On Fuzzy $\alpha$-Weakly $r$-Continuous Mappings

Won Keun Min

Department of Mathematics, Kangwon National University, Chuncheon, 200-701, Korea

Abstract

In this paper, we introduce the concept of fuzzy $\alpha$-weakly $r$-continuous mapping on a fuzzy topological space and investigate some properties of such a mapping and the relationships among fuzzy $\alpha$-weakly $r$-continuity, fuzzy $r$-continuity and fuzzy weakly $r$-continuity.

Key words: fuzzy $\alpha$-weakly $r$-continuous, fuzzy weakly $r$-semicontinuous, fuzzy $S$-weakly $r$-continuous, fuzzy weakly $r$-continuous

1. Introduction

The concept of fuzzy set was introduced by Zadeh [9]. Chang [1] defined a fuzzy topological space using fuzzy sets. In [3, 4], Chattopadhyay, Hazra and Samanta introduced the concept of smooth fuzzy topological space which is a generalization of the fuzzy topological space. Lee and Lee [8] introduced the concepts of fuzzy strongly $r$-semiopen and fuzzy strongly $r$-semicontinuous mappings in fuzzy topological spaces defined by Chattopadhyay. These concepts are generalizations of fuzzy strongly $r$-preopen sets. In this paper, we introduce the concept of fuzzy $\alpha$-weakly $r$-continuous mapping as a generalization of the fuzzy strongly $r$-semicontinuous mapping and study some properties of the mapping and the relationships among fuzzy $\alpha$-weakly $r$-continuity, fuzzy $r$-continuity and fuzzy weakly $r$-continuity.

2. Preliminaries

Let $I$ be the unit interval $[0, 1]$ of the real line. A member $\mu$ of $I^X$ is called a fuzzy set of $X$. By $0$ and $1$ we denote constant maps on $X$ with value 0 and 1, respectively. For any $\mu \in I^X$, $\mu^c$ denotes the complement $1 - \mu$. All other notations are standard notations of fuzzy set theory.

An fuzzy point $x_{\alpha}$ in $X$ is a fuzzy set $x_{\alpha}$ defined by

$$x_{\alpha}(y) = \begin{cases} \alpha, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

A fuzzy point $x_{\alpha}$ is said to belong to an fuzzy set $A$ in $X$, denoted by $x_{\alpha} \in A$, if $\alpha \leq A$ for $x \in X$.

A fuzzy set $A$ in $X$ is the union of all fuzzy points which belong to $A$.

Let $f : X \rightarrow Y$ be a mapping and $\alpha \in I^X$ and $\beta \in I^Y$.

Then $f(\alpha)$ is a fuzzy set in $Y$, defined by

$$f(\alpha)(y) = \begin{cases} \sup_{z \in f^{-1}(y)} \alpha(z), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases}$$

for $y \in Y$.

$f^{-1}(\beta)$ is a fuzzy set in $X$, defined by $f^{-1}(\beta)(x) = \beta(f(x))$, $x \in X$.

A fuzzy topology [3, 4] on $X$ is a map $T : I^X \rightarrow I$ which satisfies the following properties:

1. $T(\emptyset) = T(1) = 1$.
2. $T(\mu_1 \wedge \mu_2) \geq T(\mu_1) \wedge T(\mu_2)$ for $\mu_1, \mu_2 \in I^X$.
3. $T(\mu_1) \geq T(\mu_2)$ for $\mu_1 \in I^X$.

The pair $(X, T)$ is called a fuzzy topological space. And $\mu \in I^X$ is said to be fuzzy $r$-open (resp., $r$-closed) if $T(\mu) \geq r$ (resp., $T(\mu^c) \geq r$).

Let $A$ be a fuzzy set in an FTS $(X, T)$ and $r \in (0, 1] = I_0$.

The $r$-closure of $A$, denoted by $cl(A, r)$, is defined as $cl(A, r) = \bigcap \{B \in I^X : A \subseteq B \text{ and } B \text{ is fuzzy } r\text{-closed}\}$.

The $r$-interior of $A$, denoted by $int(A, r)$, is defined as $int(A, r) = \cup \{B \in I^X : B \subseteq A \text{ and } B \text{ is fuzzy } r\text{-open}\}$.

Definition 2.1 ([5, 6, 7]). Let $A$ be a fuzzy set in an FTS $(X, T)$ and $r \in (0, 1] = I_0$. Then $A$ is said to be

1. fuzzy $r$-semiopen if there is a fuzzy $r$-open set $B$ in $X$ such that $B \subseteq A \subseteq cl(B, r)$,
2. fuzzy $r$-preopen if $A \subseteq int(cl(A, r), r)$,
3. fuzzy $r$-regular open if $A = int(cl(A, r), r)$,
4. fuzzy $r$-strong semiopen if $A \subseteq int(cl(A, r), r), r)$.

Let $A$ be a fuzzy set in an FTS $(X, T)$ and $r \in I_0$.

The fuzzy $r$-strong semi-closure and the fuzzy $r$-strong semi-interior of $A$, denoted by $sscl(A, r)$ and $ssint(A, r)$, respectively, are defined as
### 3. Main Results

#### Definition 2.2 ([6, 7, 8]). Let \( f : X \to Y \) be a mapping from FTS’s \((X, T)\) and \((Y, U)\). Then \( f \) is said to be

1. fuzzy \( r \)-continuous if for each fuzzy \( r \)-open set \( B \) of \( Y \), \( f^{-1}(B) \) is a fuzzy \( r \)-open set in \( X \),
2. fuzzy almost \( r \)-continuous if for each fuzzy \( r \)-open set \( B \) of \( Y \), \( f^{-1}(B) \) is a fuzzy \( r \)-regular open set in \( X \),
3. fuzzy \( r \)-semicontinuous if for each fuzzy \( r \)-open set \( B \) of \( Y \), \( f^{-1}(B) \) is a fuzzy \( r \)-semiopen set in \( X \),
4. fuzzy strongly \( r \)-semiopen if for each fuzzy \( r \)-open set \( B \) of \( Y \), \( f^{-1}(B) \) is a fuzzy strongly \( r \)-semiopen set in \( X \),
5. fuzzy weakly \( r \)-continuous if for each fuzzy \( r \)-open set \( B \) of \( Y \), \( f^{-1}(B) \) is \( \text{int}(f^{-1}(cl(B,r)),r) \),

#### Remark 3.2. Every fuzzy strongly \( r \)-semiopen is fuzzy \( \alpha \)-weakly \( r \)-continuous but the converse is not always true.

#### Example 3.3. Let \( X = I \) and let \( \beta \) and \( \mu \) be fuzzy sets of \( X \) defined as

\[
\beta(x) = -\frac{1}{3}x + \frac{2}{3}, \text{ for } x \in I, \quad \mu(x) = \frac{1}{3}x, \text{ for } x \in I.
\]

Define a fuzzy topology \( T : I^X \to I \) by

\[
T(\sigma) = \begin{cases} 
1, & \text{if } \sigma = 0, 1, \\
\frac{1}{2}, & \text{if } \sigma = \beta, \\
0, & \text{otherwise};
\end{cases}
\]

and a fuzzy topology \( U : I^X \to I \) by

\[
U(\sigma) = \begin{cases} 
1, & \text{if } \sigma = 0, 1, \\
\frac{1}{2}, & \text{if } \sigma = \mu, \\
0, & \text{otherwise}.
\end{cases}
\]

Note that

\[
\text{int}(cl(int(f^{-1}(\mu),\frac{1}{2})),\frac{1}{2})) = 0;
\]

\[
\text{ssint}(f^{-1}(cl(\mu,\frac{1}{2})),\frac{1}{2}) = \mu^c.
\]

Hence the identity mapping \( f : (X, T) \to (X, U) \) is a fuzzy \( \alpha \)-weakly \( \frac{1}{2} \)-continuous mapping but it is not fuzzy strongly \( \frac{1}{2} \)-semicontinuous.

#### Theorem 3.4. Let \( f : (X, T) \to (Y, U) \) be a mapping on FTS’s \((X, T)\) and \((Y, U)\) \((r \in I_0)\). Then \( f \) is a fuzzy \( \alpha \)-weakly \( r \)-continuous mapping if and only if for every fuzzy point \( x_0 \) and each fuzzy \( r \)-open set \( V \) containing \( f(x_0) \), there exists a fuzzy \( r \)-strong semiopen set \( U \) containing \( x_0 \) such that \( f(U) \subseteq cl(V,r) \).

**Proof.** Suppose \( f \) is a fuzzy \( \alpha \)-weakly \( r \)-continuous mapping. Let \( x_0 \) be a fuzzy point in \( X \) and \( V \) a fuzzy \( r \)-strong semiopen set containing \( f(x_0) \). Then there exists a fuzzy \( r \)-open set \( B \) such that \( f(x_0) \in B \subseteq V \). Since \( f \) is a fuzzy \( \alpha \)-weakly \( r \)-continuous mapping,

\[
f^{-1}(B) \subseteq \text{ssint}(f^{-1}(cl(B,r)),r) \subseteq \text{ssint}(f^{-1}(cl(V,r)),r).
\]

Set \( U = \text{ssint}(f^{-1}(cl(V,r)),r) \). Then \( U \) is a fuzzy \( r \)-strong semiopen set such that \( f^{-1}(B) \subseteq U \). So \( f(U) \subseteq cl(V,r) \).

For the converse, let \( V \) be a fuzzy \( r \)-open set in \( Y \). For each \( x_0 \in f^{-1}(V) \), by hypothesis, there exists a fuzzy \( r \)-strong semiopen set \( U_{x_0} \) containing \( x_0 \) such that \( f(U_{x_0}) \subseteq cl(V,r) \). Now we can say for each \( x_0 \in f^{-1}(V) \), \( x_0 \in U_{x_0} \subseteq f^{-1}(cl(V,r)) \).

Thus \( \cup \{U_{x_0} : x_0 \in f^{-1}(V)\} \subseteq f^{-1}(cl(V,r)) \). Since \( \cup \{U_{x_0} : x_0 \in f^{-1}(V)\} \) is fuzzy \( r \)-strong semiopen, we have \( f^{-1}(V) \subseteq \text{ssint}(f^{-1}(cl(V,r)),r) \).

#### Theorem 3.5. Let \( f : (X, T) \to (Y, U) \) be a mapping on FTS’s \((X, T)\) and \((Y, U)\) \((r \in I_0)\). Then the following statements are equivalent:

1. \( f \) is fuzzy \( \alpha \)-weakly \( r \)-continuous.
2. \( \text{sscl}(f^{-1}(\text{int}(F,r)),r) \subseteq f^{-1}(F) \) for each fuzzy \( r \)-closed set \( F \) in \( Y \).
3. \( \text{sscl}(f^{-1}(\text{int}(cl(B,r)),r)) \subseteq f^{-1}(cl(B,r)) \) for each fuzzy set \( B \) in \( Y \).
4. \( f^{-1}(\text{int}(B,r)) \subseteq \text{ssint}(f^{-1}(\text{int}(cl(B,r)),r),r) \) for each fuzzy set \( B \) in \( Y \).
5. \( \text{sscl}(f^{-1}(V,r)) \subseteq f^{-1}(cl(V,r)) \) for a fuzzy \( r \)-open set \( V \) in \( Y \).

**Proof.** (1) \(\Rightarrow\) (2) Let \( F \) be any fuzzy \( r \)-closed set of \( Y \). Then since \( \tilde{I} - F \) is a fuzzy \( r \)-open set in \( Y \), from (1), it follows

\[
f^{-1}(\tilde{I} - F) \subseteq \text{ssint}(f^{-1}(cl(\tilde{I} - int(F,r)),r),r) = \text{ssint}(f^{-1}(\tilde{I} - int(F,r)),r) = \text{ssint}(\tilde{I} - f^{-1}(int(F,r)),r) = \tilde{I} - \text{sscl}(f^{-1}(int(F,r)),r).
\]
Hence we have $sscl(f^{-1}(int(F,r)),r) \subseteq f^{-1}(F)$.

(2) $\Rightarrow$ (3) Let $B \in I^r$. Since $cl(B,r)$ is a fuzzy $r$-closed set in $Y$, by (2), $sscl(f^{-1}(int(cl(B,r),r)) \subseteq f^{-1}(cl(B,r))$.

(3) $\Rightarrow$ (4) For $B \in I^r$, from (3), it follows

\[
f^{-1}(int(B,r)) = \overline{1} - f^{-1}(cl(\overline{1} - B,r))
\subseteq \overline{1} - sscl(f^{-1}(int(\overline{1} - B,r),r),r)
= ssint(f^{-1}(cl(int(B,r),r),r)).
\]

Hence we have

\[
f^{-1}(int(B,r)) \subseteq sscl(f^{-1}(cl(int(B,r),r),r)).
\]

(4) $\Rightarrow$ (5) Let $V$ be any fuzzy $r$-open set of $Y$. Then from (4) and $(V,r) \subseteq int(cl(V,r),r)$, it follows

\[
\begin{aligned}
&\overline{1} - f^{-1}(cl(V,r))
= f^{-1}(\overline{1} - cl(V,r))
\subseteq sscl(f^{-1}(cl(int(\overline{1} - V,r),r),r)
= ssint(\overline{1} - f^{-1}(cl(int(V,r),r),r))
\subseteq \overline{1} - sscl(f^{-1}(cl(V,r),r)).
\end{aligned}
\]

Hence $sscl(f^{-1}(V),r) \subseteq f^{-1}(cl(V,r))$.

(5) $\Rightarrow$ (1) Let $V$ be a fuzzy $r$-open set in $Y$. From $(V,r) \subseteq int(cl(V,r))$ and (5), we have

\[
\begin{aligned}
f^{-1}(V) &\subseteq f^{-1}(int(cl(V,r),r))
= \overline{1} - f^{-1}(cl(\overline{1} - cl(V,r),r))
\subseteq sscl(f^{-1}(cl(V,r),r),r)
= ssint(f^{-1}(cl(V,r),r)).
\end{aligned}
\]

Hence $f$ is a fuzzy $\alpha$-weakly $r$-continuous mapping.

\[\square\]

\textbf{Theorem 3.6.} Let $f : (X,T) \rightarrow (Y,U)$ be a mapping on FTS's $(X,T)$ and $(Y,U)$ ($r \in I_0$). Then $f$ is fuzzy $\alpha$-weakly $r$-continuous if and only if $sscl(f^{-1}(int(cl(V,r),r)),r) \subseteq f^{-1}(cl(V,r))$ for each fuzzy $r$-preopen set $V$.

\textit{Proof.} Assume $f$ is fuzzy $\alpha$-weakly $r$-continuous. Let $V$ be a fuzzy $r$-preopen of $Y$. Then $V \subseteq int(cl(V,r),r)$. Set $A = int(cl(V,r),r)$. Since $A$ is a fuzzy $r$-open set, by Theorem 3.5 (5), we have

\[
sscl(f^{-1}(int(cl(A,r),r),r),s) \subseteq f^{-1}(cl(A,r)).
\]

From $cl(A,r) = cl(V,r)$, it follows

\[
sscl(f^{-1}(int(cl(V,r),r)),r) \subseteq f^{-1}(cl(V,r)).
\]

For the converse, let $G$ be a fuzzy $r$-open set of $Y$. Then since $G$ is a fuzzy $r$-preopen set, we have

\[
sscl(f^{-1}(G),r) \subseteq sscl(f^{-1}(int(cl(G,r),r)),r) \subseteq f^{-1}(cl(G,r)).
\]

Hence, by Theorem 3.5 (5), $f$ is a fuzzy $\alpha$-weakly $r$-continuous mapping.

\[\square\]

\textbf{Theorem 3.7.} Let $f : (X,T) \rightarrow (Y,U)$ be a mapping on FTS's $(X,T)$ and $(Y,U)$ ($r \in I_0$). Then $f$ is fuzzy $\alpha$-weakly $r$-continuous if and only if $sscl(f^{-1}(int(cl(G,r),r)),r) \subseteq f^{-1}(cl(G,r))$ for each fuzzy $r$-semiopen set $G$ in $Y$.

\textit{Proof.} Assume $f$ is fuzzy $\alpha$-weakly $r$-continuous. Let $V$ be a fuzzy $r$-semiopen in $Y$. Then $V \subseteq cl(int(V,r),r)$. Set $F = cl(int(V,r),r)$. Since $F$ is a fuzzy $r$-closed set, by Theorem 3.5 (2), we have

\[
sscl(f^{-1}(int(F,r),r)) \subseteq f^{-1}(F).
\]

From $cl(V,r) = cl(int(V,r),r) = F$, it follows

\[
sscl(f^{-1}(int(cl(F,r),r),r)) \subseteq f^{-1}(cl(F,r)).
\]

For the converse, let $G$ be a fuzzy $r$-open set of $Y$. Then since $G$ is a fuzzy $r$-semiopen set, by hypothesis and Theorem 3.5 (5), $f$ is a fuzzy $\alpha$-weakly $r$-continuous mapping.

\[\square\]

\textbf{Theorem 3.8} ([8]). Let $f : (X,T) \rightarrow (Y,U)$ be a mapping on FTS's $(X,T)$ and $(Y,U)$ ($r \in I_0$). Then $f$ is fuzzy strongly $r$-semicontinuous if and only if $cl(int(f^{-1}(cl(G,r)),r),r),r) \subseteq f^{-1}(cl(F,r))$ for each fuzzy set $G$ in $Y$.

\textbf{Theorem 3.9.} Let $f : (X,T) \rightarrow (Y,U)$ be a mapping on FTS's $(X,T)$ and $(Y,U)$ ($r \in I_0$). Then if $f$ is fuzzy strongly $r$-semicontinuous, then it is fuzzy weakly $r$-continuous.

\textit{Proof.} Let $B$ be fuzzy $r$-open in $Y$. Since $f$ is fuzzy strongly $r$-semicontinuous, $f^{-1}(B)$ is fuzzy strongly $r$-semiopen and $f^{-1}(cl(B,r))$ is fuzzy strongly $r$-semiclosed in $X$. Thus from Theorem 3.8, it follows

\[
f^{-1}(B) \subseteq int(cl(f^{-1}(B),r),r)
\subseteq cl(int(cl(f^{-1}(cl(B),r),r),r),r)
\subseteq f^{-1}(cl(B,r)).
\]

This implies $f^{-1}(B) \subseteq int(f^{-1}(cl(B,r),r))$. Hence $f$ is fuzzy weakly $r$-continuous.

\[\square\]

\textbf{Example 3.10.} Let $X = I$ and let $A$ and $B$ be fuzzy sets defined as follows

\[
A(x) = \frac{1}{4}x + \frac{3}{4}, \text{ for all } x \in I;
\]

\[
B(x) = \frac{1}{4}x + \frac{1}{4}, \text{ for all } x \in I.
\]

Define fuzzy topologies $T_1$ and $T_2$ on $X$ as follows.
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Finally we get the following implications:

fuzzy $r$–continuous $\Rightarrow$ fuzzy strongly $r$–semicontinuous $\Rightarrow$ fuzzy weakly $r$–continuous $\Rightarrow$ fuzzy $\alpha$-weakly $r$–continuous

References


Won Keun Min
Professor of Kangwon National University
Research Area: Fuzzy topology, General topology
E-mail: wkmin@kangwon.ac.kr