Interval-Valued Fuzzy $m\beta$-continuous mappings on Interval-Valued Fuzzy Minimal Spaces

Won Keun Min

Department of Mathematics, Kangwon National University, Chuncheon, 200-701, Korea

Abstract

We introduce the concepts of interval-valued fuzzy $m\beta$-open sets and interval-valued fuzzy $m\beta$-continuous mappings. And we study some characterizations and properties of such concepts.

Key words: interval-valued fuzzy minimal spaces, interval-valued fuzzy $m\beta$-open sets, interval-valued fuzzy $m\beta$-continuous mappings

1. Introduction

Zadeh [9] introduced the concept of fuzzy set and several researchers were concerned about the generalizations of the concept of fuzzy sets, intuitionistic fuzzy sets [1] and interval-valued fuzzy sets [3]. In [2], Alimohammady and Roohi introduced fuzzy minimal structures and fuzzy minimal spaces and gave some results. In [5], the author introduced the concept of interval-valued fuzzy minimal space (simply, IVF minimal space) as a generalization of interval-valued fuzzy topology introduced by Mondal and Samanta [8]. The concept of interval-valued fuzzy $m\beta$-continuous mappings on between interval-valued fuzzy minimal spaces, which are generalizations of interval-valued fuzzy continuous mappings. The concepts of interval-valued fuzzy $m\alpha$-open set, interval-valued fuzzy $m$-semiopen set and interval-valued fuzzy $m$-preopen set are introduced and investigated. In this paper, we introduce the concepts of interval-valued fuzzy $m\beta$-open sets and interval-valued fuzzy $m\beta$-continuous mappings defined on interval-valued fuzzy minimal spaces. These concepts are generalizations of interval-valued fuzzy $m\beta$-open sets and interval-valued fuzzy $m$-continuity, respectively. We also study characterizations and some basic properties of such concepts.

2. Preliminaries

Let $D[0,1]$ be the set of all closed subintervals of the interval $[0,1]$. The elements of $D[0,1]$ are generally denoted by capital letters $M,N,\cdots$ and note that $M = [M^L, M^U]$, where $M^L$ and $M^U$ are the lower and the upper end points of $M$, respectively. Especially, we denote $\emptyset = [0, 0]$, $I = [1, 1]$, and $a = [a, a]$ for $a \in (0, 1)$. We also note that

\begin{align*}
(1) \quad (M,N \in D[0,1])(M = N ⇔ M^L = N^L, M^U = N^U).
(2) \quad (M,N \in D[0,1])(M \leq N ⇔ M^L \leq N^L, M^U \leq N^U).
\end{align*}

For every $M \in D[0,1]$, the complement of $M$, denoted by $M^c$, is defined by $M^c = 1 - M = [1 - M^L, 1 - M^U]$. Let $X$ be a nonempty set. A mapping $A : X \to D[0,1]$ is called an interval-valued fuzzy set(simply, IVF set) in $X$. For each $x \in X$, $A(x)$ is a closed interval whose lower and upper end points are denoted by $A(x)^L$ and $A(x)^U$, respectively. For any $[a,b] \in D[0,1]$, the IVF set whose value is the interval $[a,b]$ for all $x \in X$ is denoted by $[a,b]$. For a point $p \in X$ and for $[a,b] \in D[0,1]$ with $b > 0$, the IVF set which takes the value $[a,b]$ at $p$ and $0$ elsewhere in $X$ is called an interval-valued fuzzy point (simply, IVF point) and is denoted by $[a,b]_p$. In particular, if $b = a$, then it is also denoted by $a_p$. Denoted by IVF(X) the set of all IVF sets in $X$. An IVF point $M_x$, where $M \in D[0,1]$, is said to belong to an IVF set $A \in X$, denoted by $M_x \in A$, if $A(x)^L \geq M^L$ and $A(x)^U \geq M^U$. In [7], it has been shown that $A = \cup \{M_x : M_x \in A\}$.

For every $A,B \in IVF(X)$, we define

\begin{align*}
A = B ⇔ x \in X, ([A(x)]^L = [B(x)]^L \text{ and } [A(x)]^U = [B(x)]^U),
A \subseteq B ⇔ x \in X, ([A(x)]^L \leq [B(x)]^L \text{ and } [A(x)]^U \leq [B(x)]^U).
\end{align*}

The complement $A^c$ of $A$ is defined by, for all $x \in X$, $[A^c(x)]^L = 1 - [A(x)]^L$ and $[A^c(x)]^U = 1 - [A(x)]^U$.

For a family of IVF sets $\{A_i : i \in J\}$ where $J$ is an index set, the union $G = \cup_{i \in J} A_i$ and the meet $F = \cap_{i \in J} A_i$ are defined by
\[ x \in X, \ (|G(x)|^L = \sup_{x \in J}[A_i(x)]^L, \ [G(x)]^U = \sup_{x \in J}[A_i(x)]^U), \]
\[ x \in X, \ ([F(x)]^L = \inf_{x \in J}[A_i(x)]^L, \ [F(x)]^U = \inf_{x \in J}[A_i(x)]^U), \] respectively.

Let \( f : X \to Y \) be a mapping and let \( A \) be an IVF set in \( X \). Then the image of \( A \) under \( f \), denoted by \( f(A) \), is defined as follows
\[ f(A(y))^L = \left\{ \begin{array}{ll}
\sup_{f^{-1}(y) \neq \emptyset}[A(x)]^L, & \text{if } f^{-1}(y) \neq \emptyset, \\
0, & \text{otherwise},
\end{array} \right.
\]
\[ f(A(y))^U = \left\{ \begin{array}{ll}
\sup_{f^{-1}(y) \neq \emptyset}[A(x)]^U, & \text{if } f^{-1}(y) \neq \emptyset, \\
0, & \text{otherwise},
\end{array} \right.
\]
for all \( y \in Y \).

Let \( B \) be an IVF set in \( Y \). Then the inverse image of \( B \) under \( f \), denoted by \( f^{-1}(B) \), is defined by
\[ [(f^{-1}(B)(x)]^L = [B(f)(x)]^L, \ [f^{-1}(B)(x)]^U = [B(f)(x)]^U \] for all \( x \in X \).

**Definition 2.1 ([5]).** A family \( \mathcal{M} \) of interval-valued fuzzy sets in \( X \) is called an **interval-valued fuzzy minimal structure on** \( X \) if
\[ 0, 1 \in \mathcal{M}. \]
In this case, \((X, \mathcal{M})\) is called an **interval-valued fuzzy minimal space** (simply, **IVF minimal space**). Every member of \( \mathcal{M} \) is called an interval-valued fuzzy m-open set (simply IVF m-open set). An IVF set \( A \) is called an IVF m-closed set if the complement of \( A \) (simply, \( A^c \)) is an IVF m-open set.

Let \((X, \mathcal{M})\) be an IVF minimal space and \( A \) in IVF(X). The IVF minimal-closure and the IVF minimal-interior of \( A \) [5], denoted by \( mC(A) \) and \( mI(A) \), respectively, are defined as
\[ mC(A) = \cap\{B \in IVF(X) : B^c \in \mathcal{M} \text{ and } A \subseteq B\}, \]
\[ mI(A) = \cup\{B \in IVF(X) : B \in \mathcal{M} \text{ and } B \subseteq A\}. \]

**Theorem 2.2 ([5]).** Let \((X, \mathcal{M})\) be an IVF minimal space and \( A, B \) in IVF(X).
\[ (1) \ mI(A) \subseteq A \text{ and if } A \text{ is an IVF m-open set, then } mI(A) = A. \]
\[ (2) \ A \subseteq mC(A) \text{ and if } A \text{ is an IVF m-closed set, then } mC(A) = A. \]
\[ (3) \text{ If } A \subseteq B, \text{ then } mI(A) \subseteq mI(B) \text{ and } mC(A) \subseteq mC(B). \]
\[ (4) \ mI(A) \cap mI(B) \supseteq mI(A \cap B) \text{ and } mC(A) \cup mC(B) \subseteq mC(A \cup B). \]
\[ (5) \ mI(mI(A)) = mI(A) \text{ and } mC(mC(A)) = mC(A). \]
\[ (6) 1 - mC(A) = mI(1 - A) \text{ and } 1 - mI(A) = mC(1 - A). \]

### 3. Interval-valued fuzzy \( m\beta\)-open sets and Interval-valued fuzzy \( m\beta\)-continuous mappings

**Definition 3.1.** Let \( (X, \mathcal{M}) \) be an IVF minimal space and \( A \) in IVF(X). Then an IVF set \( A \) is called an **interval-valued fuzzy \( m\beta\)-open set** (simply IVF \( m\beta\)-open set) in X if
\[ A \subseteq mC(mI(mC(A))). \]

And an IVF set \( A \) is called an **interval-valued fuzzy \( m\beta\)-closed set** (simply IVF \( m\beta\)-closed set) if the complement of \( A \) is an IVF \( m\beta\)-open set.

**Remark 3.2.** Let \((X, \mathcal{M})\) be an IVF minimal space and \( A \) in IVF(X). If \( \mathcal{M} \) is an IVF topology on \( X \), then an IVF \( m\beta\)-open set is IVF semi-preopen [4]

Let \((X, \mathcal{M})\) be an IVF minimal space and \( A \) in IVF(X). Then an IVF set \( A \) is called an IVF \( ma\)-open [7] (resp. **IVF \( m\)-semiopen, IVF \( m\)-preopen** set) in X if \( A \subseteq mI(mC(mI(A))) \) (resp. \( A \subseteq mC(mI(A)), A \subseteq mI(mC(A)) \)).

From the definitions of several types of IVF m-open sets, obviously the following are obtained:

**Lemma 3.3.** Let \((X, \mathcal{M})\) be an IVF minimal space. Then the statements are hold.
\[ (1) \text{ Every IVF } m\text{-semiopen set is IVF } m\beta\text{-open.} \]
\[ (2) \text{ Every IVF } m\text{-preopen set is IVF } m\beta\text{-open.} \]
\[ (3) \text{ A is an IVF } m\beta\text{-closed set if and only if } mI(mC(mI(A))) \subseteq A. \]

**Example 3.4.** Let \( X = [0, 1] \) and let \( A, B \) and \( C \) be IVF sets defined as follows
\[ A(x) = [\frac{1}{2}, \frac{1}{2}(x + 1)] \text{ for all } x \in X; \]
\[ B(x) = [\frac{1}{2}, -\frac{1}{2}(x - 2)] \text{ for all } x \in X; \]
\[ C(x) = [\frac{1}{2}, \frac{1}{2}]; \quad D(x) = [\frac{1}{2}, \frac{1}{2}]; \]
\[ E(x) = [\frac{1}{2}, \frac{1}{2}]; \quad F(x) = [\frac{1}{2}, \frac{1}{2}]. \]
Consider \( \mathcal{M} = \{0, A, B, C, D, E\} \) as an IVF minimal structure on \( X \). Then IVF sets \( E, F \) are IVF \( m\beta\)-open. But \( E \) is not IVF \( m\text{-preopen} \) and \( F \) is not IVF \( m\text{-semiopen}. \)

**IVF m-semiopen**

IVF \( m\)-open \to** IVF \( m\beta\)-open**

**IVF \( m\beta\)-open**

**IVF \( m\)-preopen**

**Theorem 3.5.** Let \((X, \mathcal{M})\) be an IVF minimal space. Any union of IVF \( m\beta\)-open sets is IVF \( m\beta\)-open.
Proof. Let \( A_i \) be an IVF \( m \beta \)-open set for \( i \in J \). Then
\[
A_i \subseteq mC(mI(mC((A_i)))) \subseteq mC(mI(mC(\cup A_i)))).
\]
Thus \( \cup A_i \subseteq mC(mI(mC(\cup A_i))) \).
\( \square \)

Remark 3.6. Let \((X, \mathcal{M})\) be an IVF minimal space. The intersection of any two IVF \( m \beta \)-open sets may not be IVF \( m \beta \)-open set as seen in the next example.

Example 3.7. In Example 3.4, the IVF sets \( A, B \) are IVF \( m \beta \)-open but \( A \cap B \) is not IVF \( m \beta \)-open.

Definition 3.8. Let \((X, \mathcal{M})\) be an IVF minimal space. For \( A \in IVF(X) \), the \( \beta \)-closure and the \( \beta \)-interior, denoted by \( \beta mC(A) \) and \( \beta mI(A) \) in \((X, \mathcal{M})\), respectively, are defined as the following:
\[
\beta mC(A) = \cap \{ F \in IVF(X) : A \subseteq F; F \text{ is IVF } m \beta \text{-closed in } X \},
\]
\[
\beta mI(A) = \cup \{ U \in IVF(X) : U \subseteq A, \text{ U is IVF } m \beta \text{-open in } X \}.
\]

Theorem 3.9. Let \((X, \mathcal{M})\) be an IVF minimal space and \( A, B \in IVF(X) \). Then
\begin{enumerate}
  \item \( \beta mI(A) \subseteq A \).
  \item If \( A \subseteq B \), then \( \beta mI(A) \subseteq \beta mI(B) \).
  \item \( A \) is IVF \( m \beta \)-open iff \( \beta mI(A) = A \).
  \item \( \beta mI(\beta mI(A)) = \beta mI(A) \).
  \item \( \beta mC(1 - A) = 1 - \beta mI(A) \) and \( \beta mI(1 - A) = 1 - \beta mC(A) \).
\end{enumerate}

Proof. (1), (2) Obvious.
(3) It follows from Theorem 3.5.
(4) It follows from (3).
(5) For \( A \in IVF(X) \),
\[
1 - \beta mI(A) = 1 - \cup \{ U : U \subseteq A, \text{U is IVF } m \beta \text{-open} \}
\]
\[
= \cap \{ 1 - U : U \subseteq A, \text{U is IVF } m \beta \text{-open} \}
\]
\[
= \cap \{ 1 - U : X - A \subseteq 1 - U, \text{U is IVF } m \beta \text{-open} \}
\]
\[
= \beta mC(1 - A).
\]
Similarly, it is proved \( \beta mI(1 - A) = 1 - \beta mC(A) \).
\( \square \)

Theorem 3.10. Let \((X, \mathcal{M})\) be an IVF minimal space and \( A, B, F \in IVF(X) \). Then
\begin{enumerate}
  \item \( A \subseteq \beta mC(A) \).
  \item If \( A \subseteq B \), then \( \beta mC(A) \subseteq \beta mC(B) \).
  \item \( F \) is IVF \( m \beta \)-closed iff \( \beta mC(F) = F \).
  \item \( \beta mC(\beta mC(A)) = \beta mC(A) \).
\end{enumerate}

Proof. It is similar to the proof of Theorem 3.9.
\( \square \)

Lemma 3.11. Let \((X, \mathcal{M})\) be an IVF minimal space and \( A, U, V \in IVF(X) \). Then
\begin{enumerate}
  \item \( M_x \in \beta mC(A) \) if and only if \( A \cap V \neq 0 \) for every IVF \( m \beta \)-open set \( V \) containing \( M_x \).
  \item \( M_x \in \beta mI(A) \) if and only if there exists an IVF \( m \beta \)-open set \( U \) such that \( U \subseteq A \).
\end{enumerate}

Proof. (1) Suppose there is an IVF \( m \beta \)-open set \( V \) containing \( M_x \) such that \( A \cap V = 0 \). Then \( X - V \) is an IVF \( m \beta \)-closed set such that \( A \subseteq 1 - V \) and \( M_x \not\in 1 - V \). Thus \( M_x \not\in \beta mC(A) \).

The converse is proved obviously.
(2) Obvious.
\( \square \)

Definition 3.12. Let \((X, \mathcal{M}_X)\) and \((Y, \mathcal{M}_Y)\) be two IVF minimal spaces. Then \( f : (X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y) \) is said to be interval-valued fuzzy \( m \beta \)-continuous (simply, IVF \( m \beta \)-continuous) if for each IVF point \( M_x \) and each IVF \( m \beta \)-open set \( V \) containing \( f(M_x) \), there exists an IVF \( m \beta \)-open set \( U \) containing \( M_x \) such that \( f(U) \subseteq V \).

Let \( f : X \rightarrow Y \) be a mapping on two IVF minimal spaces \((X, \mathcal{M}_X)\) and \((Y, \mathcal{M}_Y)\). Then
\begin{enumerate}
  \item \( f \) is said to be \( m \beta \)-continuous if for every \( A \in \mathcal{M}_Y \), \( f^{-1}(A) \) is in \( \mathcal{M}_X \).
  \item \( f \) is said to be \( m \beta \)-precontinuous if for each IVF point \( M_x \) and each IVF \( m \beta \)-open set \( V \) containing \( f(M_x) \), there exists an IVF \( m \beta \)-open set \( U \) containing \( M_x \) such that \( f(U) \subseteq V \).
  \item \( f \) is said to be \( m \beta \)-continuous if for each IVF point \( M_x \) and each IVF \( m \beta \)-open set \( V \) containing \( f(M_x) \), there exists an IVF \( m \beta \)-open set \( U \) containing \( M_x \) such that \( f(U) \subseteq V \).
\end{enumerate}

We have the following implications but the converses are not always true as shown in the next example.

\( \text{IVF } m \beta \)-cont. \( \rightarrow \) \( \text{IVF } m \beta \)-precont.
\( \text{IVF } m \beta \)-precont. \( \rightarrow \) \( \text{IVF } m \beta \)-cont.

Example 3.13. In Example 3.4, consider \( N = \{0, E, F, 1\} \) and the identity mapping \( f : (X, \mathcal{M}) \rightarrow (X, \mathcal{M}) \).
Theorem 3.14. Let \( f: X \to Y \) be a mapping on IVF minimal spaces \((X, M_X)\) and \((Y, M_Y)\). Then the following are equivalent:

1. \( f \) is IVF \( m\beta\)-continuous.
2. \( f^{-1}(V) \) is an IVF \( m\beta\)-open set for each IVF \( m\) -open set \( V \) in \( Y \).
3. \( f^{-1}(B) \) is an IVF \( m\beta\)-closed set for each IVF \( m\) -closed set \( B \) in \( Y \).
4. \( f(\beta mC(A)) \subseteq mC(f(A)) \) for \( A \in IVF(X) \).
5. \( \beta mC(f^{-1}(B)) \subseteq f^{-1}(mC(B)) \) for \( B \in IVF(Y) \).
6. \( f^{-1}(mI(B)) \subseteq \beta mI(f^{-1}(B)) \) for \( B \in IVF(Y) \).

Proof. (1) \( \Rightarrow \) (2) Let \( V \) be an IVF \( m\) -open set in \( Y \) and \( M_{\bar{X}} \subseteq f^{-1}(V) \). Then there exists an IVF \( m\beta\)-open set \( U_{M_{\bar{X}}} \) containing \( M_{\bar{X}} \) such that \( U_{M_{\bar{X}}} \subseteq V \). So \( M_{\bar{X}} \subseteq f^{-1}(V) \) for every \( M_{\bar{X}} \subseteq f^{-1}(V) \). Hence \( f^{-1}(V) \) is IVF \( m\beta\)-open.

(2) \( \Rightarrow \) (3) Obvious.

(3) \( \Rightarrow \) (4) For any IVF set \( A \) in \( X \),

\[
f^{-1}(mC(f(A))) = f^{-1}(\{F \in IVF(Y) : f(A) \subseteq F \text{ and } F \text{ is IVF } m\text{-closed}\}) = \cap\{f^{-1}(F) : A \subseteq f^{-1}(F) \text{ and } f^{-1}(F) \text{ is IVF } m\beta\text{-closed}\} \supseteq \{K \in IVF(X) : A \subseteq K \text{ and } K \text{ is IVF } m\beta\text{-closed}\} = \beta mC(A).
\]

Hence \( f(\beta mC(A)) \subseteq mC(f(A)) \).

(4) \( \Rightarrow \) (5) Obvious.

(5) \( \Rightarrow \) (6) For \( B \in IVF(Y) \), from \( mI(B) = 1 - mC(1 - B) \) and Theorem 3.9, it follows

\[
f^{-1}(mI(B)) = f^{-1}(1 - mC(1 - B)) = 1 - f^{-1}(mC(1 - B)) \subseteq 1 - \beta mC(f^{-1}(1 - B)) = \beta mI(f^{-1}(B)).
\]

Hence (6) is obtained.

(6) \( \Rightarrow \) (1) Let \( M_{\bar{X}} \) be an IVF point in \( X \) and \( V \) an IVF \( m\) -open set containing \( f(M_{\bar{X}}) \). Since \( V = mI(V) \),

\[
M_{\bar{X}} \subseteq f^{-1}(V) = f^{-1}(mI(V)) \subseteq \beta mI(f^{-1}(V)).
\]

Thus from Lemma 3.11, there exists an IVF \( m\beta\)-open set \( U \) containing \( M_{\bar{X}} \) such that \( M_{\bar{X}} \subseteq f^{-1}(V) \). Hence \( f \) is IVF \( m\beta\)-continuous.

Lemma 3.15. Let \((X, M_X)\) be an IVF minimal space and \( A \in IVF(X) \). Then

1. \( mI(mC(mI(A))) \subseteq mI(mC(\beta mC(A))) \subseteq \beta mC(A) \).
2. \( \beta mI(A) \subseteq mC(mC(\beta mC(A))) \subseteq mC(mI(mI(A))) \).

Proof. The proof is obvious from Theorem 3.9 and Theorem 3.10.

Theorem 3.16. Let \( f: X \to Y \) be a function on IVF minimal spaces \((X, M_X)\) and \((Y, M_Y)\). Then the following are equivalent:

1. \( f \) is IVF \( m\beta\)-continuous.
2. \( f^{-1}(V) \subseteq mC(mI(f^{-1}(V))) \) for each IVF \( m\) -open set \( V \) in \( Y \).
3. \( mI(mC(mI(f^{-1}(F)))) \subseteq f^{-1}(F) \) for each IVF \( m\) -closed set \( F \) in \( Y \).
4. \( f(mI(mI(mI(A)))) \subseteq mC(f(A)) \) for \( A \in IVF(X) \).
5. \( mI(mC(mI(f^{-1}(B)))) \subseteq f^{-1}(mC(B)) \) for \( B \in IVF(Y) \).
6. \( f^{-1}(mI(B)) \subseteq mC(mC(mI(f^{-1}(B)))) \) for \( B \in IVF(Y) \).

Proof. (1) \( \Leftrightarrow \) (2) Obvious.

(2) \( \Leftrightarrow \) (3) Obvious.

(3) \( \Leftrightarrow \) (4) Let \( A \in IVF(X) \). Then from Theorem 3.14 and Lemma 3.15, it follows \( mI(mI(mI(A))) \subseteq \beta mC(A) \subseteq f^{-1}(f(\beta mC(A))) \subseteq f^{-1}(mC(f(A))) \), and so \( f(mI(mI(mI(A)))) \subseteq mC(f(A)) \).

(4) \( \Leftrightarrow \) (5) Obvious.

(5) \( \Leftrightarrow \) (6) From (5), it follows

\[
f^{-1}(mI(B)) = f^{-1}(1 - mC(1 - B)) = 1 - f^{-1}(mC(1 - B)) \subseteq 1 - mI(mC(mI(f^{-1}(1 - B)))) = mC(mI(mI(mI(f^{-1}(1 - B))))).
\]

Hence, (6) is obtained.

(6) \( \Leftrightarrow \) (1) Let \( V \) an IVF \( m\) -open set in \( Y \). Then \( f^{-1}(V) = f^{-1}(mI(V)) \subseteq mC(mI(mC(f^{-1}(V)))) \). This implies \( f^{-1}(V) \) is an IVF \( m\beta\)-open set. Hence \( f \) is IVF \( m\beta\)-continuous.
References


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**Won Keun Min**

He received his Ph.D. degree in the Department of Mathematics from Korea University, Seoul, Korea, in 1987. He is currently a professor in the Department of Mathematics, Kangwon National University. His research interests include general topology and fuzzy topology.

E-mail : wkmin@kangwon.ac.kr