Various Connections and Their Relations

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Abstract

We investigate the properties of Galois, dual Galois, residuated, and dual residuated connections on posets. In particular, we show that their connections are related to relations.

Key Words: Galois, dual Galois, residuated, dual residuated

1. Introduction and Preliminaries

Galois, dual Galois, residuated and dual residuated connections are defined by relationship between posets. Wille [11] introduced the formal concept lattices by allowing some uncertainty in data as examples as Galois, dual Galois, residuated and dual residuated connections. Galois connection analysis is an important mathematical tool for data analysis and knowledge processing [1-5,8,9,11].

In this paper, we investigate the properties of Galois, dual Galois, residuated and dual residuated connections. We find generating functions which induce Galois, dual Galois, residuated and dual residuated connections. In particular, we show that their connections related to relations.

Let X be a set. A relation $e_X \subseteq X \times X$ is called a partially ordered set (simply, poset) if it is reflexive, transitive and anti-symmetric. We can define a poset $P(X) \times P(X)$ as $A, B \in P(X)$ if $A \subseteq B$ for $A, B \in P(X)$. If $(X, e_X)$ is a poset and we define a function $(x, y) \in e_X^1$ if $(y, x) \in e_X$, then $(X, e_X^1)$ is a poset.

Definition 1.1. [10] Let $(X, e_X)$ and $(Y, e_Y)$ be posets and $f : X \rightarrow Y$ and $g : Y \rightarrow X$ maps.

(1) $(e_X, f, g, e_Y)$ is called a Galois connection if for all $x \in X, y \in Y, (y, f(x)) \in e_Y$ if $x, (g(y)) \in e_X$.

(2) $(e_X, f, g, e_Y)$ is called a Galois connection if for all $x \in X, y \in Y, (f(x), y) \in e_Y$ if $(g(y), x) \in e_X$.

(3) $(e_X, f, g, e_Y)$ is called a residuated connection if for all $x \in X, y \in Y, (f(x), y) \in e_Y$ if $x, (g(y)) \in e_Y$.

(4) $(e_X, f, g, e_Y)$ is called a residuated connection if for all $x \in X, y \in Y, (y, f(x)) \in e_Y$ if $(g(y), x) \in e_Y$.

Remark 1.2. [10] Let $(X, e_X)$ and $(Y, e_Y)$ be a poset and $f : X \rightarrow Y$ and $g : Y \rightarrow X$ maps.

(1) $(e_X, f, g, e_Y)$ is a Galois (resp. dual Galois) connection iff $(e_Y, g, f, e_X)$ is a Galois (resp. dual Galois) connection.

(2) $(e_X, f, g, e_Y)$ is a Galois (resp. residuated) connection if $(e_X^{-1}, f, g, e_Y^{-1})$ is a dual (resp. residuated) Galois connection.

(3) $(e_X, f, g, e_Y)$ is a residuated (resp. dual residuated) connection if $(e_Y^{-1}, g, f, e_X^{-1})$ is a residuated (resp. dual residuated) connection.

(4) $(e_X, f, g, e_Y)$ is a Galois connection if $(e_Y, g, f, e_X)$ is a residuated connection.

(5) $(e_X, f, g, e_Y)$ is a residuated connection if $(e_Y, g, f, e_X)$ is a dual residuated connection.

Theorem 1.3. [10] Let $(X, e_X)$ and $(Y, e_Y)$ be a poset and $f : X \rightarrow Y$ and $g : Y \rightarrow X$ maps.

(1) $(e_X, f, g, e_Y)$ is a Galois connection if $f, g$ are antitone maps and $(y, f(g(y))) \in e_Y$ and $(x, g(f(x))) \in e_X$.

(2) $(e_X, f, g, e_Y)$ is a dual Galois connection if $f, g$ are antitone maps and $(g(f(y)), y) \in e_Y$ and $(f(g(x)), x) \in e_X$.

(3) $(e_X, f, g, e_Y)$ is a residuated connection if $f, g$ are isotope maps and $(f(g(y)), y) \in e_Y$ and $(x, g(f(x))) \in e_X$.

(4) $(e_X, f, g, e_Y)$ is a residuated connection if $f, g$ are isotope maps and $(g(f(y)), y) \in e_Y$ and $(g(f(x)), x) \in e_X$.

(5) $(e_X, f, g, e_Y)$ is a Galois connection if $(e_X^{-1}, f, g, e_Y^{-1})$ is a dual residuated Galois connection.

2. Various Connection and Their Relations

Theorem 2.1. Let $(P(X), e_{P(X)})$ and $(P(Y), e_{P(Y)})$ be a poset and $F : P(X) \rightarrow P(Y)$ and $G : P(Y) \rightarrow P(X)$ maps.

(1) $(e_{P(X)}, F, G, e_{P(Y)})$ is a Galois connection iff $F(\bigcup_{i \in \Gamma} A_i) = \bigcap_{i \in \Gamma} F(A_i)$. 

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Let \( \mathcal{P}(X), \mathcal{P}(Y) \) and \( \mathcal{P}(Y) \) be a poset and \( F : \mathcal{P}(X) \to \mathcal{P}(Y) \) and \( G : \mathcal{P}(Y) \to \mathcal{P}(X) \) maps.

1. \((\mathcal{P}(X), F, G, \mathcal{P}(Y))\) is a Galois connection iff there exists \( R \subset X \times Y \) such that

\[
F(A) = \{ y \in Y \mid (\exists x \in X)(x \in A \to (x, y) \in R) \}
\]

\[
G(B) = \{ x \in X \mid (\forall y \in Y)(y \in B \to (x, y) \in R) \}
\]

2. \((\mathcal{P}(X), F, G, \mathcal{P}(Y))\) is a residuated connection iff there exists \( R \subset X \times Y \) such that

\[
F(A) = \{ y \in Y \mid (\forall x \in X)(x \in A \to y \in R) \}
\]

\[
G(B) = \{ x \in X \mid (\exists y \in Y)(y \in B \to x \in R) \}
\]
\[ x \in G(B) \]

iff \( \{y \in F(x) \mid y \in B \} \in e_{P(Y)} \)

iff \( \forall y \in Y [(y \in F(x) \iff y \in B)] \)

iff \( \forall y \in Y [(x, y) \in R \iff y \in B] \)

(\( \Leftarrow \))

\[ (F(A), B) \in e_{P(Y)} \]

iff \( \forall y \in Y [(y \in F(A) \iff y \in B)] \)

iff \( \forall y \in Y [(\exists x \in X)(x \in A \iff (x, y) \in R \iff y \in B)] \)

iff \( \forall y \in X [(\exists x \in X)(x \in A \iff (x, y) \in R \iff y \in B)] \)

iff \( \forall y \in X [(x \in A \iff (y, y) \in R \iff x \in A)] \)

iff \( (A, G(B)) \in e_{P(X)} \).

(3)(\( \Rightarrow \))

\[ y \in F(\{x\}^c)^c \iff (F(\{x\}^c), \{y\}^c) \in e_{P(Y)} \]

iff \( (G(\{y\}^c), \{x\}^c) \in e_{P(X)} \) iff \( x \in G(\{y\}^c)^c \).

Put \( (x, y) \in R \) iff \( y \in F(\{x\}^c) \). By Theorem 2.1 (3), since \( F(\cap_{x \in A} A) = \cap_{x \in A} F(A) \), then

\[ y \in F(A) \]

iff \( y \in F(\bigcup_{x \in A} \{x\}^c) \) iff \( y \in \bigcup_{x \in A} F(\{x\}^c) \)

iff \( (\exists x \in X)(x \in A \iff (x, y) \in R \iff y \in B) \)

iff \( (\forall y \in X [(\exists x \in X)(x \in A \iff (x, y) \in R \iff y \in B)] \)

iff \( (\forall y \in X [(x \in A \iff (y, y) \in R \iff x \in A)] \)

iff \( (A, G(B)) \in e_{P(X)} \).

(\( \Leftarrow \))

\[ (B, F(A)) \in e_{P(Y)} \]

iff \( \forall y \in Y [(y \in B \iff y \in F(A))] \)

iff \( \forall y \in Y [(y \in B \iff (\exists z \in X)(z \in A \iff (z, y) \in R \iff z \in A))] \)

iff \( \forall y \in Y [(\exists z \in X)(z \in A \iff (y, y) \in R \iff z \in A)] \)

iff \( (G(B), A) \in e_{P(X)} \).

Example 2.3. Let \( X = \{a, b, c\} \) and \( Y = \{x, y, z\} \) be posets with relation

\[ R = \{(a, x), (a, z), (b, x), (b, y), (c, z)\}. \]

(1) From Theorem 2.3 (1), \( (e_{P(X)}, F, G, e_{P(Y)}) \) is a Galois connection with

\[ F(\emptyset) = Y, F(\{a\}) = \{x, z\}, F(\{b\}) = \{x, y\}, \]

\[ F(\{c\}) = \{z\}, F(\{a, b\}) = \{x\}, F(\{a, c\}) = \{z\}, \]

\[ F(\{b, c\}) = F(X) = \emptyset, \]

\[ G(\emptyset) = X, G(\{x\}) = \{a, b\}, G(\{y\}) = \{b\}, \]

\[ G(\{z\}) = \{a, c\}, G(\{x, y\}) = \{b\}, G(\{z, x\}) = \{a\}, \]

\[ G(\{y, z\}) = G(Y) = \emptyset. \]

(2) From Theorem 2.3 (2), \( (e_{P(X)}, F, G, e_{P(Y)}) \) is a residuated connection with

\[ F(\emptyset) = \emptyset, F(\{a\}) = \{x, z\}, F(\{b\}) = \{x, y\}, \]

\[ F(\{c\}) = \{z\}, F(\{a, b\}) = \{x\}, F(\{a, c\}) = \{z\}, \]

\[ F(\{b, c\}) = F(X) = Y, \]

\[ G(\emptyset) = G(\{x\}) = G(\{y\}) = \emptyset, \]

\[ G(\{z\}) = \{c\}, G(\{x, y\}) = \{b\}, G(\{z, x\}) = \{a, c\}, \]

\[ G(\{y, z\}) = \{b\}, G(\{y, z\}) = \{a, b\}. \]

(3) From Theorem 2.3 (3), \( (e_{P(X)}, F, G, e_{P(Y)}) \) is a dual Galois connection with

\[ F(\emptyset) = Y, F(\{c\}) = F(\{c\}) = Y, \]

\[ F(\{b\}) = \{x, z\}, F(\{a, c\}) = \{x, y\}, \]

\[ F(\{b, c\}) = \{x, z\}, F(\{a, c\}) = \{x, y\}, \]

\[ G(\emptyset) = G(\{x\}) = G(\{y\}) = X, G(\{z\}) = \{a, b\}, \]

\[ G(\{y, z\}) = \{a, c\}, G(\{x, y\}) = \{b\}, \]

\[ G(\{z, x\}) = \{b\}, G(\{y, z\}) = \{a, b\}. \]

(4) From Theorem 2.3 (4), \( (e_{P(X)}, F, G, e_{P(Y)}) \) is a dual residuated connection with

\[ F(\emptyset) = \emptyset, F(\{a\}) = F(\{c\}) = \emptyset, \]

\[ F(\{b\}) = \{y\}, F(X) = \emptyset, F(\{a, c\}) = \{z\}, \]

\[ F(\{b, c\}) = \{x, y\}, F(\{b, c\}) = \{y\}, \]

\[ G(\emptyset) = \emptyset, G(\{x\}) = \{a, b\}, G(\{y\}) = \{b\}, \]

\[ G(\{z\}) = \{a, c\}, G(\{x, y\}) = \{a, b\}, \]

\[ G(\{z, x\}) = G(\{y, z\}) = G(Y) = X. \]
Thus \( F \) is a Galois connection iff there exists \( F : P(X) \to P(Y) \) with \( F(\{x\}) = \gamma_x \) such that
\[
F(A) = \{ y \in Y \mid (\forall x \in X)(x \in A \to y \in \gamma_x) \}.
\]

(2) \((\Rightarrow)\) Since \( (\{x\}, F(\{x\})) \in \epsilon_{P(Y)} \) iff \((\{x\}, G(\{y\})) \in \epsilon_{P(Y)} \), then \( y \in (F(\{x\})^{c} \) iff \( x \in (G(\{y\})^{c}) \). Thus,
\[
y \in F(A)^{c} \quad \text{iff} \quad (F(A), y^{c}) \in \epsilon_{P(Y)}
\]
\[
A, G(\{y^{c}\}) \in \epsilon_{P(X)}
\]
\[
\vdash (\forall x \in X)(x \in A \to y \in G(\{y^{c}\})
\]
\[
\vdash (\forall x \in X)(x \in A \to y \in F(\{x\}))
\]
Thus, \( y \in F(A)^{c} \) iff \( (F(A), y^{c}) \in \epsilon_{P(Y)} \).

\((\Leftarrow)\) Since \( F(\{x\}), y^{c} \in \epsilon_{P(Y)} \) iff \( G(\{y\}), x^{c} \in \epsilon_{P(X)} \), \( y \in F(\{x\})^{c} \) iff \( x \in G(\{y\})^{c} \). Thus,
\[
y \in F(A)^{c} \quad \text{iff} \quad (F(A), y^{c}) \in \epsilon_{P(Y)}
\]
\[
A, G(\{y^{c}\}) \in \epsilon_{P(X)}
\]
\[
\vdash (\forall x \in X)(x \in A \to y \in G(\{y^{c}\})
\]
\[
\vdash (\forall x \in X)(x \in A \to y \in F(\{x\}))
\]
Hence \( y \in F(A)^{c} \) if \( (\forall x \in X)(x \in A \to y \in F(\{x\})) \).

\((\Leftarrow)\) Since \( F(\{x\}), y^{c} \in \epsilon_{P(Y)} \) iff \( G(\{y\}), x^{c} \in \epsilon_{P(X)} \), \( y \in F(\{x\})^{c} \) iff \( x \in G(\{y\})^{c} \). Thus,
\[
y \in F(A)^{c} \quad \text{iff} \quad (F(A), y^{c}) \in \epsilon_{P(Y)}
\]
\[
A, G(\{y^{c}\}) \in \epsilon_{P(X)}
\]
\[
\vdash (\forall x \in X)(x \in A \to y \in G(\{y^{c}\})
\]
\[
\vdash (\forall x \in X)(x \in A \to y \in F(\{x\}))
\]
\[
\vdash (\forall x \in X)(x \in A \to y \in G(\{y^{c}\})
\]
\[
\vdash (\forall x \in X)(x \in A \to y \in F(\{x\}))
\]
Hence \( y \in F(A)^{c} \) if \( (\forall x \in X)(x \in A \to y \in F(\{x\})) \).

\((\Rightarrow)\) Since \( (\{x\}, F(\{x\})) \in \epsilon_{P(Y)} \) iff \((\{x\}, G(\{y\})) \in \epsilon_{P(Y)} \), then \( y \in (F(\{x\})^{c} \) iff \( x \in (G(\{y\})^{c}) \). Thus,
\[
y \in F(A)^{c} \quad \text{iff} \quad (F(A), y^{c}) \in \epsilon_{P(Y)}
\]
\[
A, G(\{y^{c}\}) \in \epsilon_{P(X)}
\]
\[
\vdash (\forall x \in X)(x \in A \to y \in G(\{y^{c}\})
\]
\[
\vdash (\forall x \in X)(x \in A \to y \in F(\{x\}))
\]
\[
\vdash (\forall x \in X)(x \in A \to y \in G(\{y^{c}\})
\]
\[
\vdash (\forall x \in X)(x \in A \to y \in F(\{x\}))
\]
Hence \( y \in F(A)^{c} \) if \( (\forall x \in X)(x \in A \to y \in F(\{x\})) \).

\((\Leftarrow)\) Since \( F(\{x\}), y^{c} \in \epsilon_{P(Y)} \) iff \( G(\{y\}), x^{c} \in \epsilon_{P(X)} \), \( y \in F(\{x\})^{c} \) iff \( x \in G(\{y\})^{c} \). Thus,
\[
y \in F(A)^{c} \quad \text{iff} \quad (F(A), y^{c}) \in \epsilon_{P(Y)}
\]
\[
A, G(\{y^{c}\}) \in \epsilon_{P(X)}
\]
\[
\vdash (\forall x \in X)(x \in A \to y \in G(\{y^{c}\})
\]
\[
\vdash (\forall x \in X)(x \in A \to y \in F(\{x\}))
\]
\[
\vdash (\forall x \in X)(x \in A \to y \in G(\{y^{c}\})
\]
\[
\vdash (\forall x \in X)(x \in A \to y \in F(\{x\}))
\]
(F(A), B) \in e_{P(Y)}
\iff (\forall y \in Y)((\forall z \in X)(z \in A^c \& y \in F\{z\}^c)
\rightarrow y \in B)
\iff (\forall y \in Y)((\forall z \in X)(z \in A^c \rightarrow (y \in F\{z\}^c)
\rightarrow y \in B)
\iff (\forall y \in Y)((\forall z \in X)((z \in A^c \rightarrow (y \in F\{z\}^c)
\rightarrow y \in B)
\iff (\forall y \in Y)((\forall z \in X)(y \in F\{z\}^c) \rightarrow z \in A)
\iff (G(B), A) \in e_{P(X)}.

(4) \Rightarrow Since (\{y\}, F\{x\}^c) \in e_{P(Y)} \iff
(G\{y\}, \{x\}^c) \in e_{P(X)}, y \in F\{x\}^c \iff x \in G\{y\}^c).
Thus,
y \in F(A) \iff (\{y\}, F(A)) \in e_{P(Y)}
\iff (G\{y\}, A) \in e_{P(X)}
\iff (\forall x \in X)(x \in G\{y\} \rightarrow (x \in A))
\iff (\forall x \in X)(x \in A^c \rightarrow y \in G\{y\}^c)
\iff (\forall x \in X)(x \in A^c \rightarrow y \in F\{x\}^c)
\iff (\forall x \in X)(y \in F\{x\}^c) \rightarrow z \in A)

(\Leftarrow) Since F(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} F(A_i), we define
G(B)
= \bigcap_{i \in I} C_i \mid B \rightarrow F(C_i)
= \bigcap_{i \in I} C_i \mid (\forall y \in Y)(y \in B \rightarrow (\forall z \in X)(z \in C_i)\rightarrow y \in F\{z\}^c)
= \bigcap_{i \in I} C_i \mid (\forall y \in Y)((\forall z \in X)(y \in B \& y \in F\{z\}^c)
\rightarrow z \in C_i)\rightarrow C_i)
= \bigcap_{i \in I} C_i \mid (\forall z \in X)((\exists y \in Y)(y \in B \& y \in F\{z\}^c)
\rightarrow z \in C_i)
= \{x \in X \mid (\exists y \in Y)(y \in B \& y \in F\{x\}^c)\}.

(B, F(A)) \in e_{P(Y)}
\iff (\forall y \in Y)(y \in B \rightarrow y \in F(A))
\iff (\forall y \in Y)(y \in B \rightarrow (\forall z \in X)(z \in F\{z\}^c)
\rightarrow z \in A))
\iff (G(B), A) \in e_{P(X)}.

\square

Example 2.5. Let (X = \{a, b, c\}, e_{P(X)}) and (Y =
\{x, y, z\}, e_{P(Y)}) be posets. Define F_i : P(X) \rightarrow P(Y)
for i = 1, 2 with
F_1(\{a\}) = \{x\}, F_1(\{b\}) = \{x, y\}, F_1(\{c\}) = \{y, z\},
F_2(\{a, b\}) = \{x\}, F_2(\{a, c\}) = \{x, y\}, F_2(\{b, c\}) = \{y, z\}.

(1) From Theorem 2.4 (1), (e_{P(X)}, F_1, G_1, e_{P(Y)}) is a
Galois connection with
F_1(\emptyset) = Y, F_1(\{a\}) = \{x\}, F_1(\{b\}) = \{x, y\},
F_1(\{c\}) = \{y, z\}, F_1(\{a, b\}) = \{x\}, F_1(\{b, c\}) = \{y\},
F_1(\{a, c\}) = F(X) = \emptyset
G_1(\emptyset) = X, G_1(\{x\}) = \{a, b\}, G_1(\{y\}) = \{b, c\},
G_1(\{z\}) = \{c\}, G_1(\{x, y\}) = \{b\}, G_1(\{y, z\}) = \{c\},
G_1(\{x, z\}) = G_1(Y) = \emptyset.

(2) From Theorem 2.4 (2), (e_{P(X)}, F_1, G_1, e_{P(Y)}) is a
residuated connection with
F_1(\emptyset) = 0, F_1(\{a\}) = \{a\}, F_1(\{b\}) = \{x, y\},
F_1(\{c\}) = \{y, z\}, F_1(\{a, b\}) = \{a\}, \emptyset,
F_1(\{a, c\}) = F_1(\{b, c\}) = F_1(\{x\}) = Y,
G_1(\emptyset) = G_1(\{x\}) = G_1(\{z\}) = 0,
G_1(\{x\}) = \{a\}, G_1(\{x, y\}) = \{a, b\}, G_1(\{z, x\}) = \{a\},
G_1(\{y, z\}) = \{c\}, G_1(Y) = X.

(3) From Theorem 2.4 (3), (e_{P(X)}, F_2, G_2, e_{P(Y)}) is a
dual Galois connection with
F_2(\emptyset) = Y, F_2(\{a\}) = \{a\}, F_2(\{b\}) = \{x\}, Y,
F_2(\{c\}) = \{y\}, F_2(\{a, b\}) = \{x, y\}, F_2(\{a, b\}) = \{x\},
F_2(\{b, c\}) = \{y\}, F_2(\{x\}) = \emptyset,
G_2(\emptyset) = G_2(\{y\}) = G_2(\{z\}) = X,
G_2(\{x\}) = \{a\}, G_2(\{y, x\}) = \{a, b\}, G_2(\{z, x\}) = \{a, b\},
G_2(\{y, z\}) = \{b, c\}, G_2(Y) = \emptyset.

(4) From Theorem 2.4 (4), (e_{P(X)}, F, G, e_{P(Y)}) is a
residuated connection with
F_2(\emptyset) = 0, F_2(\{b\}) = \emptyset, F_2(\{a\}) = \{x\},
F_2(\{c\}) = \{y\}, F_2(\{a, c\}) = \{x, y\}, F_2(\{a, b\}) = \{x\},
F_2(\{b, c\}) = \{y\}, F_2(\{x\}) = Y,
G_2(\emptyset) = \emptyset, G_2(\{x\}) = \{a\}, G_2(\{y\}) = \{c\},
G_2(\{z\}) = \{b, c\}, G_2(\{x, y\}) = \{a, c\}, G_2(\{y, z\}) = \{b, c\},
G_2(\{z, x\}) = X, G_2(Y) = X.

References

lattices," J. Logic and Computation, vol. 10, no

connections," Math. Logic Quart., vol. 47, pp.111-116,

2004.

Publisher, New York, 2002.


