Fuzzy Functions and Fuzzy Partially Ordered Sets

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Abstract

By using the notion of fuzzy functions introduced by Dib and Youssef, we obtain fuzzy analogues of some results concerning ordinary functions. In particular, we give the definition different from one of invertible fuzzy function introduced by Dib and Youssef. And we show that the two definitions are equivalent. Furthermore, we introduce the concepts of fuzzy increasing functions and fuzzy isomorphisms, and we obtain fuzzy analogues of many results concerning ordinary increasing functions and isomorphisms.

Key words: fuzzy function, fuzzy partially ordered set.

1. Introduction

In the usual set theory, functions are special types of relations and relations are subsets of Cartesian product. Thus the concept of Cartesian product plays an important role in the usual theory of relations and functions. Almost all authors have worked with fuzzy relations without referring to what may be called fuzzy Cartesian product (See [1,3,7,8]).

However, in 1991, by using J-fuzzy sets, Dib and Youssef introduced the notion of fuzzy Cartesian product and they defined a fuzzy relation as a subset of the fuzzy Cartesian product (See [2]). This definition is different from all known definitions of fuzzy relations. Also they defined a fuzzy function as a special type of a fuzzy relation. In particular, they defined a fuzzy partially ordered set and investigated some of its properties. We can see that this definition generalizes Zadeh’s definition and is different from those in [3]. Moreover, Hur et al. [5], and Lee [6] obtained fuzzy analogues of many results concerning ordinary equivalence relations and functions in the sense of Dib and Youssef.

In section 2, by using the definition of fuzzy functions introduced by Dib and Youssef, we obtain fuzzy analogues of many results concerning ordinary functions. In particular, we give the definition from one of invertible fuzzy function introduced by Dib and Youssef, and consequently prove that the two definitions are equivalent.

In section 3, we investigate fuzzy analogues of some results concerning ordinary partially ordered sets.

In section 4, we introduce the concepts of fuzzy increasing functions and fuzzy isomorphisms. And we study some of their properties.

2. Preliminaries

The totally ordered set \(I = [0, 1]\) is a distributive but not complemented lattice under the operations of infimum \(\land\) and supremum \(\lor\). On \(J = I \times I\) we define a partial order \(\leq\), in terms of the partial order on \(I\), as follows: For every \((r_1, r_2), (s_1, s_2) \in J\),

(i) \((r_1, r_2) \leq (s_1, s_2)\) if and only if \(r_1 \leq s_1, r_2 \leq s_2\) whenever \(s_1 \neq 0\) and \(s_2 \neq 0\),
(ii) \((0, 0) = (s_1, s_2)\) whenever \(s_1 = 0\) or \(s_2 = 0\).

It is clear that \(J\) is a distributive but not complemented vector lattice. The operations of infimum and supremum in \(J\) are given respectively by: For every \((r_1, r_2), (s_1, s_2) \in J\),

\((r_1, r_2) \land (s_1, s_2) = (r_1 \land s_1, r_2 \land s_2)\)
and
\[(r_1, r_2) \vee (s_1, s_2) \leq (r_1 \vee s_1, r_2 \vee s_2),\]
where the equality holds in the last relation when \(r_i \neq 0 \neq s_i\).

For sets \(X, Y\) and \(Z, f = (f_1, f_2) : X \rightarrow Y \times Z\) is called a complex mapping if \(f_1 : X \rightarrow Y\) and \(f_2 : X \rightarrow Z\) are mappings, where \(f(x) = (f_1(x), f_2(x))\) for each \(x \in X\).

**Definition 2.2**. Let \(X\) be a nonempty set. A complex mapping \(A = (\mu_A, \eta_A) : X \rightarrow J\) is called a fuzzy set in \(X\), where \(A(x) = (\mu_A(x), \eta_A(x))\) for each \(x \in X\). In particular, \(\emptyset\) and \(X\) denote the fuzzy empty set and fuzzy whole set in \(X\) defined by \(\emptyset(x) = (0, 0)\) and \(X(x) = (1, 1)\) for each \(x \in X\), respectively.

The notation \(\{(x, A(x)) : x \in X\}\) or simply \(\{(x, r)\}\), where \(r = A(x)\), will be used to denote a fuzzy set in \(X\) (see [7]). Similarly, a J-fuzzy set in \(X\), a fuzzy set in \(X \times Y\) and a J-fuzzy set in \(X \times Y\) will be denoted respectively by \(\{(x, (r_1, r_2))\}, \{(x, (y, r))\}\) and \(\{(x, (y, (r_1, r_2)))\}\). To each fuzzy set \(\{(x, r_1)\}\) in \(X\) and fuzzy set \(\{(y, r_2)\}\) in \(Y\) there corresponds a J-fuzzy set \(\{(x, (y, (r_1, r_2)))\}\) in \(X \times Y\). Throughout this thesis, the notation \((x, r) \in A\) means that \(A(x) = r\), where \(A \in I^X\) and \(X, Y, Z\), etc denote ordinary sets.

**Definition 2.2**. Let \(X\) and \(Y\) be two ordinary sets. Then the collection of all J-fuzzy sets in \(X \times Y\) is called the fuzzy Cartesian product of \(X\) and \(Y\) and is denoted by \(X \times Y\). Hence \(X \times Y = J^{X \times Y}\).

The fuzzy Cartesian product of a fuzzy set \(A = \{(x, r)\}\) in \(X\) and a fuzzy set \(B = \{(y, s)\}\) in \(Y\) is the J-fuzzy set \(A \times B\) in \(X \times Y\) defined by:

\[A \times B = \{(x, y, (r, s)) : x \in X, y \in Y\} \equiv \{(x, (y, (r, s)))\}\]

It is clear that \(A \times B \in J^{X \times Y}\) for each \(A \in I^X\) and \(B \in I^Y\). The above definitions can be generalized for any finite number of sets. Furthermore, the above definitions can be generalized in an obvious way by replacing the unit interval \(I\) by an arbitrary completely distributive lattice.

**Definition 2.3**. \(\rho\) is called a fuzzy relation from \(X\) to \(Y\) if \(\rho \subseteq X \times Y\). In particular, \(\rho\) is called a fuzzy relation in \(X\) if \(\rho \subseteq X \times X\).

It is clear that \(X \times Y\) is itself a fuzzy relation from \(X\) to \(Y\) and any collection of \(A \times B\), where \(A \in I^X\) and \(B \in I^Y\), is a fuzzy relation from \(X\) to \(Y\).

The fuzzy Cartesian product \(X \times X\) is called the universal fuzzy relation in \(X\). The fuzzy relation \(\emptyset \times \emptyset = \emptyset\) is called the empty fuzzy relation. Between these two extreme cases, lies the identity fuzzy relation, denoted by \(\Delta_X\), where \(\Delta_X\) is the fuzzy relation in \(X\) whose members are the \(J\)-fuzzy sets \(\{(x, (x, r)) : x \in X\}\) and \(r \in J\).

**Definition 2.4**. Let \(\rho_1\) and \(\rho_2\) be fuzzy relations from \(X\) to \(Y\).

1. We say that \(\rho_1\) is contained in \(\rho_2\) if whenever \(((x, y), (r_1, r_2)) \in A \subseteq \rho_1\), there exists \(B \subseteq \rho_2\) such that \(((x, y), (r_1, r_2)) \in B\). In this case, we write \(\rho_1 \subseteq \rho_2\).
2. We say that \(\rho_1\) and \(\rho_2\) are equal if \(\rho_1 \subseteq \rho_2\) and \(\rho_2 \subseteq \rho_1\). In this case, we write \(\rho_1 = \rho_2\).

To each \(J\)-fuzzy set \(C = \{(x, (y, (r, s)))\}\) in \(X \times Y\) we associate a \(J\)-fuzzy set \(C^{-1}\) in \(Y \times X\) defined by \(C^{-1} = \{(y, (x, (s, r)))\}\).

**Definition 2.5**. Let \(\rho\) be a fuzzy relation from \(X\) to \(Y\). Then the inverse of \(\rho\), denoted \(\rho^{-1}\), is the fuzzy relation from \(Y\) to \(X\) defined by \(\rho^{-1} = \{C^{-1} : C \in \rho\}\).

**Definition 2.6**. Let \(\rho\) be a fuzzy relation from \(X\) to \(Y\) and let \(\sigma\) be a fuzzy relation from \(Y\) to \(Z\). Then the composition of \(\rho\) and \(\sigma\), denoted \(\sigma \circ \rho\), is the fuzzy relation from \(X\) to \(Z\) whose constituting \(J\)-fuzzy sets \(C \in X \times Z\) are defined as follows:

\(((x, y), (r_1, r_3)) \in \rho\) if and only if there exists \((y, z), (r_2, r_3)) \in \sigma\) such that \(((x, y), (r_1, r_2)) \in A\) and \(((y, z), (r_2, r_3)) \in B\) for some \(A \in \rho\) and \(B \subseteq \sigma\). Hence \(\sigma \circ \rho = \{C \subseteq X \times Z : C\) is defined above\}.

It is clear that if \(\rho\) is a fuzzy relation in \(X\), then \(\Delta_X \circ \rho \subseteq \rho\) and \(\rho \circ \Delta_X \subseteq \rho\).

**Result 2.2**. Let \(\rho, \rho_1, \rho_2, \rho_3, \sigma_1, \sigma_2\) be any fuzzy relations defined on the appropriate sets. Then

1. \((\rho \circ \rho_2) \circ \rho_3 = \rho \circ (\rho_2 \circ \rho_3)\).
2. \(\rho_1 \subseteq \rho_2\) and \(\sigma_1 \subseteq \sigma_2\) \(\Rightarrow\) \(\rho_1 \circ \sigma_1 \subseteq \rho_2 \circ \sigma_2\).
3. \(\rho_1 \circ (\rho_2 \cup \rho_3) = (\rho_1 \circ \rho_2) \cup (\rho_1 \circ \rho_3)\).
4. \(\rho_1 \circ (\rho_2 \cap \rho_3) \subseteq (\rho_1 \circ \rho_2) \cap (\rho_1 \circ \rho_3)\).
5. \(\rho_1 \circ \rho_2 \Rightarrow \rho_1^{\circ^{-1}} \subseteq \rho_2^{\circ^{-1}}\).
6. \((\rho^{-1})^{-1} = \rho\) and \((\rho_1 \circ \rho_2)^{-1} = \rho_2^{-1} \circ \rho_1^{-1}\).
7. \((\rho_1 \cup \rho_2)^{-1} = \rho_1^{-1} \cup \rho_2^{-1}\).
8. \((\rho_1 \cap \rho_2)^{-1} = \rho_1^{-1} \cap \rho_2^{-1}\).

**Definition 2.7**. Let \(\rho\) be a fuzzy relation in \(X\). Then \(\rho\) is said to be:

1. (reflexive in \(X\) if for each \(x \in X\) and \(r \in I\), there exists \(A \subseteq \rho\) such that \(((x, x), (r, r)) \in A\), i.e., \(\Delta_X \subseteq \rho\).
2. (symmetric in \(X\) if whenever \(((x, y), (r, s)) \in \rho\), there exists \(B \subseteq \rho\) such that \(((y, x), (s, r)) \in B\), i.e., \(\rho^{-1} = \rho\).
(3) transitive in $X$ if whenever $((x, y), (r, s)) \in A \in \rho$ and $((y, z), (s, t)) \in B \in \rho$, there exists $C \in \rho$ such that $((x, z), (r, t)) \in C$, i.e., $\rho \circ \rho \subset \rho$.

(4) a fuzzy equivalence relation in $X$ if it is reflexive, symmetric and transitive.

We will denote the set of all fuzzy equivalence relations in $X$ as $FRel_E(X)$. It is clear that $X \sqsupseteq X \sqsubset FRel_E(X)$.

Result 2. B[2, Proposition in p.303]. Let $\rho$ and $\sigma$ be fuzzy relations on a set $X$. Then

(1) If $\rho$ is reflexive [resp. symmetric and transitive], then $\rho^{-1}$ is reflexive [resp. symmetric and transitive].

(2) If $\rho$ is reflexive [resp. symmetric and transitive], then $\rho \circ \rho$ is reflexive [resp. symmetric and transitive].

(3) If $\rho$ is reflexive, then $\rho \subset \rho \circ \rho$.

(4) If $\rho$ is symmetric, then $\rho \cup \rho^{-1}$, $\rho \cap \rho^{-1}$ are symmetric and $\rho \circ \rho^{-1} = \rho^{-1} \circ \rho$.

(5) If $\rho$ and $\sigma$ are reflexive [resp. symmetric and transitive], then $\rho \cap \sigma$ is reflexive [resp. symmetric and transitive].

(6) If $\rho$ and $\sigma$ are symmetric, then $\rho \cup \sigma$ is symmetric.

3. Fuzzy functions

In the usual set theory, functions are special types of relations. Dib and Youssef[2] defined fuzzy functions analogously as special types of fuzzy relations.

First of all, we list some definitions introduced and some results investigated by Dib and Youssef. Next, we improve another results corresponding to fuzzy functions for ordinary functions.

Definition 3.1[2]. Let $X$ and $Y$ be nonempty sets. Then a fuzzy relation $F$ from $X$ to $Y$ is called a fuzzy function from $X$ to $Y$ if $F : I^X \rightarrow I^Y$ is a function from $I^X$ to $I^Y$ characterized by the ordered pair $(F, \{f_x\}_{x \in X})$, where $F : X \rightarrow Y$ is a function from $X$ to $Y$ and $\{f_x\}_{x \in X}$ is a family of functions $f_x : I \rightarrow I$ satisfying the conditions:

\begin{itemize}
  \item[(a)] $f_x$ is nondecreasing on $I$,
  \item[(b)] $f_x(0) = 0$ and $f_x(1) = 1$,
\end{itemize}

such that the image of any fuzzy set $A$ in $X$ under $F$ is a fuzzy set $F(A)$ in $Y$ defined as follows: For each $y \in Y$,

$$F(A) = \begin{cases} \bigvee_{x \in F^{-1}(y)} f_x(A(x)) & \text{if } F^{-1}(y) \neq \emptyset, \\ 0 & \text{if } F^{-1}(y) = \emptyset. \end{cases}$$

In this case, we write $F = (F, \{f_x\}_{x \in X}) : X \rightarrow Y$ or, simply, $F = (F, f_x) : X \rightarrow Y$ to denote a fuzzy function from $X$ to $Y$ and we call the functions $f_x, x \in X$, the comembership functions associated to $F$. If there exists $A \in F$ such that $(x, F(x), r, f_x(r)) \in A \in \rho$ for each $(x, r) \in X \times I$, then we write this in the form $F(x, r) = (F(x), f_x(r))$.

Two fuzzy functions $F = (F, f_x) : X \rightarrow Y$ and $G = (G, g_y) : X \rightarrow Y$ are said to be equal, denoted by $F = G$, if $F(A) = G(A)$ for each $A \in I^X$.

Result 3. A[2, Theorem 5]. Let $F = (F, f_x) : X \rightarrow Y$ and $G = (G, g_y) : X \rightarrow Y$ be fuzzy functions. Then $F = G$ if and only if $F = G$ and $f_x = g_x$ for each $x \in X$.

Let $F = (F, f_x) : X \rightarrow Y$ be a fuzzy function. The inverse image under $F$ of a fuzzy set $B$ in $Y$, denoted by $F^{-1}(B)$, is a fuzzy set in $X$ defined as follows:

$$F^{-1}(B) = \bigcup\{C \in I^X : F(C) \subset B\}.$$

Result 3. B[2, Proposition in p.311]. Let $F = (F, f_x) : X \rightarrow Y$ be any fuzzy function whose comembership functions $f_x$ are surjective. Then for each fuzzy set $B$ in $Y$, and each $x \in X$,

$$F^{-1}(B)(x) = \bigvee f_x^{-1}[\text{sup}(F(x))],$$

where the supremum is taken over the set of values $f_x^{-1}[\text{sup}(F(x))] \subset I$.

Result 3. C[2, Theorem 6]. Let $F = (F, f_x) : X \rightarrow Y$ be a fuzzy function, let $A, B \in I^X$, $\{A_\alpha\}_{\alpha \in \Gamma} \subset I^X$ and let $C, D \in I^Y$, $\{C_\alpha\}_{\alpha \in \Gamma} \subset I^Y$. Then

\begin{itemize}
  \item[(1)] $F(\emptyset) = \emptyset$.
  \item[(2)] $F(X) = Y$ if $F$ is surjective.
  \item[(3)] If $A \subset B$, then $F(A) \subset F(B)$.
  \item[(4)] $F(A \cup B) = F(A) \cup F(B)$.
  \item[(5)] $F(A \cap B) \subset F(A) \cap F(B)$ (equality holds if $F$ is injective).
\end{itemize}

If $f_x$ is surjective for each $x \in X$, then

\begin{itemize}
  \item[(6)] $F^{-1}(Y) = X$.
  \item[(7)] If $C \subset D$, then $F^{-1}(C) \subset F^{-1}(D)$.
  \item[(8)] $A \subset F^{-1}(F(A))$ (equality holds if $F$ is bijective).
\end{itemize}

If $f_x$ is surjective for each $x \in X$, then

\begin{itemize}
  \item[(9)] $F(\bigcup_{\alpha \in \Gamma} A_\alpha) = \bigcup_{\alpha \in \Gamma} F(A_\alpha)$.
  \item[(10)] $F(\bigcap_{\alpha \in \Gamma} A_\alpha) \subset \bigcap_{\alpha \in \Gamma} F(A_\alpha)$ (equality holds if $F$ is injective).
\end{itemize}

If $f_x$ is surjective and $f_x(1 - r) = 1 - f_x(r)$ for each $(x, r) \in X \times I$, then

\begin{itemize}
  \item[(11)] $F^{-1}(\bigcup_{\alpha \in \Gamma} C_\alpha) = \bigcup_{\alpha \in \Gamma} F^{-1}(C_\alpha)$.
  \item[(12)] $F^{-1}(\bigcap_{\alpha \in \Gamma} C_\alpha) = \bigcap_{\alpha \in \Gamma} F^{-1}(C_\alpha)$.
  \item[(13)] $F^{-1}(C) \subset C$ (equality holds if $F$ is surjective).
\end{itemize}

If $f_x(1 - r) = 1 - f_x(r)$ for each $(x, r) \in X \times I$, then

$$F^{-1}(D^c) = (F^{-1}(D))^c.$$
\( \mathcal{G} \circ \mathcal{F} : X \to Z \) defined by \((\mathcal{G} \circ \mathcal{F})(A) = \mathcal{G}(\mathcal{F}(A))\) for each \( A \in \mathcal{I}^X \).

Let \( \mathcal{F} = (F, f_x) : X \to Y \) be a fuzzy function. Then \( \mathcal{F} \) is said to be injective or one-to-one if for any fuzzy sets \( A_1 \) and \( A_2 \) in \( X \), \( \mathcal{F}(A_1) = \mathcal{F}(A_2) \) implies \( A_1 = A_2 \).

**Surjective** and **bijective** fuzzy functions can be defined similarly in an obvious manner. It is clear that the **fuzzy identity function** \( \mathcal{id}_X \) is a fuzzy function. The inverse \( \mathcal{F}^{-1} \) of \( \mathcal{F} \) is a fuzzy function if and only if \( \mathcal{F} \) is bijective.

**Proposition 3.3.** Let \( \mathcal{F} = (F, f_x) : X \to Y \) and \( \mathcal{G} = (G, g_y) : Y \to Z \) be fuzzy functions. Let \( g_y \) be surjective for each \( y \in Y \). Then

1. The composition \( \mathcal{G} \circ \mathcal{F} : X \to Z \) is defined by \((\mathcal{G} \circ \mathcal{F})(x) = \mathcal{G}(\mathcal{F}(x)) = (G \circ F)(x)\).
2. \( \mathcal{F} = (F, f_x) \) is surjective [resp. injective] if and only if \( F \) is surjective [resp. injective].

**Result 3.D[2, Theorem 7].** Let \( \mathcal{F} = (F, f_x) : X \to Y \) and \( \mathcal{G} = (G, g_y) : Y \to Z \) be fuzzy functions. Let \( g_y \) be surjective for each \( y \in Y \). Then

1. The composition \( \mathcal{G} \circ \mathcal{F} : X \to Z \) is defined by \((\mathcal{G} \circ \mathcal{F})(x) = \mathcal{G}(\mathcal{F}(x)) = (G \circ F)(x)\).
2. \( \mathcal{F} = (F, f_x) \) is surjective [resp. injective] if and only if \( F \) is surjective [resp. injective].

**Definition 3.5.** A fuzzy function \( \mathcal{F} = (F, f_x) : X \to Y \) is said to be **invertible** if \( \mathcal{F}^{-1} = (F^{-1}, f_{z^{-1}}) : Y \to X \) is a fuzzy function.

Let \( \mathcal{F} = (F, f_x) : X \to Y \) be an invertible fuzzy function. Then \((x, y, r, s) \in \mathcal{F}(x) \) if and only if \( (y, x, s, r) \in \mathcal{F}^{-1}(y, s) \). Thus \( \mathcal{F}(x, r) = (y, s) \) if and only if \( \mathcal{F}^{-1}(y, s) = (x, r) \).

The following is the immediate result of Definition 3.5.

**Proposition 3.6.** Let \( \mathcal{F} = (F, f_x) : X \to Y \) be an invertible fuzzy function. Then for each \( x \in X \), \( y \in Y \), \( r \in I \), \( s \in \mathcal{I}^X \) and \( g_y \) is surjective, \((y, x, s, r) \in I \) if and only if \( (y, x, s, r) \in \mathcal{F}^{-1}(y, s) \). Thus \( \mathcal{F}(x, r) = (y, s) \) if and only if \( \mathcal{F}^{-1}(y, s) = (x, r) \).

The next two lemmas give a necessary and sufficient condition for a fuzzy function to be invertible.

**Lemma 3.7.** If \( \mathcal{F} = (F, f_x) : X \to Y \) is a bijective fuzzy function, then \( \mathcal{F}^{-1} = (F^{-1}, f_{z^{-1}}) : Y \to X \) is a bijective fuzzy function.

**Proof.** Suppose \( \mathcal{F} = (F, f_x) \) is bijective. Then, by Result 3.D(2), \( F \) and \( f_x \), \( x \in X \), are bijective. Thus \( \mathcal{F}^{-1} : Y \to X \) and \( f_{z^{-1}} \) are bijective for each \( x \in X \). So, by Result 3.D(2), \( \mathcal{F}^{-1} = (F^{-1}, f_{z^{-1}}) \) is bijective.

**Lemma 3.8.** If \( \mathcal{F} = (F, f_x) : X \to Y \) is invertible, then \( \mathcal{F} \) is bijective.
Proof. Suppose \( F = (F, f_x) : X \to Y \) is invertible. Then \( F^{-1} = (F^{-1}, f_x^{-1}) : Y \to X \) is a fuzzy function. Thus, by Definition 2.1, \( F^{-1} : Y \to X \) and \( f_x^{-1} : I \to I \) are functions. So \( F : X \to Y \) and \( f_x : I \to I \) are invertible. Hence \( F \) and \( f_x \) are bijective. Therefore, by Result 3.3(2), \( F \) is bijective.

The following is the immediate result of Definition 4.1.

Proposition 4.2. Let \( \rho \) be a fuzzy relation in \( X \). If \( \rho \) is antisymmetric, then \( \rho^{-1} \) is antisymmetric.

Definition 4.3[2]. A fuzzy relation \( \rho \) in \( X \) is called a fuzzy partial order or, simply, a fuzzy order if it is reflexive, antisymmetric and transitive.

A nonempty set \( X \) in which a fuzzy partial order is defined is called a fuzzy partially ordered set or, simply, a fuzzy ordered set.

Let \( X \) be a fuzzy partially ordered set with the fuzzy order \( \rho \). Then we will write \((x, r) \leq \rho (y, t)\), simply, \((x, r) \leq \rho (y, t)\) to denote the fact that there exist \( A \in \rho \) such that \((x, y, (r, t)) \in A \) for any \( x, y \in X \) and \( r, t \in I \). We further agree that \((y, t) \geq \rho (x, r)\) has the same meaning as \((x, r) \leq \rho (y, t)\) and that \((x, r) \not\leq \rho (y, t)\) means that there is no \( A \in \rho \) such that \((x, y, (r, t)) \in A \).

If \((x, r), (y, t) \in X \times I \) and \((x, r) \leq \rho (y, t)\), then we say that \((x, r) \) is fuzzy less than or fuzzy equal to \((y, t)\).

We agree that \((x, r) < (y, t)\) is an abbreviation for \((x, r) \leq \rho (y, t)\) and \((x, r) \neq \rho (y, t)\). If \((x, r) < (y, t)\), then we say that \((x, r) \) is strictly fuzzy less than \((y, t)\).

Example 4.3. Let \( X = \{a, b, c, d, e\} \) be a set and let \( \rho = \Delta_X \cup \{A_1, A_2, A_3\} \) be the fuzzy relation in defined \( X \) as follows:

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Proposition 4.4. Let \( \rho \) be a fuzzy partial order in \( X \). Then \( \rho^{-1} \) is a fuzzy partial order in \( X \).

Definition 4.5[2]. A fuzzy relation \( \rho \) in \( X \) is called a fuzzy total (or linear) order if for any \( x,y \in X \) and \( r,t \in I \) there exists \( A \in \rho \) such that either \( ((x,y),(r,t)) \in A \) or \( (y,x),(t,r)) \in A \).

A nonempty set \( X \) in which a fuzzy total order is defined is called a fuzzy totally (or linearly) ordered set.

Result 4.4[A, Theorem 4]. Let \( \rho \) be a fuzzy order in \( X \). Then

1. For each \( x_0 \in X \), \( \rho \) induces an (ordinary) order \( \rho_1(x_0) \) in \( I \) defined by
   \[
   \rho_1(x_0) = \{ (r,s) \in I : \exists A \in \rho \text{ s.t. } ((x_0,x_0),(r,s)) \in A \}.
   \]

2. For each \( r_0 \in I \), \( \rho \) induces an (ordinary) order \( \rho_x(r_0) \) in \( X \) defined by
   \[
   \rho_x(r_0) = \{ (x,y) \in X \times X : \exists A \in \rho \text{ s.t. } ((x,y),(r_0,r_0)) \in A \}.
   \]

Example 4.A. Let \( \rho \) be the fuzzy partial order in \( X \) in Example 4.3. Then
   \[
   \rho_x(r_0) = \{ (a,a), (b,b), (c,c), (d,d), (e,e), (c,a), (c,b), (a,c), (a,d), (b,d), (d,e), (c,d), (c,e), (e,d) \}
   \]
   and
   \[
   \rho_x(r_1) = \{ (a,a), (b,b), (c,c), (d,d), (e,e), (a,b), (a,c), (a,d), (a,e), (b,d), (b,e), (c,d), (c,e), (d,e) \}
   \]
are ordinary partial orders in \( X \) with following diagrams:

Furthermore,
   \[
   \rho_1(a) = \rho_1(b) = \rho_1(c) = \rho_1(d) = \rho_1(e) = \{ (r,r) : r \in I \} \cup \{ (r_0,r_1) \}
   \]
are ordinary partial orders in \( I \).

Result 4.B[2, Corollary in p.308]. (1) To each fuzzy partial order \( \rho \) in \( X \) there are associated an ordinary partial order \( \rho_1 \) in \( I \) and an ordinary partial order \( \rho_x \) in \( X \). In fact, \( \rho_1 = \bigcap_{x \in X} \rho_1(x) \) and \( \rho_x = \bigcap_{r \in I} \rho_x(r) \). In this case, \( \rho_1 \) [resp. \( \rho_x \)] is called the partial order in \( I \) [resp. \( X \)] associated to the fuzzy partial order \( \rho \).

(2) If \( \rho \) is a fuzzy total order in \( X \), then \( \rho_1(x) \) [resp. \( \rho_x(r) \)] is a total order in \( I \) [resp. \( X \)] for each \( x \in X \) [resp. \( r \in I \)] in this case, \( \rho_1 \) and \( \rho_x \) are evidently total order relations.

Proposition 4.6. Let \( \{ \rho_\alpha \}_{\alpha \in \Gamma} \) be an indexed family of fuzzy partial orders in \( X \). Then \( \bigcap_{\alpha \in \Gamma} \rho_\alpha \) is a fuzzy partial order in \( X \).

Proof. Let \( \rho = \bigcap_{\alpha \in \Gamma} \rho_\alpha \) and let \( (x,r) \in X \times I \). Since \( \Delta_X \subseteq \rho_\alpha \) for each \( \alpha \in \Gamma \), there exists \( A_\alpha \in \rho_\alpha \) such that \( ((x,x),(r,r)) \in A_\alpha \) for each \( \alpha \in \Gamma \). Then \( ((x,x),(r,r)) \in \bigcap_{\alpha \in \Gamma} A_\alpha \). Thus \( \Delta_X \subseteq \rho \). So \( \rho \) is reflexive. Suppose \( ((y,x),(r,t)) \in A \in \rho \) and \( ((y,x),(t,r)) \in A \in \rho \) for some \( A \in \rho \). Since \( \rho = \bigcap_{\alpha \in \Gamma} \rho_\alpha \), there exists \( A_\alpha \in \rho_\alpha \) such that \( A = \bigcap_{\alpha \in \Gamma} A_\alpha \). Then \( ((y,x),(r,t)) \in A_\alpha \) or \( ((y,x),(t,r)) \in A_\alpha \). Since \( \rho_\alpha \) is antisymmetric, \( x = y \) and \( r = t \). Thus \( \rho \) is antisymmetric. Suppose \( ((x,y),(r,s)) \in A \in \rho \) and \( ((y,z),(s,t)) \in B \in \rho \). Then there exist \( A_\alpha \in \rho_\alpha \) and \( B_\alpha \in \rho_\alpha \) such that \( ((x,y),(r,s)) \in A_\alpha \) and \( ((y,z),(s,t)) \in B_\alpha \) for each \( \alpha \in \Gamma \). Since \( \rho_\alpha \) is transitive for each \( \alpha \in \Gamma \), there exists \( C_\alpha \in \rho_\alpha \) such that \( ((x,z),(r,t)) \in C_\alpha \) for each \( \alpha \in \Gamma \). Let \( C = \bigcap_{\alpha \in \Gamma} C_\alpha \). Then clearly \( ((x,z),(r,t)) \in C \in \rho \). Thus \( \rho \) is transitive. Hence \( \rho \) is a fuzzy partial order in \( X \).

Theorem 4.7. Let \( \rho \) be a fuzzy relation in \( X \). Then \( \rho \) is a fuzzy partial order in \( X \) if and only if \( \rho \cap \rho^{-1} = \Delta_X \) and \( \rho \circ \rho = \rho \).

Proof. \( (\Rightarrow): \) Suppose \( \rho \) is a fuzzy partial order. Since \( \rho \) is reflexive, by Result 1.A(1), \( \rho^{-1} \) is reflexive. Then \( \Delta_X \subseteq \rho \) and \( \Delta_X \subseteq \rho^{-1} \). Thus \( \Delta_X \subseteq \rho \cap \rho^{-1} \). Let \( ((x,y),(r,t)) \in A \in \rho \cap \rho^{-1} \). Then there exist \( B_\alpha \in \rho \) such that \( A = B \cap C \). Thus \( ((x,y),(r,t)) \in B \) and \( ((y,x),(t,r)) \in C \), i.e., \( ((y,x),(t,r)) \in C \). Since \( \rho \) is
5. Fuzzy order preserving fuzzy functions

Definition 5.1. Let $X$ and $Y$ be fuzzy partially ordered sets. Then a fuzzy function $F = (F,x) : X \to Y$ is said to be fuzzy increasing or fuzzy order preserving if it satisfies the following condition: For any $(x,r),(y,t) \in X \times I$,

$$(x,r) \lesssim (y,t) \Rightarrow F(x,r) \lesssim F(y,t).$$

Theorem 5.2. Let $(X,\rho)$ and $(Y,\sigma)$ be fuzzy partially ordered sets, let $F = (F,x) : X \to Y$ be a fuzzy function and let $f_x$ be identical for all $x \in X$. Then $F$ is fuzzy increasing if and only if $F : (X,\rho_x) \to (Y,\sigma_y)$ and $f_x : (I,\rho_I) \to (I,\sigma_I)$ are increasing.

Proof. $(\Rightarrow)$: Suppose $F$ is fuzzy increasing. For any $x,y \in X$, let $(x,y) \in \rho_x$ and let $r_0 \in I$. Then, by Results 4.A(2) and 4.B(1), there exists $A \in \rho$ such that $(x,y),(r_0,r_0) \in A$. Thus $(x,r_0) \lesssim (y,r_0)$.

Since $F$ is fuzzy increasing, $F(x,r_0) \lesssim F(y,r_0)$, i.e., $(F(x),f_x(r_0)) \lesssim (F(y),f_y(r_0))$. Since $f_x$ are identical for all $x \in X$, $f_x(r_0) = f_y(r_0)$. Thus there exists $B \in \sigma$ such that $(F(x),F(y),(f_x(r_0),f_y(r_0)) \in B$.

Now for any $r,t \in I$, let $(r,t) \in \rho_x$ and let $x_0 \in X$. Then, by Results 4.A(1) and 4.B(1), there exists $C \in \rho$ such that $(x_0,x_0), (r,t) \in C$. Thus $(x_0,r) \lesssim (x_0,t)$.

Since $F$ is fuzzy increasing, $F(x_0,r) \lesssim F(x_0,t)$, i.e., $(F(x_0),f_{x_0}(r)) \lesssim (F(x_0),f_{x_0}(t))$. Thus there exists $D \in \sigma$ such that $(F(x_0),F(x_0),(f_{x_0}(r),f_{x_0}(t)) \in D$. So $(f_{x_0}(r),f_{x_0}(t)) \in \sigma_I$. Hence $f_{x_0} : (I,\rho_I) \to (I,\sigma_I)$ is increasing.

$(\Leftarrow)$: Obvious.

Proposition 5.3. Let $F = (F,x) : X \to Y$ and $G = (G,g_y) : Y \to Z$ be fuzzy increasing. If $g_y : I \to I$ is surjective for each $y \in Y$, then $G \circ F : X \to Z$ is fuzzy increasing.

Proof. Suppose $(x,r) \lesssim (y,t)$ for any $(x,r),(y,t) \in X \times I$. Since $F$ is fuzzy increasing, $F(x,r) \lesssim F(y,t)$, i.e., $(F(x),f_x(r)) \lesssim (F(y),f_y(t))$ in $Y$. Since $G$ is fuzzy increasing and $g_y : I \to I$ is surjective for each $y \in Y$, $G((F(x),f_x(r))) \lesssim G((F(y),f_y(t)))$, i.e., $(G(F(x)),g_y(f_x(r))) \lesssim (G(F(y)),g_y(f_y(t)))$ in $Z$. Thus $(G \circ F)(x) \lesssim (G \circ F)(y,t)$. Hence $G \circ F$ is fuzzy increasing.

Definition 5.4. Let $(X,\rho)$ be a fuzzy partially ordered set and let $Y \subseteq P(X)$, where $P(X)$ denote the set of all subsets of $X$. Then $\sigma$ is called a fuzzy partial suborder of $\rho$ if $\sigma = \rho \cap \rho$ is a fuzzy partial order in $Y$, where $\rho \cap \rho = \{A \mid A \subseteq \rho \times \rho : A \in \rho\}$. In this case, $(Y,\sigma)$ is called a fuzzy partially ordered subset of $X$.

Let $Y$ be a fuzzy partially ordered subset of a fuzzy partially ordered set $X$. If $(x,r) \in Y \times I$ and $(y,t) \in Y \times I$, then we let $(x,r) \lesssim (y,t)$ in $Y$ if and only if $(x,r) \lesssim (y,t)$ in $X$.

Let $X$ be a fuzzy partially ordered set. Two elements $(x,r)$ and $(y,t)$ in $Y \times I$ are said to be fuzzy comparable if either $(x,r) \lesssim (y,t)$ or $(y,t) \lesssim (x,r)$; otherwise, they are said to be fuzzy incomparable.

Definition 5.5. Let $X$ be a fuzzy partially ordered set and let $Y$ be a fuzzy partially ordered subset of $X$. Then $Y$ is called a fuzzy fully ordered subset of $X$, or a fuzzy linearly ordered subset of $X$ or a fuzzy chain of $X$ if any two elements of $X \times I$ are fuzzy comparable.

Proposition 5.6. Let $F = (F,x) : X \to Y$ be an increasing fuzzy function and let $f_x : I \to I$ be surjective for each $x \in X$. If $C$ is a fuzzy chain of $X$, then $F(C)$ is a fuzzy chain of $Y$.

Proof. Let $(y,u),(y',v) \in F(C) \times I$. Then there exist $x,x' \in C$ such that $y = F(x)$ and $y' = F(x')$. Since $f_x : I \to I$ is surjective for each $x \in X$, there exist $r,t \in I$ such that $u = f_x(r)$ and $v = f_x(t)$. Since $F$ is a fuzzy function, $F(x,r) = (y,u)$ and $F(x',t) = (y',v)$. Since $C$ is a fuzzy chain of $X$, either $(x,r) \lesssim (x',t)$ or $(x',t) \lesssim (x,r)$. Since $F$ is fuzzy increasing, either $F(x,r) \lesssim F(x',t)$ or $F(x',t) \lesssim F(x,r)$. Thus either $(y,u) \lesssim (y',v)$ or $(y',v) \lesssim (y,u)$. Hence $F(C)$ is a fuzzy chain of $Y$.

Definition 5.7. Let $C$ be a fuzzy partially ordered subset of a fuzzy partially ordered set $X$. Then $C$ is said to be fuzzy convex if it satisfies the following condition: $(a,r),(b,t) \in C \times I$ and $(a,r) \lesssim (x,s)$ imply $(x,s) \in C \times I$.

Proposition 5.8. Let $X$ and $Y$ be fuzzy ordered
Proof. Clearly $F^{-1}(C)$ is a fuzzy ordered subset of $X$. Suppose $(a, r), (b, t) \in F^{-1}(C) \times I$ and $(a, r) \preceq (x, s) \preceq (b, t)$. Since $F = (F, f_x) : X \rightarrow Y$ is an increasing fuzzy function, $F(a, r) \preceq F(x, s) \preceq F(b, t)$, i.e., $(F(a), f_a(r)) \preceq (F(x), f_x(s)) \preceq (F(b), f_b(t))$. Since $(a, r), (b, t) \in F^{-1}(C) \times I$, $(F(a), f_a(r)) \in C \times I$ and $(F(b), f_b(t)) \in C \times I$. Since $C$ is a fuzzy convex ordered subset of $Y$, $(F(x), f_x(s)) \in C \times I$. Thus $(x, s) \in F^{-1}(C) \times I$. Hence $F^{-1}(C)$ is a fuzzy convex ordered subset of $X$. \hfill \Box

Definition 5.9. Let $X$ and $Y$ be fuzzy partially ordered sets. Then a fuzzy function $F = (F, f_x) : X \rightarrow Y$ is called a fuzzy isomorphism if it is fuzzy bijective and satisfies the following condition: For any two elements $(x, r) \in X \times I$ and $(y, t) \in X \times I$,

$$(x, r) \preceq (y, t) \iff F(x, r) \preceq F(y, t).$$

We say that $X$ is fuzzy isomorphic with $Y$, denoted by $X \simeq Y$, if there exists a fuzzy isomorphism from $X$ to $Y$.

Theorem 5.10. Let $X$ and $Y$ be fuzzy partially ordered sets and let $F = (F, f_x) : X \rightarrow Y$ be a fuzzy bijective function. Then $F : X \rightarrow Y$ is a fuzzy isomorphism if and only if $F : X \rightarrow Y$ and $F^{-1} : Y \rightarrow X$ are increasing fuzzy functions.

Proof. Since $F$ is fuzzy bijective, by Theorem 3.9, $F$ is invertible. Then $F^{-1}(F(x, r)) = (x, r)$ for each $(x, r) \in X \times I$.

$(\Rightarrow)$: Suppose $F$ and $F^{-1}$ are fuzzy increasing. If $(x, r) \preceq (y, t)$ in $X$, then clearly $F(x, r) \preceq F(y, t)$. Now if $F(x, r) \preceq F(y, t)$ in $Y$, then $F^{-1}(F(x, r)) \preceq F^{-1}(F(y, t))$ in $X$ since $F^{-1}$ is fuzzy increasing. Thus $(x, r) \preceq (y, t)$ in $X$. So $F$ is a fuzzy isomorphism.

$(\Leftarrow)$: Suppose $F$ is a fuzzy isomorphism. Then clearly $F$ is fuzzy increasing. Let $F(x, r)$ and $F(y, t)$ be any elements of $Y \times I$. Then

$$F(x, r) \preceq F(y, t) \Rightarrow (x, r) \preceq (y, t) \Rightarrow F^{-1}(F(x, r)) \preceq F^{-1}(F(y, t)).$$

So $F^{-1}$ is fuzzy increasing. This completes the proof. \hfill \Box

Theorem 5.11. Let $X, Y, Z$ be fuzzy partially ordered sets.

1. If $F = (F, f_x) : X \rightarrow Y$ is a fuzzy isomorphism, then $F^{-1} = (F^{-1}, f_x^{-1}) : Y \rightarrow X$ is a fuzzy isomorphism.

2. If $F = (F, f_x) : X \rightarrow Y$ is a fuzzy isomorphism, then $F^{-1} = (F^{-1}, f_x^{-1}) : Y \rightarrow X$ is a fuzzy isomorphism.

3. If $F = (F, f_x) : X \rightarrow Y$ and $G = (G, g_y) : Y \rightarrow Z$ are fuzzy isomorphisms, then $G \circ F : X \rightarrow Z$ is a fuzzy isomorphism.

Proof. (1) It is clear that, $id_X : X \rightarrow X$ is fuzzy bijective. Moreover, $id_X$ and $id_X^{-1}$ are fuzzy increasing. So, by Theorem 5.10, $id_X$ is a fuzzy isomorphism.

(2) From Theorem 5.10 and Theorem 3.9, we can easily see that $F^{-1}$ is a fuzzy isomorphism.

(3) By Proposition 5.2, $G \circ F = (G \circ F, g_y \circ f_x) : X \rightarrow Z$ is fuzzy increasing. By Proposition 3.3(3), $(G \circ F)$ is fuzzy bijective. Moreover, we can easily see that $G \circ F^{-1}$ is fuzzy increasing. So, by Theorem 5.10, $G \circ F$ is a fuzzy isomorphism. This completes the proof. \hfill \Box

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