Fuzzy $r$-Quasi Open Set and Fuzzy $r$-Quasi Continuity

Won Keun Min

Department of Statics, Kangwon National University, Chuncheon, 200-701, Korea

Abstract

In this paper, we introduce the concept of fuzzy $r$-quasi open sets which are generalizations of fuzzy $r$-open sets, and obtain some basic properties of such fuzzy sets. Also we introduce and study the concepts of fuzzy $r$-quasi continuous mapping and fuzzy $r$-quasi open(closed) mapping.

Key Words: fuzzy quasi topological space, fuzzy $r$-quasi open set, fuzzy $r$-quasi continuous, fuzzy $r$-quasi open mapping, fuzzy $r$-quasi closed mapping

1. Introduction

Let $X$ be a set and $I = [0, 1]$ be the unit interval of the real line. $I^X$ will denote the set of all fuzzy sets of $X$, $0_X$ and $1_X$ will denote the characteristic functions of the empty set and $X$, respectively, $A^c$ will denote the complement $1_X - A$ of a fuzzy set $A$ of $X$.

A Chang's fuzzy topological space [1] is an ordered pair $(X, \tau)$, where $X$ is a non-empty set and $\tau \subseteq I^X$ satisfying the following conditions:

- (O1) $0_X, 1_X \in \tau$;
- (O2) $\forall A, B \in I^X$, if $A, B \in \tau$, then $(A \cap B) \in \tau$;
- (O3) for every subfamily $\{A_i : i \in J\} \subseteq I^X$, if $A_i \in \tau$, then $\bigcup_{i \in J} A_i \in \tau$.

A smooth topological space [4] is an ordered pair $(X, \tau)$, where $X$ is a non-empty set and $\tau : I^X \rightarrow I$ is a mapping satisfying the following conditions:

- (O1) $\tau(0_X) = \tau(1_X) = 1$;
- (O2) $\forall A, B \in I^X$, $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;
- (O3) for every subfamily $\{A_i : i \in J\} \subseteq I^X$,

Then the mapping $\tau : I^X \rightarrow I$ is called a smooth topology on $X$. The number $\tau(A)$ is called the degree of openness of $A$.

A mapping $\tau^* : I^X \rightarrow I$ is called a smooth cotopology [4] iff the following three conditions are satisfied:

- (C1) $\tau^*(0_X) = \tau^*(1_X) = 1$;
- (C2) $\forall A, B \in I^X$, $\tau^*(A \cup B) \geq \tau^*(A) \wedge \tau^*(B)$;
- (C3) for every subfamily $\{A_i : i \in J\} \subseteq I^X$,

Theorem 1.3 [3]: If $T$ is a fuzzy quasi topology on $X$, then the mapping $T^* : I^X \rightarrow I$, defined by $T^*(A) = T^*(\bar{A'})$ where $\bar{A'}$ denotes the complement of $A$, is a fuzzy quasi cotopology on $X$. And if $T^*$ is a fuzzy quasi cotopology on $X$, then the mapping $T : I^X \rightarrow I$, defined by $T(A) = T^*(\bar{A'})$, is a fuzzy quasi topology on $X$.

Definition 1.1 [3]: A fuzzy quasi topological space (simply, FQTS) is an ordered pair $(X, T)$, where $X$ is a non-empty set and $T : I^X \rightarrow I$ is a mapping satisfying the following conditions:

- (QO1) $T(0_X) = 1$;
- (QO2) $\forall A, B \in I^X$, $T(A \cap B) \geq T(A) \wedge T(B)$;
- (QO3) for every subfamily $\{A_i : i \in J\} \subseteq I^X$, $T(\bigcup_{i \in J} A_i) \geq \bigwedge_{i \in J} T(A_i)$.

Then the mapping $T : I^X \rightarrow I$ is called a fuzzy quasi topology on $X$. The number $T(A)$ is called the degree of quasi openness of $A$.

Chang's fuzzy topology $\Rightarrow$ smooth topology $\Rightarrow$ fuzzy quasi topology

Definition 1.2 [3]: A mapping $T^* : I^X \rightarrow I$ is called a fuzzy quasi cotopology if the following three conditions are satisfied:

- (QC1) $T^*(1_X) = 1$;
- (QC2) $\forall A, B \in I^X$, $T^*(A \cup B) \geq T^*(A) \wedge T^*(B)$;
- (QC3) for every subfamily $\{A_i : i \in J\} \subseteq I^X$, $T^*(\bigcap_{i \in J} A_i) \geq \bigwedge_{i \in J} T^*(A_i)$.

Then the mapping $T^* : I^X \rightarrow I$ is called a fuzzy quasi cotopology on $X$. The number $T^*(A)$ is called the degree of quasi closedness of $A$.
2. Main Results

**Definition 2.1.** Let \((X, T)\) be a FQTS and \(A \in I^X\). Then
(1) The \(r\)-closure of \(A\), denoted by \(qCl_r(A)\), is defined by
\[
qCl_r(A) = \cap \{ K \in I^X : T^+(K) \geq r, A \subseteq K \},
\]
where \(T^+(K) = T(K^c)\).
(2) The \(r\)-interior of \(A\), denoted by \(qInt_r(A)\), is defined by
\[
qInt_r(A) = \cup \{ K \in I^X : T(K) \geq r, A \subseteq K \}.
\]

A fuzzy set \(A\) is said to be fuzzy \(r\)-quasi open if \(T(A) \geq r\), and \(A\) is said to be fuzzy \(r\)-quasi closed if \(T^+(A) \geq r\).

**Theorem 2.2.** Let \((X, T)\) be a FQTS and \(A, B \in I^X\). Then
(1) \(qInt_r(0_X) = 0_X, qInt_r(1_X) = 1_X\).
(2) \(qInt_r(A) \subseteq A, A \subseteq qCl_r(A)\).
(3) \(A \subseteq B \Rightarrow qInt_r(A) \subseteq qInt_r(B), qCl_r(A) \subseteq qCl_r(B)\).

**Proof.** Obvious. \(\square\)

**Theorem 2.3.** Let \((X, T)\) be a FQTS and \(A \in I^X\). Then
(1) \(qCl_r(A)^c = qInt_r(A)^c\).
(2) \(qInt_r(A)^c = qCl_r(A)^c\).

**Proof.** (1) For \(A \in I^X\),
\[
qCl_r(A)^c = (\cap \{ K \in I^X : T^+(K) \geq r, A \subseteq K \})^c
= \cup \{ K^c : K \in I^X, \tau(K^c) \geq r, K^c \subseteq A^c \}
= \cup \{ U \in I^X : \tau(U) \geq r, U \subseteq A^c \}
= qInt_r(A)^c.
\]

(2) It is similar to the proof of (1). \(\square\)

**Lemma 2.4.** Let \((X, T)\) be a FQTS. The statements are hold:
(1) If \(T(A_i) \geq r\) for each \(i \in J\), then \(T(\cup_{i \in J} A_i) \geq r\).
(2) If \(T^+(A_i) \geq r\) for each \(i \in J\), \(T^+(\cap_{i \in J} A_i) \geq r\).

**Proof.** (1) For each \(i \in J\), if \(T(A_i) \geq r\), then \(T(\cup_{i \in J} A_i) \geq r\).
(2) It follows from definition of fuzzy quasi topology. \(\square\)

From Lemma 2.4, the next theorem is easily obtained.

**Theorem 2.5.** Let \((X, T)\) be a FQTS and \(A \in I^X\). Then
(1) \(A\) is fuzzy \(r\)-quasi open iff \(A = qInt_r(A)\).
(2) \(A\) is fuzzy \(r\)-quasi closed iff \(A = qCl_r(A)\).

**Theorem 2.6.** Let \((X, T)\) be a FQTS and \(A, B \in I^X\). Then
(1) \(qInt_r(qInt_r(A)) = qInt_r(A)\).
(2) \(qCl_r(qCl_r(A)) = qCl_r(A)\).

**Proof.** It follows from Theorem 2.5. \(\square\)

**Definition 2.7.** Let \(f : (X, T_1) \rightarrow (Y, T_2)\) be a mapping on FQTSs. Then \(f\) is said to be fuzzy \(r\)-quasi continuous if for every \(A \in I^Y\), we have
\[
T_2(A) \geq r \Rightarrow T_1(f^{-1}(A)) \geq r.
\]

**Theorem 2.8.** Let \((X, T_1)\) and \((Y, T_2)\) be FQTSs. Then the following are equivalent:
(1) \(f\) is fuzzy \(r\)-quasi continuous.
(2) For every fuzzy \(r\)-quasi open set \(A\) in \(Y\), \(f^{-1}(A)\) is fuzzy \(r\)-quasi open in \(X\).
(3) \(T_2^+(B) \geq r \Rightarrow T_1^+(f^{-1}(B)) \geq r\) for \(B \in I^Y\).
(4) For every fuzzy \(r\)-quasi closed set \(A\) in \(Y\), \(f^{-1}(A)\) is fuzzy \(r\)-quasi closed in \(X\).
(5) \(f(qCl_r(A)) \subseteq qCl_r(f(A))\) for \(A \in I^X\).
(6) \(f(qInt_r(f^{-1}(B)) \subseteq f^{-1}(qCl_r(B))\) for \(B \in I^Y\).
(7) \(f^{-1}(qInt_r(f^{-1}(B)) \subseteq qInt_r(f^{-1}(B))\) for \(B \in I^Y\).

**Proof.** (1) \(\Rightarrow (2)\) Let \(A\) be a fuzzy \(r\)-quasi open set. Then \(T_2(A) \geq r\) and so by fuzzy \(r\)-quasi continuity, \(T_1(f^{-1}(A)) \geq r\). Hence \(f^{-1}(A)\) is fuzzy \(r\)-quasi open.
(2) \(\Rightarrow (3)\) For \(B \in I^Y\), if \(T_2^+(B) \geq r\), then \(T_2(B^c) \geq r\), so \(B^c\) is fuzzy \(r\)-quasi open. By (2), \(f^{-1}(B^c)\) is fuzzy \(r\)-quasi open, and this implies\(T_1(f^{-1}(B^c)) = T_1((f^{-1}(B))^c) = T_1^+(f^{-1}(B)) \geq r\). So \(T_1^+(f^{-1}(B)) \geq r\).
(3) \(\Rightarrow (4)\) Obvious.
(4) \(\Rightarrow (5)\) For \(A \in I^X\),\n\[
f^{-1}(qCl_r(f(A)) = f^{-1}([\cap \{ F \in I^Y : f(A) \subseteq F, \quad F\text{ is fuzzy }r\text{-quasi closed}]])
= \cap \{ f^{-1}(F) \in I^X : A \subseteq f^{-1}(F), \quad f^{-1}(F)\text{ is fuzzy }r\text{-quasi closed} \}.
\]

Thus from definition of operator of closure on a FQTS, \(qCl_r(A) \subseteq f^{-1}(qCl_r(f(A)))\). So \(f(qCl_r(A)) \subseteq qCl_r(f(A))\).
(5) \(\Rightarrow (6)\) Obvious.
(6) \(\Rightarrow (7)\) Obvious.
(7) \(\Rightarrow (1)\) For \(B \in I^Y\), if \(T_2(B) \geq r\), then \(B\) is fuzzy \(r\)-quasi open, and
\[
f^{-1}(B) = f^{-1}(qInt_r(B)) \subseteq qInt_r(f^{-1}(B)).
\]
This implies \(f^{-1}(B)\) is fuzzy \(r\)-quasi open, that is, \(T_1(f^{-1}(B)) \geq r\). Hence \(f\) is fuzzy \(r\)-quasi continuous. \(\square\)
Definition 2.9. Let \( f : (X, T_1) \to (Y, T_2) \) be a mapping on FQTS's. Then \( f \) is said to be fuzzy \( r \)-quasi open if for every fuzzy \( r \)-quasi open set \( A \) in \( X \), \( f(A) \) is fuzzy \( r \)-quasi open in \( Y \).

Theorem 2.10. Let \( (X, T_1) \) and \( (Y, T_2) \) be FQTS's. Then the following are equivalent:

1. \( f \) is fuzzy \( r \)-quasi open.
2. For \( A \in I^X \), \( \overline{T_1(A)} \geq r \Rightarrow T_2(f(A)) \geq r \).
3. \( f(qInt_r(A)) \subseteq qInt_r(f(A)) \) for \( A \in I^X \).
4. \( qInt_r(f^{-1}(B)) \subseteq f^{-1}(qInt_r(B)) \) for \( B \in I^Y \).

Proof. \((1) \Leftrightarrow (2)\) It is obvious from definition of fuzzy \( r \)-quasi open set.

\((1) \Rightarrow (3)\) For \( A \in I^X \), \( qInt_r(A) \) is fuzzy \( r \)-quasi open. Since \( f \) is fuzzy \( r \)-quasi open, \( f(qInt_r(A)) \) is fuzzy \( r \)-quasi open. So

\[ f(qInt_r(A)) = qInt_r(f(qInt_r(A))) \subseteq qInt_r(f(A)). \]

\((3) \Rightarrow (4)\) Obvious.

\((4) \Rightarrow (1)\) Let \( A \) be a fuzzy \( r \)-quasi open set. Then from \( (4) \), it follows

\[ qInt_r(A) \subseteq qInt_r(f^{-1}(f(A))) \subseteq f^{-1}(qInt_r(f(A))). \]

Since \( A = qInt_r(A) \), we have \( f(A) \subseteq qInt_r(f(A)) \), and hence from Theorem 2.5, \( f(A) \) is fuzzy \( r \)-quasi open. \( \square \)

Definition 2.11. Let \( f : (X, T_1) \to (Y, T_2) \) be a mapping on FQTS's. Then \( f \) is said to be fuzzy \( r \)-quasi closed if for every fuzzy \( r \)-quasi closed set \( A \) in \( X \), \( f(A) \) is fuzzy \( r \)-quasi closed in \( Y \).

Theorem 2.12. Let \( (X, T_1) \) and \( (Y, T_2) \) be FQTS's. Then the following are equivalent:

1. \( f \) is fuzzy \( r \)-quasi closed.
2. For \( A \in I^X \), \( \overline{T_1(A)} \geq r \Rightarrow \overline{T_2(f(A))} \geq r \).
3. \( qCl_r(f(A)) \subseteq f(qCl_r(A)) \) for \( A \in I^X \).

Proof. \((1) \Leftrightarrow (2)\) Obvious.

\((1) \Rightarrow (3)\) For \( A \in I^X \), \( qCl_r(A) \) is fuzzy \( r \)-quasi closed. Since \( f \) is fuzzy \( r \)-quasi closed, \( f(qCl_r(A)) \) is fuzzy \( r \)-quasi closed. So

\[ qCl_r(f(A)) \subseteq qCl_r(f(qCl_r(A))) = f(qCl_r(A)). \]

\((3) \Rightarrow (1)\) Let \( A \) be a fuzzy \( r \)-quasi closed set. Then from \( (3) \) and \( qCl_r(A) = A \),

\[ qCl_r(f(A)) \subseteq f(qCl_r(A)) = f(A). \]

Thus \( f(A) \) is fuzzy \( r \)-quasi closed. \( \square \)

References


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Won Keun Min received the B.S. degree in mathematics from Kangwon National University, Chunchon, Korea in 1981 and the M.S. and the Ph.D. degrees in mathematics from Korea University, Seoul, Korea in 1983 and 1987, respectively. He is currently a professor in the Department of Mathematics, Kangwon National University. His research interests include general topology and fuzzy topology.