The Properties of Various Concepts

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Abstract

In this paper, we investigate the relations among formal concepts, attribute oriented concept and object oriented concepts on a complete residuated lattice.

Key Words : Complete residuated lattices, order reverse generating maps, isotone (antitone) Galois connection, formal (resp. attribute oriented, object oriented) concepts

1. Introduction and preliminaries

Wille [11] introduced the formal concept lattices by allowing some uncertainty in data. Formal concept analysis is an important mathematical tool for data analysis and knowledge processing [1-5,8,9,11]. A fuzzy context consists of $\mathcal{X} = (X, Y, R)$ where $X$ is a set of objects, $Y$ is a set of attributes and $R$ is a relation between $X$ and $Y$. Bělohlávek [1-4] developed the notion of formal concepts with $\mathcal{B}$elohlávek [1-4] developed the notion of formal concepts.

Let $\mathcal{L} = (L, \leq, \odot)$ be a complete residuated lattice. A order reversing map $\sigma: \mathcal{L} \rightarrow \mathcal{L}$ is called an isotone Galois connection if for each $a, b \in \mathcal{L}$ hold:

1. $\sigma(a \odot b) = \sigma(a) \odot \sigma(b)$
2. $\sigma(0) = 0$
3. $\sigma(1) = 1$

Definition 1.2. [5,8,9] Let $X$ and $Y$ be two sets. Let $\omega^{-}, \phi^{-}, \xi^{-} : L_X \rightarrow L_Y$ and $\omega^{-}, \phi^{-}, \xi^{-} : L_Y \rightarrow L_X$ be operators.

1. The pair $(\omega^{-}, \phi^{-})$ is called antitone Galois connection between $X$ and $Y$ if for $\mu \in L_X$ and $\rho \in L_Y$, $\rho \leq \omega^{-}(\mu)$ if and only if $\omega^{-}(\mu) \leq \rho$.
2. The pair $(\phi^{-}, \xi^{-})$ is called an isotone Galois connection between $X$ and $Y$ if for $\mu \in L_X$ and $\rho \in L_Y$, $\phi^{-}(\mu) \leq \rho$ if and only if $\phi^{-}(\rho) \leq \mu$.

Definition 1.3. [5,8,9] Let $\omega^{-}, \phi^{-}, \xi^{-} : L_X \rightarrow L_Y$ and $\omega^{-}, \phi^{-}, \xi^{-} : L_Y \rightarrow L_X$ be functions. A pair $(\mu, \rho) \in L_X \times L_Y$ is called:

1. formal concept if $\rho = \omega^{-}(\mu)$ and $\mu = \omega^{-}(\rho)$
2. attribute oriented concept if $\rho = \phi^{-}(\mu)$ and $\mu = \phi^{-}(\rho)$
3. object oriented concept if $\rho = \xi^{-}(\mu)$ and $\mu = \xi^{-}(\rho)$

Theorem 1.4. [9] Let $\omega^{-} : L_X \rightarrow L_Y$ and $\omega^{-} : L_Y \rightarrow L_X$ be operators. Let $(\omega^{-}, \omega^{-})$ be an antitone Galois connection between $X$ and $Y$. Then the following properties hold:

1. For each $\mu \in L_X$ and $\rho \in L_X$, $\rho \leq \omega^{-}(\omega^{-}(\mu))$.
2. If $\rho_1 \leq \rho_2$, then $\omega^{-}(\rho_1) \geq \omega^{-}(\rho_2)$.
3. For each $\mu \in L_X$ and $\rho \in L_Y$, $\omega^{-}(\omega^{-}(\rho)) = \omega^{-}(\rho)$ and $\omega^{-}(\omega^{-}(\mu)) = \omega^{-}(\mu)$.
4. For each $\mu_i \in L_X$ and $\rho_j \in L_Y$, $\omega^{-}(\bigvee_{i \in I} \mu_i) = \bigwedge_{i \in I} \omega^{-}(\mu_i)$ and $\omega^{-}(\bigvee_{j \in J} \rho_j) = \bigwedge_{j \in J} \omega^{-}(\rho_j)$.
Theorem 1.5. [8] Let $\phi^- : L^X \to L^Y$ and $\phi^- : L^Y \to L^X$ be operators. Let $(\phi^- , \phi^-)$ be an isotope Galois connection between $X$ and $Y$. For each $\mu, \mu \in L^X$ and $\rho, \rho \in L^Y$, the following properties hold:

1. $\mu \leq \phi^- (\phi^- (\mu))$ and $\phi^- (\phi^- (\mu)) \leq \mu$.
2. If $\mu_1 \leq \mu_2$, then $\phi^- (\mu_1) \leq \phi^- (\mu_2)$. Moreover, if $\mu_1 \leq \mu_2$, then $\phi^- (\mu_1) \leq \phi^- (\mu_2)$.
3. $\phi^- (\phi^- (\mu)) = \phi^- (\mu)$ and $\phi^- (\phi^- (\mu)) = \phi^- (\mu)$.
4. $\phi^- (\bigvee_{\lambda \in \Gamma} \lambda_i) = \bigvee_{\lambda \in \Gamma} \phi^- (\lambda_i)$ and $\phi^- (\bigwedge_{\lambda \in \Gamma} \lambda_i) = \bigwedge_{\lambda \in \Gamma} \phi^- (\lambda_i)$.

Definition 1.6. [8, 9] An operator $\phi^- : L^X \to L^Y$ is called a join-generating operator, denoted by $\phi^- \in J(X,Y)$, if $\phi^- (\bigvee_{\lambda \in \Gamma} \lambda_i) = \bigvee_{\lambda \in \Gamma} \phi^- (\lambda_i)$, for $\{\lambda_i\} \subseteq L^X$.

An operator $\psi^- : L^Y \to L^X$ is called a meet-generating operator, denoted by $\psi^- \in M(X,Y)$, if $\psi^- (\bigwedge_{\lambda \in \Gamma} \lambda_i) = \bigwedge_{\lambda \in \Gamma} \psi^- (\lambda_i)$, for $\{\lambda_i\} \subseteq L^X$.

An operator $\omega^- : L^X \to L^Y$ is called an order reverse-generating operator, denoted by $\omega^- \in \Omega(X,Y)$, if $\omega^- (\bigvee_{\lambda \in \Gamma} \lambda_i) = \bigwedge_{\lambda \in \Gamma} \omega^- (\lambda_i)$, for $\{\lambda_i\} \subseteq L^X$.

Theorem 1.7. [9] For $\omega^- \in K(X,Y)$. Define functions $\phi^- , \xi^- : L^X \to L^Y$. Then the following properties hold:

1. $\omega^- (\lambda) \geq \phi^- (\omega^- (\lambda)) \geq \lambda$ for all $\lambda \in L^X$ and $\rho \in L^Y$.
2. The pair $(\omega^- , \omega^-)$ is an antitone Galois connection and $(\omega^- (\omega^- (\lambda)), \omega^- (\lambda))$ for all $\lambda \in L^X$ are formal concepts.
3. The pair $(\phi^- , \xi^-)$ is an isotope Galois connection and $(\phi^- (\phi^- (\lambda)), \phi^- (\lambda)) = (\omega^- (\omega^- (\lambda)), \omega^- (\lambda))$ for all $\lambda \in L^X$ are attribute oriented concepts.
4. $\xi^- : L^X \to L^Y$ is a join-generating function such that $\xi^- (\mu) = (\omega^- (\mu))^+$ and $\xi^- : L^X \to L^Y$ is a meet-generating function such that $\xi^- (\lambda) = \omega^- (\lambda)^*$.
5. The pair $(\xi^- , \xi^-)$ is an isotope Galois connection and $(\xi^- (\phi^- (\lambda)), \xi^- (\phi^- (\lambda))) = (\omega^- (\phi^- (\lambda))^+, \omega^- (\phi^- (\lambda)))$ for all $\rho \in L^Y$ are object oriented concepts.

Theorem 1.8. [9] Let $(X, Y, R)$ be a fuzzy context. Define a function $\omega^- : L^X \to L^Y$ as follows:

$\omega^- (\lambda) (y) = \bigvee_{x \in X} (\lambda (x) \to R(x, y))$.

Then the following properties hold:

1. $\omega^- \in K(X,Y)$, $(\omega^- , \omega^-)$ is an isotope Galois connections and $(\omega^- (\omega^- (\lambda)), \omega^- (\lambda))$ for all $\lambda \in L^X$ are formal concepts.
2. $\omega^- (\mu) = (\omega^- (\mu))^*$ and $\omega^- (\mu) = \omega^- (\mu)$ where $\omega^- (\mu) = (\omega^- (\mu))^*$ and $\omega^- (\mu) = \omega^- (\mu)$.
3. $\omega^- (\mu) = (\omega^- (\mu))^*$ and $\omega^- (\mu) = \omega^- (\mu)$ for all $\mu \in L^Y$.
4. $\omega^- (\mu) = (\omega^- (\mu))^*$ and $\omega^- (\mu) = \omega^- (\mu)$.
5. $\omega^- (\mu) = (\omega^- (\mu))^*$ and $\omega^- (\mu) = \omega^- (\mu)$. Moreover, the pair $(\omega^- , \omega^-)$ is an isotope Galois connection and $(\omega^- (\omega^- (\lambda)), \omega^- (\lambda))$ are attribute concepts.
6. The pair $(\xi^- , \xi^-)$ is an isotope Galois connection and $(\xi^- (\phi^- (\lambda)), \xi^- (\phi^- (\lambda))) = (\omega^- (\phi^- (\lambda))^+, \omega^- (\phi^- (\lambda)))$ for all $\rho \in L^Y$ are attribute oriented concepts.

Theorem 1.9. [8] For $\phi^- \in J(X,Y)$. Define functions $\omega^- , \xi^- : L^X \to L^Y$. Then the following properties hold:

1. $\phi^- (\lambda) \leq \omega^- (\phi^- (\lambda)) \leq \lambda$ for all $\lambda \in L^X$ and $\rho \in L^Y$.
2. The pair $(\phi^- , \phi^-)$ is an antitone Galois connection and $(\phi^- (\phi^- (\lambda)), \phi^- (\lambda)) = (\omega^- (\omega^- (\lambda)), \omega^- (\lambda))$ for all $\lambda \in L^X$ are attribute oriented concepts.
3. The pair $(\xi^- , \xi^-)$ is an isotope Galois connection and $(\xi^- (\phi^- (\lambda)), \xi^- (\phi^- (\lambda))) = (\omega^- (\phi^- (\lambda))^+, \omega^- (\phi^- (\lambda)))$ for all $\rho \in L^Y$ are object oriented concepts.
2. The properties of various concepts

**Theorem 2.1.** Let \( \mathcal{F}(X, Y) = \{(\mu, \rho) \in L^X \times L^Y\} \) be a family of formal concepts. For all \((\mu_i, \rho_i) \in \mathcal{F}(X, Y)\), we define

\[
\bigvee_{i \in \Gamma} (\mu_i, \rho_i) = (\omega^- (\bigwedge_{i \in \Gamma} \rho_i), \bigwedge_{i \in \Gamma} \rho_i),
\]

\[
\bigwedge_{i \in \Gamma} (\mu_i, \rho_i) = (\bigwedge_{i \in \Gamma} \mu_i, \omega^- (\bigwedge_{i \in \Gamma} \rho_i)).
\]

Then \((\mathcal{F}(X, Y), \vee, \wedge)\) is a complete lattice.

**Proof.** Since \( \bigwedge_{i \in \Gamma} \rho_i \leq \bigwedge_{i \in \Gamma} \rho_i \) by Theorem 1.4(3), \( \omega^- (\bigwedge_{i \in \Gamma} \rho_i) = \omega^- (\bigwedge_{i \in \Gamma} \rho_i) \),

\[
(\bigvee_{i \in \Gamma} (\mu_i, \rho_i)) = (\bigwedge_{i \in \Gamma} \mu_i, \omega^- (\bigwedge_{i \in \Gamma} \rho_i)),
\]

\[
(\bigwedge_{i \in \Gamma} (\mu_i, \rho_i)) = (\bigwedge_{i \in \Gamma} \mu_i, \omega^- (\bigwedge_{i \in \Gamma} \rho_i)).
\]

Hence, \((\bigvee_{i \in \Gamma} (\mu_i, \rho_i), \bigwedge_{i \in \Gamma} (\mu_i, \rho_i)) \in \mathcal{F}(X, Y)\).

**Theorem 2.2.** Let \( \mathcal{G}(X, Y) = \{(\mu, \rho) \in L^X \times L^Y\} \) be a family of attribute oriented concepts. For all \((\mu_i, \rho_i) \in \mathcal{G}(X, Y)\), we define

\[
\bigvee_{i \in \Gamma} (\mu_i, \rho_i) = (\phi^- (\bigwedge_{i \in \Gamma} \rho_i), \bigwedge_{i \in \Gamma} \rho_i),
\]

\[
\bigwedge_{i \in \Gamma} (\mu_i, \rho_i) = (\bigwedge_{i \in \Gamma} \mu_i, \phi^- (\bigwedge_{i \in \Gamma} \rho_i)).
\]

Then \((\mathcal{G}(X, Y), \vee, \wedge)\) is a complete lattice.

**Proof.** Since \( \bigwedge_{i \in \Gamma} \rho_i \leq \bigwedge_{i \in \Gamma} \rho_i \) by Theorem 1.5(3,4), \( \phi^- (\bigwedge_{i \in \Gamma} \rho_i) = \phi^- (\bigwedge_{i \in \Gamma} \rho_i) \),

\[
(\bigvee_{i \in \Gamma} (\mu_i, \rho_i)) = (\bigwedge_{i \in \Gamma} \mu_i, \phi^- (\bigwedge_{i \in \Gamma} \rho_i)),
\]

\[
(\bigwedge_{i \in \Gamma} (\mu_i, \rho_i)) = (\bigwedge_{i \in \Gamma} \mu_i, \phi^- (\bigwedge_{i \in \Gamma} \rho_i)).
\]

Hence, \((\bigvee_{i \in \Gamma} (\mu_i, \rho_i), \bigwedge_{i \in \Gamma} (\mu_i, \rho_i)) \in \mathcal{G}(X, Y)\).

**Theorem 2.3.** Let \( \mathcal{H}(X, Y) = \{(\mu, \rho) \in L^X \times L^Y\} \) be a family of object oriented concepts. For all \((\mu_i, \rho_i) \in \mathcal{H}(X, Y)\), we define

\[
\bigvee_{i \in \Gamma} (\mu_i, \rho_i) = (\bigwedge_{i \in \Gamma} \mu_i, \xi^- (\bigwedge_{i \in \Gamma} \rho_i)),
\]

\[
\bigwedge_{i \in \Gamma} (\mu_i, \rho_i) = (\xi^- (\bigwedge_{i \in \Gamma} \mu_i), \bigwedge_{i \in \Gamma} \rho_i).
\]

Then \((\mathcal{H}(X, Y), \vee, \wedge)\) is a complete lattice.
\[ U(\mu_1, \phi(\mu_1)) \vee U(\mu_2, \phi(\mu_2)) = (\mu_1, \phi(\mu_1)) \vee (\mu_2, \phi(\mu_2)) \]
\[ = (\mu_1 \wedge \mu_2, \phi(\mu_1 \vee \mu_2)) \]
\[ = U(\mu_1 \wedge \mu_2, \phi^{-1}(\mu_1 \vee \mu_2)) \]
\[ = U(\mu_1, \phi^{-1}(\mu_1)) \wedge (\mu_2, \phi^{-1}(\mu_2)) \]
\[ \text{(by Theorem 2.2).} \]

Hence, \( G(X, Y) \) and \( H(X, Y) \) are anti-isomorphic.

**Theorem 2.6.** Let \( \mathcal{F}(X, Y) = \{ (\mu, \rho) \in L^X \times L^Y \} \) and \( G(X, Y) = \{ (\mu, \rho) \in L^X \times L^Y \} \) be two families of formal concepts and object oriented concepts. We define \( V : \mathcal{F}(X, Y) \to G(X, Y) \) as follows:
\[
V(\mu, \rho) = \begin{cases} 
(\phi_{\mu}(\rho^*), \rho), & \text{if } \mu = \omega^{-}(\rho), \\
(\mu, \phi_{\rho}(\mu)), & \text{if } \rho = \omega^{-}(\mu),
\end{cases}
\]
where \( \phi_{\mu}(\rho) = (\omega^{-}(\mu))^* \phi_{\mu}(\rho^*) \) is an anti-tame Galois connection. Then \( \mathcal{F}(X, Y) \) and \( G(X, Y) \) are isomorphic where \( (\phi_{\mu}, \phi_{\rho}) \) is an isotope Galois connection.

**Proof.** Since \( \phi_{\mu}(\rho) \leq \rho \iff (\omega^{-}(\mu))^* \leq \rho \iff \omega^-(\rho^*) \leq \mu \iff \mu \geq \omega^-(\rho) \), for all \( \mu \in L^X, \rho \in L^Y \), \( (\phi_{\mu}, \phi_{\rho}) \) is an isotope Galois connection. Since \( \phi_{\mu}(\phi_{\rho}(\rho^*)) = \phi_{\mu}(\omega^{-}(\rho)) = (\omega^{-}(\omega^{-}(\rho))^*) = \rho^* \) and \( \mu = \omega^{-}(\rho) = \phi_{\rho}(\rho \rho^*) \), then \( (\phi_{\mu}(\rho^*), \rho^*) = (\phi_{\mu}(\rho^*), \phi_{\mu}(\phi_{\rho}(\rho^*))) \in \mathcal{G}(X, Y) \) and \( (\mu, \phi_{\rho}(\mu)) = (\phi_{\mu}(\rho^*), \phi_{\mu}(\phi_{\rho}(\rho^*))) \in G(X, Y) \). Hence \( V \) is well defined. For \( (\phi_{\mu}(\rho^*), \rho) \in \mathcal{G}(X, Y) \), since \( \rho = \phi_{\mu}(\phi_{\rho}(\rho^*)) = (\omega^{-}(\omega^{-}(\rho))^*) = \omega^{-}(\omega^{-}(\rho))^* \), \( (\omega^{-}(\rho^*), \rho^*) = (\omega^{-}(\rho^*), \omega^{-}(\omega^{-}(\rho))^*) \in \mathcal{F}(X, Y) \) such that \( V(\omega^{-}(\rho^*), \rho^*) = (\phi_{\mu}(\rho^*), \rho) \). For \( (\mu, \phi_{\rho}(\mu)) \in \mathcal{G}(X, Y) \), since \( \mu = \phi_{\mu}(\phi_{\rho}(\mu)) = \phi_{\mu}(\omega^{-}(\mu)^*) = \omega^{-}(\omega^{-}(\mu)), (\mu, \omega^{-}(\mu)) = (\omega^{-}(\omega^{-}(\mu)), \omega^{-}(\mu)) \in \mathcal{F}(X, Y) \) such that \( V(\mu, \omega^{-}(\mu)) = (\mu, \phi_{\rho}(\mu)) \). Hence \( V \) is surjective. Trivially, \( V \) is injective.

\[
W(\omega^{-}(\rho_1), \rho_1) \vee W(\omega^{-}(\rho_2), \rho_2) = (\phi_{\mu}(\rho_1^*), \rho_1) \vee (\phi_{\mu}(\rho_2^*), \rho_2) = (\phi_{\mu}(\rho_1^* \vee \rho_2^*), \rho_1 \vee \rho_2) = W(\omega^{-}(\rho_1^* \vee \rho_2^*), \omega^{-}(\rho_1^* \vee \rho_2^*)) \in \mathcal{F}(X, Y) \]

Then \( \mathcal{F}(X, Y) \) and \( \mathcal{H}(X, Y) \) are anti-isomorphic.
Then $\omega^-(\lambda) = \bigvee\{\alpha \in L^X | \omega^-(\alpha) \geq \rho\}$

For $\rho \in L^Y$, we denote $(\rho, \omega^-(\rho)) = ((\rho(u), (\rho(v), (\omega^-(\rho)(a), \omega^-(\rho)(b)))).$

We obtain a formal concept family as follows:

$$\mathcal{F}(X, Y) = \{(0, 0, 0), (1, 1, 1), (0, 1, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 1, 0, 0)\}$$

Let $v$ be dispersable for $\mathcal{F}(X, Y)$ because $\mathcal{F}(X, Y) \nsubseteq \mathcal{F}(X, Y - \{v\})$. $B = \{u, w\}$ is a consistent set for $\mathcal{F}(X, B)$ because $\mathcal{F}(X, Y) \nsubseteq \mathcal{F}(X, B)$ where

$$B = \{(0, 0, 0), (1, 1, 1), (0, 0, 0, 0), (0, 0, 0, 0), (0, 0, 0, 0, 0)\}$$

Moreover, $B$ is a reduct for $\mathcal{F}(X, Y)$ because $B$ is a consistent set and $\mathcal{F}(X, Y) \neq \mathcal{F}(X, B - \{u\})$ and $\mathcal{F}(X, Y) \neq \mathcal{F}(X, B - \{w\})$.

For $\lambda \in L^X$, we denote $(\lambda, \omega^-_{\omega^-(\lambda)}) = ((\lambda(a), \lambda(b), \omega^-_{\omega^-(\lambda)}(a), \omega^-_{\omega^-(\lambda)}(b)), \omega^-_{\omega^-(\lambda)}(w))$, we obtain:

$$((0, 0, 0), (0, 0, 0), (0, 0, 0, 0), (0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0))$$

For $\rho \in L^Y$, we denote $(\rho, \omega^-_{\omega^-(\rho)}) = ((\rho(u), (\rho(v), (\omega^-_{\omega^-(\rho)}(a), \omega^-_{\omega^-(\rho)}(b)))).$
We obtain an object oriented concept logic family as follows:

\[ \mathcal{H}(X, Y) = \{(0, 0), (0, 0, 0), (0, 0, 0, 0), (0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0)\} \]

Then \( \mathcal{F}(X, Y) \cong \mathcal{H}(X, Y) \).

References


