_3$; (b)For any $r_5 \in (0,1)$, there exist $r_6, r_7 \in (0,1)$ such that $r_6 * r_5 \geq r_5$ and $r_6 r_7 \leq r_5$.

Definition 2.1. ([1]) The 5--tuple $(X, M, N, *, \circ)$ is said to be an intuitionistic fuzzy metric space if $X$ is an arbitrary set, $*$ is a continuous $t$--norm, $\circ$ is a continuous $t$--conorm and $M, N$ are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions; for all $x, y, z \in X$,

(a)$M(x, y, t) > 0$,
(b)$M(x, y, t) = 1$ if and only if $x = y$,
(c)$M(x, y, t) = M(y, x, t)$,
(d)$M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s)$,
(e)$M(x, y, \cdot) : (0, \infty) \rightarrow [0,1]$ is continuous,
(f)$N(x, y, t) > 0$,
(g)$N(x, y, t) = 0$ if and only if $x = y$,
(h)$N(x, y, t) = N(y, x, t)$,
(i)$N(x, y, t) \circ N(y, z, s) \geq N(x, z, t + s)$,
(j)$N(x, y, \cdot) : (0, \infty) \rightarrow [0,1]$ is continuous.

Note that $(M, N)$ is called an intuitionistic fuzzy metric on $X$. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between $x$ and $y$ with respect to $t$, respectively.

Lemma 2.2. ([3]) Let $X$ be an intuitionistic fuzzy metric space. If there exists a number $k \in (0,1)$ such that for all $x, y \in X$ and $t > 0$, $M(x, y, kt) \geq M(x, y, t)$, $N(x, y, kt) \leq N(x, y, t)$, then $x = y$.

Definition 2.3. Let $X$ be an intuitionistic fuzzy metric space.

(a) A self mapping $T$ on $X$ is said to be f-contraction if there exists a real number $0 < k < 1$ such that

$$M(Tx, Ty, kt) \geq M(fx, fy, t),$$

$$N(Tx, Ty, kt) \leq M(fx, fy, t).$$

Manuscript received Apr. 14, 2008; revised Jun. 1, 2008.

*Corresponding Author: Jong Seo Park, parkjs@cue.ac.kr

**This paper is supported by the Chinju National University of Education Research Fund in 2009
for all $x, y \in X$. If $k = 1$, then $T$ is said to be $\xi$-nonexpansive.

(b) A mapping $T$ on $X$ is said to be asymptotically $\xi$-nonexpansive if there exists a sequence $\{\mu_n\}$ of real numbers with $\mu_n \geq 1$ and $\lim_{n \to \infty} \mu_n = 1$ such that

$$M(T^n x, T^n y, \mu_n) \geq M(f x, f y, t),$$

$$N(T^n x, T^n y, \mu_n) \leq N(f x, f y, t)$$

for all $x, y \in X$ and $n = 1, 2, 3, \cdots, \infty$.

(c) $T$ is said to be uniformly asymptotically regular on $X$ if for each $r > 0$, there exists $N(e) = N$ such that

$$M(T^n x, T^n y, \mu_n) > 1 - r,$$

$$N(T^n x, T^n y, \mu_n) < r$$

for all $n \geq N$ and $x \in X$.

(d) Two self mappings $T$ and $f$ on $X$ are said to be commuting if $T f x = f T x$ for all $x \in X$.

**Definition 2.5.** Let $T$ be a self mapping of an intuitionistic fuzzy metric space $X$. Then $T$ is called a Banach operator of type $k$ if

$$M(T^{n+1} x, T^{n+1} y, \mu_n t) \geq M(T^n x, T^n y, t),$$

$$N(T^{n+1} x, T^{n+1} y, \mu_n t) \leq N(T^n x, T^n y, t)$$

for some $k \geq 0$ and for all $x \in X$.

**Proposition 2.6.** If $T$ and $f$ are two self mappings of an intuitionistic fuzzy metric space $X$, then $(T, f)$ is a Banach operator pair if any one of the following conditions is satisfied

(a) $T(F(f)) \subseteq F(f)$ (the set of fixed points of $f$),

(b) $T f x = T x$ for each $x \in F(f)$,

(c) $f T x = T f x$ for $x \in F(f)$,

(d) $M(T f x, f x, f t) \geq M(f x, x, t), N(T f x, f x, f t) \leq N(f x, x, t)$ for some $k \geq 0$.

**Proposition 2.7.** If $T$ and $f$ are two continuous self mappings of an intuitionistic fuzzy metric space $X$, then $(T, f)$ is a Banach operator pair if and only if for each $x_n$ in $X$ such that $\lim_{n \to \infty} x_n = \lim_{n \to \infty} T x_n = x$, it follows that

$$\lim_{n \to \infty} M(T f x_n, T x_n, k t) = 1$$

and

$$\lim_{n \to \infty} N(T f x_n, T x_n, k t) = 0$$

or

$$\lim_{n \to \infty} M(T f x_n, f T x_n, k t) = 1$$

and

$$\lim_{n \to \infty} N(T f x_n, f T x_n, k t) = 0.$$

**Proof.** Let $T$ and $f$ be two continuous self mappings of an intuitionistic fuzzy metric space $X$. If $(T, f)$ is a Banach operator pair, and for each $\{x_n\} \subset X$ such that $\lim_{n \to \infty} x_n = \lim_{n \to \infty} T x_n = x$, then $f T x_n = T x_n$ from Definition 2.5. Also, By continuity of $T, f$, Proposition 2.6 and $T f x_n = f T x_n$,

$$\lim_{n \to \infty} M(T f x_n, T x_n, k t) = 1$$

and

$$\lim_{n \to \infty} N(T f x_n, T x_n, k t) = 0.$$

Conversely, for each $\{x_n\} \subset X$ such that $\lim_{n \to \infty} x_n = \lim_{n \to \infty} T x_n = x$, if

$$\lim_{n \to \infty} M(T f x_n, f T x_n, k t) = 1$$

and

$$\lim_{n \to \infty} N(T f x_n, f T x_n, k t) = 0,$$

then from Definition 2.5, $(T, f)$ is a Banach operator pair.

**3. Some Results**

Now, we prove some fixed point theorems satisfying some conditions on intuitionistic fuzzy metric space.

**Theorem 3.1.** Let $Y$ be a nonempty closed subset of an intuitionistic fuzzy metric space $X$ with $f s t t \geq t, t o t \leq t$ for all $t \in [0, 1]$ and let $f, T : Y \to Y$ be commuting self mappings on $Y - \{q\}$ for some $q \in X$ such that
Some Properties and Theorems on Intuitionistic Fuzzy Metric Space

\( T(Y - \{ q \}) \subset f(Y) - \{ q \} \). Suppose that there exists \( k \in (0, 1) \) such that

\[
M(Tx, Ty, kt) \\
\geq \min\{M(fx, fx, t), M(fx, Tx, t), M(fy, Ty, t), M(fy, Tx, t)\},
\]

\[
N(Tx, Ty, kt) \\
\leq \max\{N(fx, fy, t), N(fx, Tx, t), N(fy, Ty, t), N(fy, Tx, t)\}
\]

for all \( x, y \in Y \). Further, if \( T \) is continuous and \( (T(Y - \{ q \})) \) is complete, then \( F(f) \cap F(T) \) has a unique point in \( Y \).

Proof. Let \( x_0 \in Y \). Since \( T(Y - \{ q \}) \subset f(Y) - \{ q \} \), define a sequence \( \{ x_n \} \subset Y \) as \( f x_n = T x_{n-1} \) for each \( n \geq 1 \). Then we have

\[
M(f x_{n+1}, f x_n, kt) \\
= M(T x_n, T x_{n+1}, kt) \\
\geq \min\{M(f x_n, f x_{n-1}, t), M(f x_n, T x_n, t), M(f x_{n-1}, T x_n, t), M(f x_{n-1}, T x_{n-1}, t)\},
\]

\[
N(f x_{n+1}, f x_n, kt) \\
\leq \max\{N(f x_n, f x_{n-1}, t), N(f x_n, T x_n, t), N(f x_{n-1}, T x_n, t), N(f x_{n-1}, T x_{n-1}, t)\}
\]

for all \( n \in \mathbb{N} \). Therefore \( \{ x_n \} \) is a Cauchy sequence in \( Y \). So, \( \{ x_n \} \) is a Cauchy sequence in \( Y \) and since \( T(Y - \{ q \}) \) is complete, \( \lim_{n \to \infty} x_n = y \in Y \) and consequently, \( \lim_{n \to \infty} f x_n = y \). Since \( T \) and \( f \) are commuting on \( Y - \{ q \}, T f x_n = f T x_n \). As \( T \) is continuous, \( \lim_{n \to \infty} f T x_n = \lim_{n \to \infty} f x_n = Ty \). Now

\[
M(T x_n, TT x_n, kt) \\
\geq \min\{M(f x_n, f T x_n, t), M(f x_n, T x_n, t), M(f T x_n, T x_n, t), M(f T x_n, TT x_n, t)\},
\]

\[
N(T x_n, TT x_n, kt) \\
\leq \max\{N(f x_n, f T x_n, t), N(f x_n, T x_n, t), N(f T x_n, T x_n, t), N(f T x_n, TT x_n, t)\}
\]

Taking the limit as \( n \to \infty \) in above equation, we obtain

\[
M(y, Ty, kt) \\
\geq \min\{M(y, Ty, t), M(y, Ty, t), M(Ty, Ty, t), M(Ty, Ty, t)\},
\]

\[
N(y, Ty, kt) \\
\leq \max\{N(y, Ty, t), N(y, Ty, t), N(Ty, Ty, t), N(Ty, Ty, t)\}.
\]

Thus since \( a \ast a \geq a \) and \( a \circ a \leq a \) for all \( a \in [0, 1], M(y, Ty, kt) \geq M(y, Ty, t), N(y, Ty, kt) \leq N(y, Ty, t) \). Thus by Lemma 2.2, \( y = Ty \in T(Y) \) and \( T(Y) \subset f(Y) \), there exists \( z \in Y \) such that \( y = Ty = fz \).

Now, we prove that \( Tz = fz \). Since

\[
M(TT x_n, Tz, kt) \\
\geq \min\{M(TT x_n, fz, t), M(TT x_n, TT x_n, t), M(fz, Tz, t), M(TT x_n, Tz, t)\},
\]

\[
N(TT x_n, Tz, kt) \\
\leq \max\{N(TT x_n, fz, t), N(TT x_n, TT x_n, t), N(fz, Tz, t), N(fz, Tz, t)\}.
\]

Taking the limit as \( n \to \infty \) in above equation, we obtain

\[
M(Ty, Tz, kt) \\
\geq \min\{M(Ty, fz, t), M(Ty, Tz, t), M(fz, Tz, t), M(Ty, Tz, t)\},
\]

\[
N(Ty, Tz, kt) \\
\leq \max\{N(Ty, fz, t), N(Ty, Tz, t), N(fz, Tz, t), N(fz, Tz, t)\}.
\]

Since \( y = Ty \) and \( a \ast a \geq a \) and \( a \circ a \leq a \) for all \( a \in [0, 1], \) therefore

\[
M(y, Tz, kt) \geq M(Ty, Tz, t), \quad N(y, Tz, kt) \leq N(y, Tz, t).
\]

From Lemma 2.2, \( y = Tz \). Hence \( y = Tz = Ty = fz \). Also, since \( T f z = f T z, y = Ty = f y \). That is, \( y \) is a unique point of \( F(f) \cap F(T) \).

\[\square\]

**Corollary 3.2.** Let \( T \) and \( f \) be two self mappings of a nonempty closed subset \( Y \) of an intuitionistic fuzzy metric space \( X \) with \( t \ast t \geq t, t \circ t \leq t \) for all \( t \in [0, 1] \) such that \( T(Y - \{ q \}) \) is complete for some \( q \in X \). Suppose that \( (T, f) \) is a Banach operator pair on \( Y - \{ q \} \) satisfying inequality (1) for all \( x, y \in Y \) and \( k \in [0, 1] \). If \( f \) is continuous and \( F(f) \) is nonempty, then there is a unique common fixed point of \( T \) and \( f \).
Proof. Since $F(f)$ is the fixed point set of $f$, $f(F(f)) = F(f)$. Also, since $(T,f)$ is a Banach operator pair on $Y - \{q\}$, $T(F(f) - \{q\}) \subseteq F(f) - \{q\}$ and $T(F(f)) \subseteq f(F(f))$. Also, $T(F(f) - \{q\})$ is complete. Furthermore, since $(T,f)$ satisfies inequality (1) for all $x, y \in Y$ and by Theorem 3.1, $T$ and $f$ have a unique common fixed point $z$ in $F(f)$.

References


Jong Seo Park
Professor of Chinju National University of Education
Research Area: Fuzzy mathematics, Fuzzy fixed point theory, Fuzzy differential equation
E-mail : parkjs@cue.ac.kr