On Fuzzy Weak $r$-minimal Continuity Between Fuzzy Minimal Spaces and Fuzzy Topological Spaces

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Abstract

In this paper, we introduce the concept of fuzzy weakly $r$-minimal continuous function between a fuzzy $r$-minimal space and a fuzzy topological space. We investigate characterizations and some properties for the continuity.

Key Words: $r$-minimal structure, fuzzy $r$-minimal continuous, fuzzy weakly $r$-minimal continuous

1. Introduction

The concept of fuzzy set was introduced by Zadeh [12]. Chang [2] defined fuzzy topological spaces using fuzzy sets. In [3], Chattopadhyay, Hazra and Samanta introduced a smooth topological space which is a generalization of fuzzy topological space. In [11], Yoo et al. introduced a smooth topological space which is a generalization of fuzzy topological space. In [3], Chattopadhyay, Hazra and Samanta introduced the concept of fuzzy $r$-minimal space which is an extension of the smooth topological space. The concept of fuzzy $r$-M continuity was also introduced and investigated in [11]. In [9], the author introduced the concepts of fuzzy $r$-minimal continuous function and fuzzy $r$-minimal open function between fuzzy $r$-minimal spaces and fuzzy topological spaces, and investigate characterizations for such functions. The purpose of this paper is to generalize the concept of fuzzy $r$-minimal continuous function. So, in this paper, we introduce the concept of fuzzy weakly $r$-minimal continuous function between a fuzzy $r$-minimal space and a fuzzy topological space. We investigate characterizations and some properties for the continuity.

2. Preliminaries

Let $I$ be the unit interval $[0,1]$ of the real line. A member $A$ of $I^X$ is called a fuzzy set of $X$. By $\bar{0}$ and $\bar{1}$ we denote constant maps on $X$ with value 0 and 1, respectively. For any $A \in I^X$, $A^c$ denotes the complement $\bar{1} - A$. All other notations are standard notations of fuzzy set theory.

A fuzzy point $x_\alpha$ in $X$ is a fuzzy set $x_\alpha$ defined as follows

$$x_\alpha(y) = \begin{cases} \alpha, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

A fuzzy point $x_\alpha$ is said to belong to a fuzzy set $A$ in $X$, denoted by $x_\alpha \in A$, if $\alpha \leq A(x)$ for $x \in X$. A fuzzy set $A$ in $X$ is the union of all fuzzy points which belong to $A$.

Let $f : X \to Y$ be a function and $A \in I^X$ and $B \in I^Y$. Then $f(A)$ is a fuzzy set in $Y$, defined by

$$f(A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} A(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise}. \end{cases}$$

for $y \in Y$, and $f^{-1}(B)$ is a fuzzy set in $X$, defined by

$$f^{-1}(B)(x) = B(f(x)), x \in X.$$
Then the \((X, M)\) is called a fuzzy \(r\)-minimal space (simply \(r\)-FMS) if \(M\) has a fuzzy \(r\)-minimal structure. Every member of \(M\), is called a fuzzy \(r\)-minimal open set. A fuzzy set \(A\) is called a fuzzy \(r\)-minimal closed set if the complement of \(A\) (simply, \(A^c\)) is a fuzzy \(r\)-minimal open set.

Let \((X, M)\) be an \(r\)-FMS and \(r \in I_0\). The fuzzy \(r\)-minimal closure and the fuzzy \(r\)-minimal interior of \(A\) denoted by \(mC(A, r)\) and \(mI(A, r)\), respectively, are defined as

\[
mC(A, r) = \cap \{ B \in I^X : B^c \in M, A \subseteq B \}
\]
\[
mI(A, r) = \cup \{ B \in I^X : B \in M, B \subseteq A \}
\]

**Theorem 2.3 (11).** Let \((X, M)\) be an \(r\)-FMS and \(A, B \in I^X\).

1. \(mI(A, r) \subseteq A\) and if \(A\) is a fuzzy \(r\)-minimal open set, then \(mI(A, r) = A\).
2. If \(A \subseteq B\), then \(mI(A, r) \subseteq mI(B, r)\) and \(mC(A, r) \subseteq mC(B, r)\).
3. If \(A \subseteq B\), then \(mI(A, r) \subseteq mI(B, r)\) and \(mC(A, r) \subseteq mC(B, r)\).
4. \(mI(A, r) \cap mI(B, r) \subseteq mI(A \cap B, r)\) and \(mC(A, r) \cup mC(B, r) \subseteq mC(A \cup B, r)\).
5. \(mI(mI(A, r), r) = mI(A, r)\) and \(mC(mC(A, r), r) = mC(A, r)\).
6. \(\tilde{1} - mI(A, r) = mI(\tilde{1} - A, r)\) and \(\tilde{1} - mI(A, r) = mC(\tilde{1} - A, r)\).

**Definition 2.4 (9).** Let \((X, M_X)\) be an \(r\)-FMS and \((Y, \sigma)\) a fuzzy topological space. Then \(f : X \rightarrow Y\) is said to be fuzzy \(r\)-minimal continuous if for every fuzzy \(r\)-open set \(A\) in \(Y\), \(f^{-1}(A)\) is fuzzy \(r\)-minimal open in \(X\).

**Theorem 2.5 (9).** Let \(f : X \rightarrow Y\) be a function between an \(r\)-FMS \((X, M_X)\) and a fuzzy topological space \((Y, \sigma)\). Then the following statements are equivalent:

1. \(f\) is fuzzy \(r\)-minimal continuous.
2. \(f^{-1}(B)\) is a fuzzy \(r\)-minimal closed set for each fuzzy \(r\)-open set \(B\).
3. \(f(mC(A, r)) \subseteq cl(f(A), r)\) for \(A \in I^X\).
4. \(f^{-1}(cl(B, r)) \subseteq cl(f^{-1}(B, r))\) for \(B \in I^Y\).
5. \(f^{-1}(cl(B, r)) \subseteq mI(f^{-1}(cl(B, r), r))\) for \(B \in I^Y\).

Then \((1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5)\).

### 3. Fuzzy weakly \(r\)-minimal continuous function

**Definition 3.1.** Let \((X, M_X)\) be an \(r\)-FMS and \((Y, \sigma)\) a fuzzy topological space. Then \(f : X \rightarrow Y\) is said to be fuzzy weakly \(r\)-minimal continuous if for each fuzzy point \(x_a\) and for each fuzzy \(r\)-open set \(V\) with \(f(x_a) \in V\), there exists a fuzzy \(r\)-minimal open set \(U\) such that \(x_a \in U\) and \(f(U) \subseteq cl(V, r)\).

**Remark 3.2.** Let \(f : X \rightarrow Y\) be a function between an \(r\)-FMS \((X, M_X)\) and a fuzzy topological space \((Y, \sigma)\). Then every fuzzy \(r\)-minimal continuous mapping \(f\) is clearly fuzzy weakly \(r\)-minimal continuous but the converse is not always true as shown in the next example.

**Example 3.3.** Let \(X = I\) and let us consider two fuzzy sets \(A, B\) defined as

\[
A(x) = \frac{1}{2} x, \quad x \in I;
\]
\[
B(x) = -\frac{1}{2} (x - 1), \quad x \in I.
\]

Consider a fuzzy family

\[
M_X(U) = \begin{cases} \frac{1}{3}, & \text{if } U = \emptyset, \tilde{1}, \\ \frac{1}{2}, & \text{if } U = A, B, \\ 0, & \text{otherwise}, \end{cases}
\]

and a fuzzy topology

\[
\sigma(U) = \begin{cases} 1, & \text{if } U = \emptyset, \tilde{1}, A, B, \\ \frac{1}{3}, & \text{if } U = A \cap B, A \cup B, \\ 0, & \text{otherwise}. \end{cases}
\]

Then the identity function \(f : (X, M_X) \rightarrow (X, \sigma)\) is fuzzy weakly \(\frac{1}{3}\)-minimal continuous but not fuzzy \(\frac{1}{5}\)-minimal continuous.

**Theorem 3.4.** Let \(f : X \rightarrow Y\) be a function between an \(r\)-FMS \((X, M_X)\) and a fuzzy topological space \((Y, \sigma)\). Then the following statements are equivalent:

1. \(f\) is fuzzy weakly \(r\)-minimal continuous.
2. \(f^{-1}(B) \subseteq mI(f^{-1}(cl(B, r), r))\) for each fuzzy \(r\)-open set \(B\) of \(Y\).
3. \(mC(f^{-1}(int(F, r), r)) \subseteq f^{-1}(F)\) for each fuzzy \(r\)-closed set \(F\) in \(Y\).
4. \(mC(f^{-1}(cl(B, r), r)) \subseteq f^{-1}(cl(B, r))\) for each \(B \in I^Y\).
5. \(f^{-1}(int(B, r)) \subseteq mI(f^{-1}(cl(int(B, r), r))\) for each \(B \in I^Y\).
6. \(mC(f^{-1}(V, r)) \subseteq f^{-1}(cl(V, r))\) for each fuzzy \(r\)-open set \(V\) in \(Y\).

**Proof.**

1. \(\Rightarrow (2)\) Let \(B\) be a fuzzy \(r\)-open set in \(Y\). Since \(f\) is fuzzy weakly \(r\)-minimal continuous, for each \(x_a \in f^{-1}(B)\), there exists a fuzzy \(r\)-minimal open set \(U_{x_a}\) of \(x_a\) such that \(f(U_{x_a}) \subseteq cl(B)\). Now we can say for each \(x_a \in f^{-1}(B)\), there exists a fuzzy \(r\)-minimal open set \(U_{x_a}\) such that

\[
x_a \in U_{x_a} \subseteq f^{-1}(f(U_{x_a})) \subseteq f^{-1}(cl(B, r)).
\]

This implies \(x_a \in mI(f^{-1}(cl(B, r), r))\). Hence

\[
f^{-1}(B) \subseteq mI(f^{-1}(cl(B, r), r)).
\]

2. \(\Rightarrow (1)\) Let \(x_a\) be a fuzzy point in \(X\) and \(V\) a fuzzy \(r\)-open set containing \(f(x_a)\). Then since \(x_a \in \)
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$f^{-1}(V) \subseteq mI(f^{-1}(cl(V, r)), r)$, there exists a fuzzy $r$-minimal open set $U$ containing $x_a$ such that $x_a \in U \subseteq f^{-1}(cl(V, r))$. This implies $f(U) \subseteq f^{-1}(cl(V, r)) \subseteq cl(V, r)$. Hence $f$ is fuzzy weakly $r$-minimal continuous.

(1) $\Rightarrow$ (3) Let $F$ be any fuzzy $r$-closed set of $Y$. Then $\tilde{F} = F$ is a fuzzy $r$-open set in $Y$, from Theorem 2.2 and Theorem 2.4, it follows

$$f^{-1}(\tilde{F} - F) \subseteq mI(f^{-1}(\tilde{F} - F), r) = mI(f^{-1}(\tilde{F} - int(F, r)), r) = \tilde{1} - mC(f^{-1}(int(F, r)), r).$$

Hence we have $mC(f^{-1}(int(F, r)), r) \subseteq f^{-1}(F)$.

(3) $\Rightarrow$ (4) Let $B$ be any fuzzy set in $Y$. Since $cl(B, r)$ is a fuzzy $r$-closed set in $Y$, by (3),

$$mC(f^{-1}(int(cl(B, r)), r), r) \subseteq f^{-1}(cl(B, r)).$$

(4) $\Rightarrow$ (5) For $B \in Y^r$, $f^{-1}(int(B, r)) = \tilde{1} - f^{-1}(cl(\tilde{1} - B, r)) \subseteq \tilde{1} - mC(f^{-1}(int(\tilde{1} - B, r)), r) = mI(f^{-1}(cl(\tilde{1} - B, r)), r), r).$

Thus $f^{-1}(int(B, r)) \subseteq mI(f^{-1}(cl(int(B, r)), r), r)$.

(5) $\Rightarrow$ (6) Let $V$ be any fuzzy $r$-open set of $Y$. Then by (5),

$$\tilde{1} - f^{-1}(cl(V, r)) = f^{-1}(int(\tilde{1} - V, r)) \subseteq mI(f^{-1}(cl(int(\tilde{1} - V, r)), r), r) = mI(\tilde{1} - f^{-1}(cl(V, r)), r), r) \subseteq \tilde{1} = mC(f^{-1}(cl(V, r)), r), r) \subseteq mC(f^{-1}(V), r) \subseteq f^{-1}(cl(V, r)).$$

Hence we have $mC(f^{-1}(V), r) \subseteq f^{-1}(cl(V, r))$.

(6) $\Rightarrow$ (1) Let $V$ be a fuzzy $r$-open set containing $f(x_a)$. By (6),

$$f^{-1}(V) \subseteq f^{-1}(int(cl(V, r), r)) = \tilde{1} - f^{-1}(cl(\tilde{1} - cl(V, r), r)) \subseteq \tilde{1} - mC(f^{-1}(cl(V, r), r), r) \subseteq f^{-1}(cl(V, r), r).$$

It implies $x_a \in mI(f^{-1}(cl(V, r), r), r)$. Thus there exists a fuzzy $r$-minimal open set $U$ such that $M_x \in U \subseteq f^{-1}(cl(V, r))$. Hence $f(U) \subseteq cl(V, r)$.

$$\Box$$

Theorem 3.5. Let $f : X \rightarrow Y$ be a function between an $r$-FMS $(X, M_X)$ and a fuzzy topological space $(Y, \sigma)$. Then the following statements are equivalent:

(1) $f$ is fuzzy weakly $r$-minimal continuous.

(2) $mC(f^{-1}(int(cl(G, r), r)), r) \subseteq f^{-1}(cl(G, r))$ for each fuzzy $r$-open set $G$ in $Y$.

(3) $mC(f^{-1}(int(cl(V, r), r)), r) \subseteq f^{-1}(cl(V, r))$ for each fuzzy $r$-preopen set $V$ in $Y$.

(4) $mC(f^{-1}(int(K, r), r)) \subseteq f^{-1}(K)$ for each fuzzy $r$-regular closed set $K$ in $Y$.

(5) $mC(f^{-1}(int(cl(G, r), r)), r) \subseteq f^{-1}(cl(G, r))$ for each fuzzy $r$-$\beta$-open set $G$ in $Y$.

(6) $mC(f^{-1}(int(cl(G, r), r)), r) \subseteq f^{-1}(cl(G, r))$ for each fuzzy $r$-semiopen set $G$ in $Y$.

Proof. (1) $\Rightarrow$ (2) Let $G$ be a fuzzy $r$-open set of $Y$; then by Theorem 3.4 (3), we have $mC(f^{-1}(int(cl(G, r), r)), r) \subseteq f^{-1}(cl(G, r))$.

(2) $\Rightarrow$ (3) Let $V$ be a fuzzy $r$-preopen of $Y$. Then $V \subseteq int(cl(V, r), r)$. Set $A = int(cl(V, r), r)$. Thus $A$ is a fuzzy $r$-open set, from (2), it follows

$$mC(f^{-1}(int(cl(A, r), r), r)) \subseteq f^{-1}(cl(A, r)).$$

Since $cl(A, r) = cl(V, r)$, we have

$$mC(f^{-1}(int(cl(A, r), r), r)) \subseteq f^{-1}(cl(V, r)).$$

(3) $\Rightarrow$ (4) Let $K$ be a fuzzy $r$-regular closed set of $Y$. Then since $int(K, r)$ is an a fuzzy $r$-preopen set, by (3),

$$mC(f^{-1}(int(cl(int(K, r), r), r)), r) \subseteq f^{-1}(int(cl(K, r), r)).$$

Since $K$ is fuzzy $r$-regular closed and $int(K, r) = int(cl(K, r), r), r)$, we have

$$mC(f^{-1}(int(cl(K, r), r)), r) \subseteq f^{-1}(K, r).$$

(4) $\Rightarrow$ (5) Let $G$ be a fuzzy $r$-$\beta$-open set. Then $G \subseteq cl(int(cl(G, r), r), r) = cl(cl(G, r), r), r)$. So $cl(G)$ is a fuzzy $r$-regular closed set. Hence by (4), we have

$$mC(f^{-1}(int(cl(G, r), r)), r) \subseteq f^{-1}(cl(G, r)).$$

(5) $\Rightarrow$ (6) It is obvious.

(6) $\Rightarrow$ (1) Let $V$ be a fuzzy $r$-open set; then since $V$ is a fuzzy $r$-semiopen set, by (6) and $V \subseteq int(cl(V, r), r)$, we have

$$mC(f^{-1}(V), r) \subseteq mC(f^{-1}(int(cl(V, r), r)), r) \subseteq f^{-1}(cl(V, r)).$$

Hence, by Theorem 3.4 (6), $f$ is fuzzy weakly $r$-minimal continuous.

$\Box$

Definition 3.6. Let $f : X \rightarrow Y$ be a mapping between an $r$-FMS $(X, M_X)$ and a fuzzy topological space $(Y, \sigma)$. Then $f$ is to be fuzzy co-$r$-minimal open if for every fuzzy $r$-minimal open set $A$ in $X$, $f(A)$ is fuzzy $r$-open in $Y$.

Theorem 3.7. Let $f : X \rightarrow Y$ be a function between an $r$-FMS $(X, M_X)$ and a fuzzy topological space $(Y, \sigma)$. Then the following are equivalent:

(1) $f$ is fuzzy co-$r$-minimal open.

(2) $f(mI(A, r)) \subseteq int(f(A, r))$ for $A \in I_X$.

(3) $mI(f^{-1}(B), r) \subseteq f^{-1}(int(B, r))$ for $B \in I_Y$.  

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Let \( A \subseteq \bigcup \mathcal{r} \) be a fuzzy r-minimal set in \( X \). Then, if \( f(A) \) is fuzzy r-open, and hence \( f \) is fuzzy co-r-minimal.

\[
(2) \Rightarrow (3) \quad \text{For } B \in I^Y, \text{ from } (2) \text{ it follows that }
\]
\[
f(mI(f^{-1}(B), r)) \subseteq int(f^{-1}(B), r) \subseteq int(B, r).
\]
Hence we get (3).

Similarly, we get \( (3) \Rightarrow (2) \).

Definition 3.8 ([5]). Let \((X, \mathcal{M}_X)\) be an r-FMS and \( A = \{ A_i \in I^X : i \in J \} \) is called a fuzzy r-minimal cover if \( \bigcup \{ A_i : i \in J \} = \mathbb{1} \). It is a fuzzy r-minimal open cover if each \( A_i \) is a fuzzy r-minimal open set. A subcover of a fuzzy r-minimal cover \( A \) is a subfamily of it which also is a fuzzy r-minimal cover. A fuzzy set \( A \) in \( X \) is said to be fuzzy r-minimal compact (resp., almost fuzzy r-minimal compact, nearly fuzzy r-minimal compact) if for every fuzzy r-minimal open cover \( A = \{ A_i \in I^X : i \in J \} \) of \( A \), there exists \( J_0 = \{ j_1, j_2, \ldots, j_n \} \subseteq J \) such that \( A \subseteq \bigcup_{j \in J_0} A_j \) (resp., \( A \subseteq \bigcup_{j \in J_0} mC(A_j, r) \), \( A \subseteq \bigcup_{j \in J_0} mI(mC(A_j, r), r) \)).

Definition 3.9 ([4]). Let \((X, \tau)\) be a fuzzy topological space. A fuzzy set \( A \) in \( X \) is said to be r-fuzzy compact (resp., r-fuzzy almost compact, r-fuzzy nearly compact) if for every fuzzy r-open cover \( A = \{ A_i \in I^X : \tau(A_i) \geq r, i \in J \} \) of \( A \), there exists \( J_0 = \{ j_1, j_2, \ldots, j_n \} \subseteq J \) such that \( A \subseteq \bigcup_{j \in J_0} A_j \) (resp., \( A \subseteq \bigcup_{j \in J_0} mC(A_j, r) \), \( A \subseteq \bigcup_{j \in J_0} mI(mC(A_j, r), r) \)).

Let \( X \) be a nonempty set and \( \mathcal{M} : I^X \rightarrow I \) a fuzzy family on \( X \). The fuzzy family \( \mathcal{M} \) is said to have the property \((\mathcal{U})\) [11] if for \( A_i \in \mathcal{M} \) (i.e., \( A \subseteq I^X \)), \( \mathcal{M}(\bigcup A_i) \geq \wedge \mathcal{M}(A_i) \).

Theorem 3.10 ([11]). Let \((X, \mathcal{M})\) be an r-FMS with the property \((\mathcal{U})\). Then

1. \( mI(A, r) = A \) if and only if \( A \in \mathcal{M} \), for \( A \in I^X \).
2. \( mC(A, r) = A \) if and only if \( A^c \in \mathcal{M}^c \) for \( A \in I^X \).

Theorem 3.11. Let \( f : X \rightarrow Y \) be a fuzzy weakly r-minimal continuous between an r-FMS \((X, \mathcal{M}_X)\) and a fuzzy topological space \((Y, \sigma)\). If \( A \) is a fuzzy r-minimal compact set and if \( \mathcal{M}_X \) has the property \((\mathcal{U})\), then \( f(A) \) is r-fuzzy almost compact.
References


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