The Linear Discrepancy of a Fuzzy Poset

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Abstract

In 2001, the notion of a fuzzy poset defined on a set $X$ via a triplet $(L, G, I)$ of functions with domain $X \times X$ and range $[0, 1]$ satisfying a special condition $L + G + I = 1$ is introduced by J. Negger and Hee Sik Kim, where $L$ is the ‘less than’ function, $G$ is the ‘greater than’ function, and $I$ is the ‘incomparable to’ function. Using this approach, we are able to define a special class of fuzzy posets, and define the ‘skeleton’ of a fuzzy poset in view of major relation. In this sense, we define the linear discrepancy of a fuzzy poset of size $n$ as the minimum value of all maximum of $I(x, y) | f(x) - f(y) |$ for $f \in F$ and $x, y \in X$ with $I(x, y) > \frac{1}{2}$, where $F$ is the set of all injective order-preserving maps from the fuzzy poset to the set of positive integers. We first show that the definition is well-defined. Then, it is shown that the optimality appears at the same injective order-preserving maps in both cases of a fuzzy poset and its skeleton if the linear discrepancy of a skeleton of a fuzzy poset is 1.

Key Words: Partially ordered set, Fuzzy poset, Linear discrepancy.

1. Introduction

Almost all relationships can be understood by a partial order properties, namely, reflexivity, anti-symmetry and transitivity, in which a practical meaning of “$x$ is less than $y$” is not always true in the real world. In order to agree that the statement is true, it can be accompanied with a condition such as “a certain fraction of the time”. Indeed, “$x$ may be greater than $y$” during a certain period, while it is even possible that $x$ and $y$ are incomparable in a part of the time as well.

In 2001, J. Neggers and Hee Sik Kim introduced three functions $L, G, I$ from $X \times X$ to $[0, 1]$, and they constructed a sort of posets [4]. These three functions $L(x, y), G(x, y)$ and $I(x, y)$ for $(x, y) \in X \times X$ represent three fractions of time when “$x$ is less than $y$”, “$x$ is greater than $y$”, and “$x$ is incomparable to $y$”, respectively. Hence, it is natural to suppose that

$$L(x, y) + G(x, y) + I(x, y) = 1. \quad (1)$$

If “$x$ is less than $y$” is to be an acceptable statement, then we would expect the statement to hold during more than half of the time, i.e., $L(x, y) > \frac{1}{2}$. Similarly, “$x$ is greater than $y$” would require $G(x, y) > \frac{1}{2}$, and “$x$ and $y$ are incomparable” $I(x, y) > \frac{1}{2}$.

In certain situations, the relationship may be asymmetric. For example, suppose that a relation is defined as a person’s ability in a company $X$. A member $x$ of $X$ may feel that he is more able than a member $y$ of $X$ during the most of time, i.e., $L(x, y) < \frac{1}{2}$ and $G(x, y) > \frac{1}{2}$. However, $y$ may also feel that he is more able than $x$ during the most of time, i.e., $L(y, x) < \frac{1}{2}$ and $G(y, x) > \frac{1}{2}$. Hence, $L(x, y) > \frac{1}{2}$ may not be true though $G(y, x) > \frac{1}{2}$. If we accept this majority rule along with the condition that we shall always be able to decide which of these conditions holds, then we suppose that

$$\max \{L(x, y), G(x, y), I(x, y)\} > \frac{1}{2}. \quad (2)$$

Let $P = (X, \leq_P)$ be a poset with a ground set $X$ and a relation $\leq_P$. Then, by the anti-asymmetry, $x \leq_P y$ and $y \leq_P x$ imply $x = y$. In other word, there are no distinct elements $x$ and $y \in X$ such that $x \leq_P y$ and $y \leq_P x$, i.e., for distinct $x$ and $y \in X$, either $x \leq_P y$ or $y \leq_P x$. This can be interpreted, in terms of the fuzziness and the majority rule, that either $L(x, y) > \frac{1}{2}$ or $L(y, x) > \frac{1}{2}$ for distinct $x$ and $y \in X$. In order to emphasize this condition, we consider the condition

$$L(x, y) + L(y, x) \leq 1. \quad (3)$$

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Remark 1.1. For distinct $x$ and $y$ in a set $X$, suppose that $L(x, y)$ is to mean the fraction of occurrences that $x \leq_P y$ for a relation $\leq_P$ on $X$, and satisfies (3). If both $x \leq_P y$ and $y \leq_P x$ hold during more than a half of the time, i.e., $L(x, y) > \frac{1}{2}$ and $L(y, x) > \frac{1}{2}$, then $L(x, y) + L(y, x) > 1$. However, this is a contradiction to (3). Hence, there are no distinct such $x$ and $y$ in $X$, i.e., $x = y$. This means that the asymmetry holds.

Similarly, if $I(x, y) > \frac{1}{2}$ is to reflect that $x$ and $y$ are mostly incomparable, then it is a reasonable requirement to expect some symmetry as in the condition that

$$I(x, y) > \frac{1}{2} \text{ implies } I(y, x) > \frac{1}{2}.$$  

(4)

(4)

Now, we introduce some propositions as follows.

Proposition 1.2. (J. Negger, and Hee sik Kim [4]) Let $X$ be a set, and let three functions $L$, $G$ and $I$ from $X \times X$ to $[0, 1]$ satisfy (1), (2), (3), (4) and (5). Then the followings hold.

1. If $L(x, y) > \frac{1}{2}$, then $G(y, x) > \frac{1}{2}$.
2. If $G(x, y) > \frac{1}{2}$, then $L(y, x) > \frac{1}{2}$.

To introduce transitivity into the structure, we suppose the following property:

$$I(x, y) > \frac{1}{2} \text{ and } L(y, z) > \frac{1}{2} \text{ then } L(x, z) > \frac{1}{2}.$$  

(5)

Remark 1.3. For distinct $x$ and $y$ in a set $X$, suppose that $L(x, y)$ is to mean the fraction of occurrences that $x \leq_P y$ for a relation $\leq_P$ on $X$, and satisfies (5). For $x$, $y$ and $z$ be distinct elements in $X$, suppose that both $x \leq_P y$, and $y \leq_P z$ hold during more than a half of the time, i.e., $L(x, y) > \frac{1}{2}$ and $L(y, z) > \frac{1}{2}$, then from (5), we have $L(x, z) > \frac{1}{2}$. This can be interpreted as $x \leq_P z$ during more than a half of the time. Therefore, (5) represents the transitivity.

From Proposition 1.2, we obtain the following proposition.

Proposition 1.4. (J. Negger, and Hee sik Kim [4]) Let $X$ be a set, and let three functions $L$, $G$ and $I$ from $X \times X$ to $[0, 1]$ satisfy (1), (2), (3), (4) and (5). Then the followings hold.

1. $G(x, y) + G(y, x) \leq 1$.
2. If $G(x, y) > \frac{1}{2}$ and $G(y, z) > \frac{1}{2}$, then $G(x, z) > \frac{1}{2}$.

Remark 1.5. Let $P = (X, \leq_P)$ be a poset. Then we can define three functions $L$, $G$ and $I$ from $X \times X$ to $[0, 1]$ as follows.

$$L(x, y) = \begin{cases} 1, & \text{if } x \leq_P y, \\ 0, & \text{otherwise}, \end{cases}$$

$$G(x, y) = \begin{cases} 1, & \text{if } y \leq_P x, \\ 0, & \text{otherwise}, \end{cases}$$

$$I(x, y) = \begin{cases} 1, & \text{if } x \mid y, \\ 0, & \text{otherwise}. \end{cases}$$

Then, it is clear that $L$, $G$ and $I$ satisfy (1), (2), (3), (4) and (5). We can redefine the poset $P = (X, \leq_P)$ with respect to three fuzzy relations $L$, $G$ and $I$.

From (1), (2), (3), (4) and (5), we can induce the antisymmetry and the transitivity. Using all three fuzzy relations $L$, $G$ and $I$, we can induce a poset $F = (X, L, G, I)$ as follows.

Definition 1.6. Let $X$ be a set, and $L$, $G$ and $I$ maps from $X \times X$ to $[0, 1]$. Then we define a majority rule fuzzy poset induced by fuzzy relations $L$, $G$ and $I$, simply $F$-poset, as $F = (X, L, G, I)$ with the following properties: for distinct $x$, $y$, $z \in X$,

1. $L(x, y) + G(x, y) + I(x, y) = 1$,
2. $\max\{L(x, y), G(x, y), I(x, y)\} > \frac{1}{2}$,
3. $L(x, y) + L(y, x) \leq 1$,
4. $I(y, x) > \frac{1}{2}$ if $I(x, y) > \frac{1}{2}$,
5. $L(x, z) > \frac{1}{2}$ if $L(x, y) > \frac{1}{2}$ and $L(y, z) > \frac{1}{2}$.
6. $L(x, x) = \frac{1}{2}$, $G(x, x) = \frac{1}{2}$, and $I(x, x) = 0$.

Remark 1.7. For a given $P = (X, \leq_P)$, every pair $(x, y) \in X \times X$ always holds the reflexivity during the evolution of $P$ along the time. Hence, it is impossible that $x$ is incomparable to $x$. We define $I(x, x) = 0$ in Definition 1.6. Moreover, in order to satisfy the condition $L(x, x) + G(x, x) + I(x, x) = 1$, we also define $L(x, x) = \frac{1}{2}$ and $G(x, x) = \frac{1}{2}$ in Definition 1.6.

Remark 1.8. Definition 1.6 is introduced by J. Negger and Hee Sik Kim[4] in 2001. (6) in Definition 1.6 can not be found in the original definition of J.Negger and Hee Sik Kim. For the reflexivity, we include (6) in the definition.

In an $F$-poset $F$, when $L$, $G$, or $I$ is greater that $\frac{1}{2}$, the corresponding relation can be emphasized so that we can obtain a new object. The following definition explains the new object.

Definition 1.9. Let $F = (X, L, G, I)$ be an $F$-poset with a ground set $X$, functions $L$, $G$, $I$ from $X \times X$ to $[0, 1]$. Define a new function $L_{sk} : X \times X \rightarrow [0, 1]$ as

$$L_{sk}(x, y) = \begin{cases} 1, & \text{if } L(x, y) > \frac{1}{2}, \\ \frac{1}{2}, & \text{if } L(x, y) = \frac{1}{2}, \\ 0, & \text{if } L(x, y) < \frac{1}{2}. \end{cases}$$

$L_{sk}$ is called a ‘skeleton less than’ function on an $F$-poset $F$. Similarly, we can define a ‘skeleton greater than’ function $G_{sk}$ and a ‘skeleton incomparable to’ function $I_{sk}$ on $F$. In this time, $sk(F) = (X, L_{sk}, G_{sk}, I_{sk})$ is called a skeleton of an $F$-poset $F$. 

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Obviously, we note that sk(F) is also an F-poset.

**Proposition 1.10.** Let F = (X, L, G, I) be an F-poset. Then, sk(F) = (X, ≤_{sk(F)}) is a poset where ≤_{sk(F)} is a relation defined as

1. x ≤_{sk(F)} y if and only if L_{sk}(x, y) = 1 for distinct x, y ∈ X.
2. x ≤_{sk(F)} x for x ∈ X.

**Proof.** Let sk(F) = (X, ≤_{sk(F)}) where ≤_{sk(F)} is a relation defined as

1. x ≤_{sk(F)} y if and only if L_{sk}(x, y) = 1 for distinct x, y ∈ X.
2. x ≤_{sk(F)} x for x ∈ X.

Clearly, sk(F) satisfies the reflexivity.

For distinct x, y ∈ X, suppose that x ≤_{sk(F)} y and y ≤_{sk(F)} x, i.e., L_{sk}(x, y) = 1 and L_{sk}(y, x) = 1. Then we have L(x, y) > \frac{1}{2} and L(y, x) > \frac{1}{2} so that we have G(y, x) > \frac{1}{2} and G(x, y) > \frac{1}{2} by Proposition 1. This implies that L(x, y) + G(x, y) + f(x, y) > \frac{1}{2} + \frac{1}{2} + f(x, y) > 1, which is a contradiction. Hence x = y, i.e., sk(F) satisfies the anti-symmetry.

For distinct x, y, z ∈ X, suppose that x ≤_{sk(F)} y and y ≤_{sk(F)} z, i.e., L_{sk}(x, y) = 1 and L_{sk}(y, z) = 1. Then we have L(x, y) > \frac{1}{2} and L(y, z) > \frac{1}{2} so that we have L(x, z) > \frac{1}{2} since F is an F-poset. This implies that L_{sk}(x, z) = 1, i.e., x ≤_{sk(F)} z. Hence, sk(F) satisfies the transitivity. Therefore, sk(F) is a poset with a relation ≤_{sk(F)} defined as x ≤_{sk(F)} y if L_{sk}(x, y) = 1 for distinct x, y ∈ X.

**Example 1.11.** Let X = {a, b, c, d}. Suppose that a F-poset F = (X, L, G, I) is defined as follows: From the skeleton functions L_{sk}, G_{sk}, and I_{sk} as follows. Then, we can obtain that the relation ≤_{sk(F)} = \{(a, a), (a, c), (b, b), (b, d), (c, c), (d, d)\}, illustrated by a Hesse diagram as in Figure 1, i.e., sk(F) is a disjoint sum of two 2-element chains, i.e., 2 + 2.

<table>
<thead>
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<th>(a,c)</th>
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<td>0.7</td>
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</table>

Figure 1: 2 + 2

2. The Linear Discrepancy of an F-Poset

For positive integers a and b with a ≤ b, [a, b] denotes \{a, a + 1, ..., b\}. Especially, \([b]\) denotes \{1, b\}. The chain of order n, denoted by n = (X, ≤_n), is a poset such that |X| = n and x ≤_n y or y ≤_n x for all x, y ∈ X. For a given poset P = (X, ≤_P), an injective map f : X → \mathbb{Z} is called an isotone if it preserves the order-relation of P, i.e., f(x) ≤ f(y) if x ≤_P y in P. When the image of an isotone f of P = (X, ≤_P) with n = |X| is |n|, we call such f a labeling of P. The tightness of an isotone f on P, written as T_f(P), is the maximum difference between the values of f of incomparable pairs in P. We define T_f(n) = 0 for a chain n. The linear discrepancy of P = (X, ≤_P), written as ld(P), is the minimum tightness over all isotones on P, i.e.,

\[ld(P) = \min_{f ∈ \mathcal{F}} T_f(P) = \min_{f ∈ \mathcal{F}} \max \{|f(x) - f(y)| : x, y ∈ X\}\]

where \(\mathcal{F}\) is the set of all isotones of P. An isotone f on P is called optimal if T_f(P) = ld(P). For an optimal isotone f on P, if f(y) − f(x) = ld(P) for some (x, y) ∈ P, then (x, y) is called an ld-pair of P.

For a given F-poset F = (X, L, G, I), an injective map f : X → \mathbb{Z} is called an fuzzy isotone on F if f(x) ≤ f(y) whenever L(x, y) > \frac{1}{2}. When the image of an isotone f of F = (X, ≤_{sk(F)}) with n = |X| is |n|, we call such f a fuzzy labeling of F. The fuzzy tightness of an isotone f on an F-poset F, written as T_f(F), is the maximum difference between the values of f of incomparable pairs in sk(F).

We define T_f(F) = 0 for an F-poset F whose skeleton is a chain. The fuzzy linear discrepancy of F = (X, L, G, I), written as ld(F), is the minimum fuzzy tightness over all
isotones on $F$, i.e.,
\[
ld(F) = \min_{f \in \mathcal{F}} T_f(F) \\
= \min_{f \in \mathcal{F}} \max \{I(x, y) | f(x) - f(y) | : I(x, y) > \frac{1}{2} \text{ and } x, y \in X \}
\]
where $\mathcal{F}$ is the set of all fuzzy isotones of $F$. A fuzzy isotone $f$ on $F$ is called fuzzy optimal if $T_f(F) = \ld(F)$. For an optimal fuzzy isotone $f$ on $F$, if $f(y) - f(x) = \ld(F)$ for some $(x, y) \in X$, then $(x, y)$ is called an fuzzy ld-pair of $F$.

**Remark 2.1.** In the notation $\ld(R)$, if $R$ is a poset, then $\ld(R)$ means the linear discrepancy of a poset $R$. If $R$ is an F-poset, then $\ld(R)$ implies that the fuzzy linear discrepancy of an F-poset $R$. The same argument can be applied to $T_f(R)$.

The following example shows the linear discrepancy of a poset and the fuzzy linear discrepancy of an F-poset.

**Example 2.2.** Let $X = \{a, b, c, d\}$. Suppose that a F-poset $F = (X, L, G, I)$ is defined as follows: Then, from

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<td>$I$</td>
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By the definition of a skeleton of an F-poset, we can easily obtain the following lemma.

**Lemma 2.3.** For a given F-poset $F = (X, L, G, I)$ and $x, y \in X$, we have $I(x, y) > \frac{1}{2}$ if and only if $|f(x) - f(y)|$ in $\ld$.

**Proposition 2.4.** For a given F-poset $F = (X, L, G, I)$, we have $\ld(F) \leq \ld(sk(F))$.

**Proof.** Let $f$ be an isotone on $sk(X)$. Since $I(x, y) < 1$, we have
\[
I(x, y)|f(x) - f(y)| \leq |f(x) - f(y)|
\]
for all $x, y \in X$. Hence, we have $T_f(F) \leq T_f(sk(F))$. This implies that
\[
\ld(F) \leq \ld(sk(F)).
\]

**Proposition 2.5.** Let $F = (X, L, G, I)$ be an F-poset whose skeleton is an antichain. If $|X| \leq 3$, then $\ld(F) \leq \min \{I(x, y) : x, y \in X\}(|X| - 1)$.

**Proof.** Let $X = \{x_1, \ldots, x_n\}$ and let $I(x_1, x_n)$ be a minimal value of $I$. Suppose that $n \leq 3$. Firstly, if $n = 2$, then it is clear that $\ld(F) \leq \min \{I(x, y) : x, y \in X\}(|X| - 1)$. Suppose that $n = 3$, i.e., $X = \{x_1, x_2, x_3\}$, and $I(x_1, x_3)$ is a minimal value. Then $\ld(sk(X)) = 2$. Define an fuzzy isotone $f$ as $f(x_i) = i$ for $i = 1, 2, 3$. Then we have
\[
I(x_1, x_3)|f(x_1) - f(x_3)| > \frac{1}{2}(n - 1) = 1
\]
since $I(x_1, x_3) > \frac{1}{2}$. By Proposition 2.4, we have
\[
I(x_1, x_3)|f(x_1) - f(x_3)| \geq I(x, y)|f(x) - f(y)|
\]
for $x, y \in X$, i.e.,
\[
T_f(F) = \min \{I(x, y) : x, y \in X\}(|X| - 1).
\]
Since $\ld(F) \leq T_f(F)$, we conclude that
\[
\ld(F) \leq \min \{I(x, y) : x, y \in X\}(|X| - 1).
\]

In general, although $f$ is optimal on $sk(F)$, it can not be fuzzy optimal on an F-poset $F = (X, L, G, I)$. The following example shows this.

**Example 1.11.** $sk(X)$ is $2 + 2$.

(1) An isotone $f : X \rightarrow [4]$ is defined as $f(a) = 1$, $f(b) = 2$, $f(c) = 4$ and $f(d) = 3$. Then $f$ is an optimal isotone on $sk(F)$ so that $\ld(sk(X)) = 2$. Moreover, $(a, d)$ and $(b, c)$ are ld-pairs.

(2) An fuzzy isotone $f : X \rightarrow [4]$ is defined as $f(a) = 1$, $f(b) = 2$, $f(c) = 4$ and $f(d) = 3$. Then $T_f(F) = 1.6$. In fact, $f$ is a fuzzy optimal isotone on $F$ so that $\ld(F) = 1.6$. Moreover, $(a, d)$ and $(b, c)$ are fuzzy ld-pairs.

These are illustrated in Figure 2.
This is a contradiction. Therefore, if \( f \) is optimal isotone on \( F \), and vice versa.

Hence, \( f \) defined in (1) is optimal on \( sk(F) \) but not a fuzzy optimal on \( F \). These are illustrated in Figure 3.

The following theorem tell us that, in the case of \( ld(sk(F)) = 1 \), if \( f \) is a fuzzy optimal isotone on \( F \), then \( f \) is optimal, and vice versa.

**Theorem 2.7.** For a given F-poset \( F = (X, L, G, I) \) with \( ld(sk(F)) = 1 \), a map \( f \) is an optimal isotone on \( sk(F) \) if and only if \( f \) is also a fuzzy optimal isotone on \( F \).

**Proof.** Let \( f \) be a fuzzy optimal isotone on \( F \). Suppose not, i.e., \( f \) is not optimal on \( sk(F) \). Then \( T_f(sk(F)) \geq 2 \), and there are \( x_0, x'_0 \in X \) such that \( x_0 || x'_0 \) in \( sk(F) \) and

\[ |f(x_0) - f(x'_0)| \geq 2. \]

Since \( x_0 || x'_0 \) in \( sk(F) \), we have

\[ I(x_0, x'_0) > \frac{1}{2} \]

so that

\[ I(x_0, x'_0)|f(x_0) - f(x'_0)| > 1, \]

i.e., \( T_f(F) > 1 \). Since \( f \) is fuzzy optimal on \( F \), we have \( ld(F) > 1 \), however, \( ld(F) \leq ld(sk(F)) = 1 \). Hence, we have

\[ 1 < ld(F) \leq ld(sk(X)) = 1. \]

This is a contradiction. Therefore, if \( f \) is fuzzy optimal on \( F \), then \( f \) is optimal on \( sk(F) \).

Let \( f \) be an optimal isotone on \( sk(F) \). Suppose not, i.e., \( f \) is not a fuzzy optimal on \( F \). Let \( g \) be a fuzzy optimal isotone on \( F \). Then \( g \) is an optimal isotone on \( sk(F) \) and \( T_g(F) < T_f(F) \). Since \( g \) is optimal on \( sk(X) \) and \( ld(sk(F)) = 1 \), we have

\[ |g(x) - g(y)| = 1 \]

for all \( x, y \in X \) with \( x || y \) in \( sk(F) \). Similarly, we have

\[ |f(x) - f(y)| = 1 \]

for \( x, y \in X \) with \( x || y \) in \( sk(F) \) since \( f \) is optimal on \( sk(F) \). Hence, we have

\[ |g(x_0) - g(x'_0)| = |f(x_0) - f(x'_0)| \]

so that

\[ I(x, y)|g(x_0) - g(x'_0)| = I(x, y)|f(x_0) - f(x'_0)|. \]

This implies that \( T_f(F) = T_g(F) = ld(F) \). This is a contradiction. Therefore, if \( f \) is optimal on \( sk(F) \), then \( f \) is fuzzy optimal on \( F \).

From Theorem 2.7, for an F-poset \( F = (X, L, G, I) \) with \( ld(sk(F)) = 1 \), fuzzy optimal isotones are exactly same to optimal isotones.

**References**


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