Fuzzy $r$-Compactness on Fuzzy $r$-Minimal Spaces

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Abstract

In [8], we introduced the concept of fuzzy $r$-minimal structure which is an extension of smooth fuzzy topological spaces and fuzzy topological spaces in Chang’s sense. And we also introduced and studied the fuzzy $r$-$M$ continuity. In this paper, we introduce the concepts of fuzzy $r$-minimal compactness, almost fuzzy $r$-minimal compactness and nearly fuzzy $r$-minimal compactness on fuzzy $r$-minimal spaces and investigate the relationships between fuzzy $r$-$M$ continuous mappings and such types of fuzzy $r$-minimal compactness.

Keywords: fuzzy $r$-minimal spaces, fuzzy $r$-$M$ open mapping, fuzzy $r$-$M$ continuous, fuzzy $r$-minimal compact, almost fuzzy $r$-minimal compact and nearly fuzzy $r$-minimal compact

1. Introduction

The concept of fuzzy set was introduced by Zadeh [9]. Chang [2] defined fuzzy topological spaces using fuzzy sets. In [3, 7], Chattopadhyay, Hazra and Samanta introduced smooth fuzzy topological spaces which are a generalization of fuzzy topological spaces.

In [8], we introduced the concept of fuzzy $r$-minimal space which is an extension of the smooth fuzzy topological space. The concepts of fuzzy $r$-open sets, fuzzy $r$-semiopen sets, fuzzy $r$-preopen sets, $r$-fuzzy $\beta$-open sets and fuzzy $r$-regular open sets are introduced in [1, 4, 5, 6], which are a kind of fuzzy $r$-minimal structures. We also introduced and studied the concepts of fuzzy $r$-$M$ continuity, fuzzy $r$-$M$ open maps and fuzzy $r$-$M$ closed maps. In this paper, we introduce the concepts of fuzzy $r$-minimal compactness, almost fuzzy $r$-minimal compactness and nearly fuzzy $r$-minimal compactness on fuzzy $r$-minimal spaces and investigate the relationships between fuzzy $r$-$M$ continuous mappings and such types of fuzzy $r$-minimal compactness.

2. Preliminaries

Let $I$ be the unit interval $[0, 1]$ of the real line. A member $\mu$ of $I^X$ is called a fuzzy set of $X$. By $\hat{0}$ and $\hat{1}$ we denote constant maps on $X$ with value 0 and 1, respectively. For any $\mu \in I^X$, $\mu^c$ denotes the complement $\hat{1} - \mu$. All other notations are standard notations of fuzzy set theory.

A smooth fuzzy topology [7] on $X$ is a map $T : I^X \rightarrow I$ which satisfies the following properties:

1. $T(\hat{0}) = T(\hat{1}) = 1$.
2. $T(\mu_1 \wedge \mu_2) \geq T(\mu_1) \wedge T(\mu_2)$.
3. $T(\mu_1 \vee \mu_2) \geq T(\mu_1)$.

The pair $(X, T)$ is called a smooth fuzzy topological space.

Let $A$ be a fuzzy set in a smooth fuzzy topological space $(X, T)$ and $r \in I$. Then $A$ is said to be fuzzy $r$-semiopen [5] (resp., fuzzy $r$-preopen [4], $r$-fuzzy $\beta$-open [1]) if $A \subseteq \text{cl}(\text{int}(A, r), r)$ (resp., $A \subseteq \text{int}(\text{cl}(A, r), r)$, $A \subseteq \text{cl}(\text{int}(A, r), r)$).

Definition 2.1. ([8]) Let $X$ be a nonempty set and $r \in (0, 1] = I_0$. A fuzzy family $\mathcal{M} : I^X \rightarrow I$ on $X$ is said to have a fuzzy $r$-minimal structure if the family

$$\mathcal{M}_r = \{A \in I^X | \mathcal{M}(A) \geq r\}$$

contains $\hat{0}$ and $\hat{1}$.

Then the $(X, \mathcal{M})$ is called a fuzzy $r$-minimal space (simply $r$-FMS). Every member of $\mathcal{M}_r$ is called a fuzzy $r$-minimal open set. A fuzzy set $A$ is called a fuzzy $r$-minimal closed set if the complement of $A$ (simply, $A^c$) is a fuzzy $r$-minimal open set.

Let $(X, \mathcal{M})$ be an $r$-FMS and $r \in I_0$. The fuzzy $r$-minimal closure and the fuzzy $r$-minimal interior of $A [8]$,
denoted by \( mC(A, r) \) and \( ml(A, r) \), respectively, are defined as

\[
mC(A, r) = \cap \{ B \in I^X : B^c \in \mathcal{M}_r \text{ and } A \subseteq B \}, \]
\[
ml(A, r) = \cup \{ B \in I^X : B \in \mathcal{M}_r \text{ and } B \subseteq A \}.
\]

**Theorem 2.2.** (\[8\]) Let \((X, \mathcal{M})\) be an \( r\)-FMS and \( A, B \) in \( I^X \).

1. \( ml(A, r) \subseteq A \) and if \( A \) is a fuzzy \( r \)-minimal open set, then \( ml(A, r) = A \).
2. \( A \subseteq mC(A, r) \) and if \( A \) is a fuzzy \( r \)-minimal closed set, then \( mC(A, r) = A \).
3. If \( A \subseteq B \), then \( ml(A, r) \subseteq ml(B, r) \) and \( mC(A, r) \subseteq mC(B, r) \).
4. \( ml(A, r) \cap ml(B, r) \supseteq ml(A \cap B, r) \) and \( mC(A, r) \cup mC(B, r) \subseteq mC(A \cup B, r) \).
5. \( ml(ml(A, r), r) = ml(A, r) \) and \( mC(mC(A, r), r) = mC(A, r) \).
6. \( \bar{I} - mC(A, r) = ml(I - A, r) \) and \( \bar{I} - ml(A, r) = mC(I - A, r) \).

**Definition 2.3.** (\[8\]) Let \((X, \mathcal{M})\) and \((Y, \mathcal{N})\) be two \( r\)-FMS’s. Then \( f : X \to Y \) is said to be

1. **fuzzy \( r \)-M continuous** mapping if for every \( A \in \mathcal{N}_r \), \( f^{-1}(A) \) is in \( \mathcal{M}_r \).
2. **fuzzy \( r \)-M open** if for every \( A \in \mathcal{M}_r \), \( f(A) \) is in \( \mathcal{N}_r \).

**3. Fuzzy \( r \)-Minimal Compactness**

**Definition 3.1.** Let \((X, \mathcal{M})\) be an \( r\)-FMS and \( A = \{ A_i : i \in I \} \). \( \mathcal{A} \) is called a fuzzy \( r \)-minimal open cover if \( \cup \{ A_i : i \in I \} = I \). It is a fuzzy \( r \)-minimal open cover if each \( A_i \) is a fuzzy \( r \)-minimal open set. A subcover of a fuzzy \( r \)-minimal cover \( \mathcal{A} \) is a subfamily of it which also is a fuzzy \( r \)-minimal cover.

**Definition 3.2.** Let \((X, \mathcal{M})\) be an \( r\)-FMS. A fuzzy set \( A \) in \( X \) is said to be fuzzy \( r \)-minimal compact if every fuzzy \( r \)-minimal open cover \( \mathcal{A} = \{ A_i \in \mathcal{M}_r : i \in I \} \) of \( A \) has a finite subcover.

**Theorem 3.3.** Let \( f : (X, \mathcal{M}) \to (Y, \mathcal{N}) \) be a fuzzy \( r \)-M continuous mapping on two \( r\)-FMS’s. If \( A \) is fuzzy \( r \)-minimal compact set, then \( f(A) \) is also a fuzzy \( r \)-minimal compact set.

**Proof.** Let \( \{ B_i : i \in I \} \) be a fuzzy \( r \)-minimal open cover of \( f(A) \) in \( Y \). Then since \( f \) is a fuzzy \( r \)-M continuous mapping, \( \{ f^{-1}(B_i) : i \in I \} \) is a fuzzy \( r \)-minimal open cover of \( A \) in \( X \). By fuzzy \( r \)-minimal compactness, there exists \( J_0 = \{ j_1, j_2, \ldots, j_n \} \subseteq J \) such that \( A \subseteq \cup_{i \in J_0} f^{-1}(B_i) \). Hence \( f(A) \subseteq \cup_{i \in J_0} B_i \).

**Definition 3.4.** Let \((X, \mathcal{M})\) be an \( r\)-FMS. A fuzzy set \( A \) in \( X \) is said to be **almost fuzzy \( r \)-minimal compact** if for every fuzzy \( r \)-minimal open cover \( \mathcal{A} = \{ A_i \in I^X : i \in J \} \) of \( A \), there exists \( J_0 = \{ j_1, j_2, \ldots, j_n \} \subseteq J \) such that \( A \subseteq \cup_{i \in J_0} mC(A_i, r) \).

**Theorem 3.5.** Let \((X, \mathcal{M})\) be an \( r\)-FMS. If a fuzzy set \( A \) in \( X \) is fuzzy \( r \)-minimal compact, then it is also almost fuzzy \( r \)-minimal compact.

**Proof.** Obvious.

In Theorem 3.5, the converse is not always true as shown the next example.

**Example 3.6.** Let \( X = I \) and \( n \in N - \{1\} \). Let \( A_1 \) and \( A_n \) be fuzzy sets defined as follows

\[
A_n(x) = \begin{cases} 
0.8, & \text{if } x = 0, \\
n x, & \text{if } 0 < x \leq \frac{1}{n}, \\
1, & \text{if } \frac{1}{n} < x \leq 1;
\end{cases}
\]

\[
A_1(x) = \begin{cases} 
1, & \text{if } x = 0, \\
\frac{2}{n}, & \text{otherwise}.
\end{cases}
\]

Consider a fuzzy \( r \)-minimal structure \( \mathcal{M} : I^X \to I^Y \) on \( X \) as follows

\[
\mathcal{M}(A) = \begin{cases} 
\frac{4}{5}, & \text{if } A = \hat{0}, \hat{1}, \\
\frac{n}{n+1}, & \text{if } A = A_n, \\
\frac{3}{2}, & \text{if } A = A_1, \\
0, & \text{otherwise}.
\end{cases}
\]

Let \( \mathcal{A} = \{ A_n : n \in N \} \) be a fuzzy \( \frac{1}{2} \)-minimal open cover of \( X \). Then there does not exist a finite subcover of \( \mathcal{A} \). Thus \( X \) is not fuzzy \( \frac{1}{2} \)-minimal compact. But \( X \) is almost fuzzy \( \frac{1}{2} \)-minimal compact.

**Theorem 3.7.** (\[8\]) Let \( f : X \to Y \) be a mapping on two \( r\)-FMS’s \((X, \mathcal{M})\) and \((Y, \mathcal{N})\),

1. \( f \) is fuzzy \( r \)-M continuous.
2. \( f^{-1}(B) \) is a fuzzy \( r \)-minimal closed set, for each fuzzy \( r \)-minimal closed set \( B \) in \( Y \).
3. \( mC(f(A), r) \subseteq mC(f(A), r) \) for \( A \in I^X \).
4. \( mC(f^{-1}(B), r) \subseteq f^{-1}(mC(B, r)) \) for \( B \in I^Y \).
5. \( f^{-1}(mC(B, r)) \subseteq mC(f^{-1}(B), r) \) for \( B \in I^Y \).

Then (1) \( \Leftrightarrow \) (2) \( \Leftrightarrow \) (3) \( \Leftrightarrow \) (4) \( \Leftrightarrow \) (5).

**Theorem 3.8.** Let \( f : (X, \mathcal{M}) \to (Y, \mathcal{N}) \) be a fuzzy \( r \)-M continuous mapping on two \( r\)-FMS’s. If \( A \) is an almost fuzzy \( r \)-minimal compact set, then \( f(A) \) is also an almost fuzzy \( r \)-minimal compact set.

**Proof.** Let \( \{ B_i : i \in J \} \) be a fuzzy \( r \)-minimal open cover of \( f(A) \) in \( Y \). Then since \( f \) is a fuzzy \( r \)-M continuous mapping, \( \{ f^{-1}(B_i) : i \in J \} \) is a fuzzy \( r \)-minimal open cover of \( A \) in \( X \). By fuzzy \( r \)-minimal compactness, there exists \( J_0 = \{ j_1, j_2, \ldots, j_n \} \subseteq J \) such that \( A \subseteq \cup_{i \in J_0} f^{-1}(B_i) \). Hence \( f(A) \subseteq \cup_{i \in J_0} f^{-1}(B_i) \).
A \subseteq \cup_{i \in J_0} mC(f^{-1}(B_i), r). From Theorem 3.7, it follows
\[
\cup_{i \in J_0} mC(f^{-1}(B_i, r)) \subseteq \cup_{i \in J_0} mC(f(B_i, r)) = f^{-1}(\cup_{i \in J_0} mC(B_i, r)).
\]
Hence \( f(A) \subseteq \cup_{i \in J_0} mC(B_i, r). \)

**Definition 3.9.** Let \((X, M)\) be an \(r\)-FMS. A fuzzy set \(A\) in \(X\) is said to be nearly fuzzy \(r\)-minimal compact if for every fuzzy \(r\)-minimal open cover \(A = \{A_i : i \in J\}\) of \(A\), there exists \(J_0 = \{j_1, j_2, \ldots, j_n\} \subseteq J\) such that \(A \subseteq \cup_{i \in J_0} mI(mC(A_i, r), r)\).

**Example 3.10.** (1) Let \(X = I\). Consider the fuzzy minimal structure \(M\) defined in Example 3.6. The fuzzy set \(1\) is an almost fuzzy \(\frac{1}{2}\)-minimal compact set but it is not nearly fuzzy \(\frac{1}{2}\)-minimal compact in \((X, M)\).

(2) Let \(X = I\). Consider fuzzy sets for \(0 < n < 1\),
\[
\sigma_n(x) = \begin{cases} \frac{1}{n}, & \text{if } 0 \leq x \leq n, \\ \frac{1}{1-n}, & \text{if } n < x \leq 1; \end{cases}
\]
\[
\alpha(x) = \begin{cases} 1, & \text{if } x = 0, \\ \frac{1}{2}, & \text{if } 0 < x \leq 1; \end{cases}
\]
\[
\beta(x) = \begin{cases} \frac{1}{2}, & \text{if } 0 \leq x < 1, \\ 1, & \text{if } x = 1. \end{cases}
\]
And consider a fuzzy minimal structure
\[
\mathcal{N}(\mu) = \begin{cases} \max\{1-n, n\}, & \text{if } \mu = \alpha_n, \\ 1, & \text{if } \mu = \alpha, \beta, \tilde{0}, \tilde{1}, \\ 0, & \text{otherwise}. \end{cases}
\]
Then \(X\) is nearly fuzzy \(\frac{1}{2}\)-minimal compact but not fuzzy \(\frac{1}{2}\)-minimal compact.

**Theorem 3.11.** Let \((X, M)\) be an \(r\)-FMS. If a fuzzy set \(A\) in \(X\) is fuzzy \(r\)-minimal compact, then it is nearly fuzzy \(r\)-minimal compact.

**Proof.** For any a fuzzy \(r\)-minimal open set \(U\) in \(X\), from Theorem 2.2, it follows \(U = mI(U, r) \subseteq mI(mC(U, r), r)\). Thus we get the result.

In Theorem 3.11, the converse implication is not true always true as shown in the Example 3.10. Hence the following implications are obtained:

fuzzy \(r\)-minimal compact \(\Rightarrow\) nearly fuzzy \(r\)-minimal compact \(\Rightarrow\) almost fuzzy \(r\)-minimal compact

**Theorem 3.12.** ([8]) Let \(f : X \rightarrow Y\) be a mapping on two \(r\)-FMS’s \((X, M)\) and \((Y, N)\). Then
(1) \(f\) is fuzzy \(r\)-M open.
(2) \(f(mI(A, r)) \subseteq mI(f(A, r))\) for \(A \subseteq I^X\).
(3) \(mI(f^{-1}(B), r) \subseteq f^{-1}(mI(B), r)\) for \(B \subseteq I^Y\).
Then (1) \(\Rightarrow\) (2) \(\Leftrightarrow\) (3).

**Theorem 3.13.** Let a mapping \(f : (X, M) \rightarrow (Y, N)\) be fuzzy \(r\)-M continuous and fuzzy \(r\)-M open on two \(r\)-FMS’s. If \(A\) is a nearly fuzzy \(r\)-minimal compact set, then \(f(A)\) is a nearly fuzzy \(r\)-minimal compact set.

**Proof.** Let \(\{B_i \in I^Y : i \in J\}\) be a fuzzy \(r\)-minimal open cover of \(f(A)\) in \(Y\). Then \(\{f^{-1}(B_i) : i \in J\}\) is a fuzzy \(r\)-minimal open cover of \(A\) in \(X\). By nearly fuzzy \(r\)-minimal compactness, there exists \(J_0 = \{j_1, j_2, \ldots, j_n\} \subseteq J\) such that \(A \subseteq \cup_{i \in J_0} mI(mC(B_i, r), r)\). From Theorem 3.7 and Theorem 3.12, it follows
\[
f(A) \subseteq \cup_{i \in J_0} f(mI(mC(f^{-1}(B_i), r), r)) \subseteq \cup_{i \in J_0} mI(f(mC(f^{-1}(B_i), r)), r) \subseteq \cup_{i \in J_0} mI(f^{-1}(mC(B_i, r)), r) \subseteq \cup_{i \in J_0} mI(mC(B_i, r), r).
\]
Hence \(f(A)\) is a nearly fuzzy \(r\)-minimal compact set.

**Remark 3.14.** In Theorem 3.13, the fuzzy \(r\)-M continuity and fuzzy \(r\)-M openness of the mapping \(f\) are necessary conditions as shown in the next example.

**Example 3.15.** Let \(X = I\). Consider fuzzy sets for \(0 < n < 1\),
\[
\sigma_n(x) = \begin{cases} \frac{1}{n}, & \text{if } 0 \leq x \leq n, \\ \frac{1}{1-n}, & \text{if } n < x \leq 1; \end{cases}
\]
\[
\alpha(x) = \begin{cases} 1, & \text{if } x = 0, \\ \frac{1}{2}, & \text{if } 0 < x \leq 1; \end{cases}
\]
\[
\beta(x) = \begin{cases} \frac{1}{2}, & \text{if } 0 \leq x < 1, \\ 1, & \text{if } x = 1. \end{cases}
\]
And consider fuzzy minimal structures
\[
\mathcal{L}(\mu) = \begin{cases} \max\{1-n, n\}, & \text{if } \mu = \alpha_n, \\ 1, & \text{if } \mu = \alpha, \beta, \gamma, \tilde{0}, \tilde{1}, \\ 0, & \text{otherwise}; \end{cases}
\]
\[
\mathcal{M}(\mu) = \begin{cases} \max\{1-n, n\}, & \text{if } \mu = \alpha_n, \\ 1, & \text{if } \mu = \alpha, \beta, \tilde{0}, \tilde{1}, \\ 0, & \text{otherwise}; \end{cases}
\]
\[
\mathcal{N}(\mu) = \begin{cases} 
\max\{1 - n, n\}, & \text{if } \mu = \alpha_n, \\
1, & \text{if } \mu = \bar{0}, \\
0, & \text{otherwise.}
\end{cases}
\]

Let \( f : (X, \mathcal{L}) \to (X, \mathcal{M}) \) be the identity mapping. It is obvious that \( f \) is fuzzy \( \frac{1}{2} \)-\( \mathcal{M} \)-continuous. \( X \) is nearly fuzzy \( \frac{1}{2} \)-minimal compact on \( (X, \mathcal{L}) \) but \( f(X) \) is not nearly fuzzy \( \frac{1}{2} \)-minimal compact on \( (X, \mathcal{M}) \).

Now let \( f : (X, \mathcal{N}) \to (X, \mathcal{M}) \) be the identity mapping. Then \( f \) is fuzzy \( \frac{1}{2} \)-\( \mathcal{M} \)-open. Consider a fuzzy set \( A \) defined as follows

\[
A(x) = \begin{cases} 
1, & \text{if } 0 < x < 1, \\
0, & \text{if } x = 0, 1.
\end{cases}
\]

Then \( A \) is nearly fuzzy \( \frac{1}{2} \)-minimal compact on \( (X, \mathcal{N}) \) but \( f(A) \) is not nearly fuzzy \( \frac{1}{2} \)-minimal compact \( (X, \mathcal{M}) \).

References


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