Fuzzy Mappings and Fuzzy Equivalence Relations

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Abstract

Equivalence relations and mappings for crisp sets are very well known. This paper attempts an investigation of equivalence relations and mappings for fuzzy sets. We list some concepts and results related to fuzzy relations. We give some examples corresponding to the concept of fuzzy equality and fuzzy mapping introduced by Demirci [1]. In addition, we introduce the notion of preimage and quotient of fuzzy equivalence relations. Finally, we investigate relations between a fuzzy equivalence relation and a fuzzy mapping.

Key Words: fuzzy mapping, fuzzy relation, fuzzy equivalence relation, fuzzy quotient of fuzzy mapping by fuzzy equivalence relation.

1. Introduction

The notion of fuzzy sets in a set generalises that of crisp subsets, and Zadeh [9] introduced it as an approach to a mathematical representation of vagueness in everyday language. Also a fuzzy relation between $X$ and $Y$ as a fuzzy set in $X \times Y$ was proposed by Zadeh [9]. Later he studied similarity relations in [10]. Subsequently, Goguen [2], Murali [3] and Ovchinnikov [5], etc., have studied fuzzy relations in various contents. Furthermore, Nemitz [4] has investigated fuzzy relations connected with equivalence relations and fuzzy functions. In particular, more recently, Demirci [1] studied fuzzy equalities and fuzzy mappings.

Equivalence relations and mappings in crisp set theory are very well known. This paper attempts an investigation of equivalence relations and mappings in fuzzy set theory. In Section 2, we list some concepts and results related to fuzzy relations. In Section 3, we give some examples corresponding to the concept of fuzzy equality and fuzzy mapping introduced by Demirci [1]. Also, adding to his results, we obtain some another results. In Section 4, we introduce the notions of preimage and quotient of fuzzy equivalence relations. And we study some properties. In Section 5, we investigate relations between a fuzzy equivalence relation and a fuzzy mapping.

Throughout this paper, we denote the unit interval $[0, 1]$ as $I$, and $X$, $Y$, $Z$, etc., denote ordinary sets. In particular, $I^X$ denotes the set of all fuzzy sets in $X$.

2. Preliminaries

In this section, we list some basic notions and results which are needed in the later sections.

Definition 2.1 [7]. Let $f : X \to Y$ be an (ordinary) mapping, let $A \in I^X$ and let $B \in I^Y$. Then:

(i) The image of $A$ under $f$, denoted by $f(A)$, is a fuzzy set in $Y$ defined as follows: For each $y \in Y$,

\[ f(A)(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise} \end{cases} \]

(ii) The preimage of $B$ under $f$, denoted by $f^{-1}(B)$, is a fuzzy set in $X$ defined as follows: For each $x \in X$,

\[ [f^{-1}(B)](x) = (B \circ f)(x) = B(f(x)). \]

Result 2.A [8]. Let $f : X \to Y$ be an (ordinary) mapping, let $A \in I^X$ and $\{A_\alpha\}_{\alpha \in \Gamma} \subseteq I^X$, and let $B \in I^Y$ and $\{B_\alpha\}_{\alpha \in \Gamma} \subseteq I^Y$. Then:
(a) \( |f(A)|^c \subset f(A')\). In particular, \( f \) is bijective, then \( |f(A)|^c = f(A')\).

(b) \( f^{-1}(B^c) = [f^{-1}(B)]^c\).

(c) If \( A_\alpha \subset A_\beta \), for \( \alpha, \beta \in \Gamma \), then \( f(A_\alpha) \subset f(A_\beta)\).

(d) If \( B_\alpha \subset B_\beta \), for \( \alpha, \beta \in \Gamma \), then \( f^{-1}(B_\alpha) \subset f^{-1}(B_\beta)\).

(e) \( A \subset f^{-1}(f(A))\). In particular, if \( f \) is injective, then \( f^{-1}(f(A)) = A\).

(f) \( f^{-1}(f(B)) \subseteq B\). In particular, if \( f \) is surjective, then \( f(f^{-1}(B)) = B\).

(g) \( f(\bigcup_{\alpha \in \Gamma} A_\alpha) = \bigcup_{\alpha \in \Gamma} f(A_\alpha)\).

(h) \( f^{-1}(\bigcup_{\alpha \in \Gamma} B_\alpha) = \bigcup_{\alpha \in \Gamma} f^{-1}(B_\alpha)\).

(i) \( f(\bigcap_{\alpha \in \Gamma} A_\alpha) \subseteq \bigcap_{\alpha \in \Gamma} f(A_\alpha)\).

(j) \( f^{-1}(\bigcap_{\alpha \in \Gamma} B_\alpha) = \bigcap_{\alpha \in \Gamma} f^{-1}(B_\alpha)\).

(k) If \( g : Y \rightarrow Z \) is a mapping and \( C \in \mathbb{I}^2 \), then \( (g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C)) \), and \( (g \circ f)(A) = g(f(A)) \).

**Definition 2.2** [9] R is called a fuzzy relation from \( X \) to \( Y \) (or a fuzzy relation on \( X \times Y \)) if \( R \in \mathbb{I}^{X \times Y} \), i.e., \( R \) is a fuzzy set in \( X \times Y \). In particular, if \( R \in \mathbb{I}^{X \times X} \), then \( R \) is called a fuzzy relation on \( (or \ in \ ) X \).

We will denote the set of all fuzzy relations on \( X \) as \( FR(X) \).

**Definition 2.3** [10] Let \( R \in \mathbb{I}^{X \times Y} \) and \( S \in \mathbb{I}^{Y \times Z} \). Then:

(i) The sup-min composition of \( R \) and \( S \), denoted by \( S \circ R \), is a fuzzy relation on \( X \times Z \) defined as follows:

\[ (S \circ R)(x, z) = \bigvee_{y \in Y} \big[ R(x, y) \land S(y, z) \big] \]

(ii) The inverse of \( R \), denoted by \( R^{-1} \), is a fuzzy relation on \( Y \times X \) defined by \( R^{-1}(y, x) = R(x, y) \), \( \forall (x, y) \in X \times Y \).

**Definition 2.4** [5,10] Let \( R \in FR(X) \). Then \( R \) is said to be:

(i) reflexive if \( R(x, x) = 1, \forall x \in X \).

(ii) symmetric if \( R(x, y) = R(y, x), \forall x, y \in X \), i.e., \( R = R^{-1} \).

(iii) transitive if \( R \circ R \subset R \).

(iv) a fuzzy equivalence relation on \( X \) if it satisfies (i), (ii) and (iii).

We will denote the set of all fuzzy equivalence relations on \( X \) as \( FER(X) \).

Let \( R \) be a fuzzy equivalence relation on \( X \) and let \( a \in X \). We defined the mapping \( Ra : X \rightarrow I \) as follows:

\[ \forall x \in X, Ra(x) = R(a, x) \]

Then clearly \( Ra \in \mathbb{I}^X \). In this case, \( Ra \) is called a fuzzy equivalence class of \( R \) containing \( a \). The set \( \{ Ra : a \in X \} \) is called the fuzzy quotient set of \( X \) by \( R \) and denoted by \( X/R \) (See[5]).

**Result 2.2** [5, Lemma 2, Corollary and Theorem 1]. Let \( R \) be a fuzzy equivalence relation on \( X \). Then

(a) \( Ra = Rb \) if and only if \( R(a, b) = 1 \), \( \forall a, b \in X \).

(b) \( Ra \cap Rb = \emptyset \) if and only if \( R(a, b) = 0 \), \( \forall a, b \in X \).

(c) \( \bigcup_{a \in X} Ra = X \).

3. Fuzzy mappings

In this section, we list some concepts and their properties by Demirci [1]. And we give some examples and obtain some results.

**Definition 3.1**[1]. A mapping \( E_X : X \times X \rightarrow I \) is called a fuzzy equality on \( X \) if it satisfies the following conditions:

(e.1) \( E_X(x, y) = 1 \leftrightarrow x = y, \forall x, y \in X \),

(e.2) \( E_X(x, y) = E_Y(x, y), \forall x, y \in X \),

(e.3) \( E_X(x, y) \land E_X(y, z) \leq E_X(x, z), \forall x, y, z \in X \).

We will denote the set of all fuzzy equalities as \( E(X) \).

**Example 3.1.** Let \( X = \{ \top, \bot, \varepsilon \} \) and let \( E_X : X \times X \rightarrow I \) be the mapping defined as following matrix:

\[
\begin{array}{ccc}
\top & \bot & \varepsilon \\
\top & 1 & 0.3 & 0.3 \\
\bot & 0.3 & 1 & 0.8 \\
\varepsilon & 0.3 & 0.8 & 1 \\
\end{array}
\]

Then we can easily see that \( E_X \in E(X) \).

**Definition 3.2**[1]. Let \( f \) be a fuzzy relation on \( X \times Y \). Then \( f \) is called a fuzzy mapping with respect to \( (\text{in short, w.r.t.}) E_X \in E(X) \) and \( E_Y \in E(Y) \), denoted by \( f : X \rightarrow Y \), if it satisfies the following condition:

(f.1) \( \forall x \in X, y \in Y, \text{such that } f(x, y) > 0 \),

(f.2) \( \forall x_1, x_2 \in X, \forall y_1, y_2 \in Y, f(x_1, y_1) \land f(x_2, y_2) \land E_X(x_1, x_2) \leq E_Y(y_1, y_2) \)

**Example 3.2.** Let \( X \) and \( E_X \in E(X) \) be same as in Example 3.1. Let \( Y = \{ a, b \} \) and \( E_Y : Y \times Y \rightarrow I \) be
the mapping defined as follows:

<table>
<thead>
<tr>
<th>$E_Y$</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>1</td>
<td>0.7</td>
</tr>
<tr>
<td>$b$</td>
<td>0.7</td>
<td>1</td>
</tr>
</tbody>
</table>

Then it is easily seen that $E_Y \in E(Y)$. Now define the fuzzy relation $f$ on $X \times Y$ follows:

<table>
<thead>
<tr>
<th>$f$</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>$\tau$</td>
<td>0.4</td>
<td>0.7</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>1</td>
<td>0</td>
</tr>
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</table>

Then we can easily prove that $f : X \rightarrow Y$ is a fuzzy mapping w.r.t. $E_X$ and $E_Y$.

**Definition 3.3[1].** Let $f : X \rightarrow Y$ be a fuzzy mapping w.r.t. $E_X \in E(X)$ and $E_Y \in E(Y)$. Then $f$ is said to be:

(i) strong if $\forall x \in X, \exists y \in Y$ such that $f(x, y) = 1$,
(ii) surjective if $\forall y \in Y, \exists x \in X$ such that $f(x, y) > 0$,
(iii) strong surjective if $\forall y \in Y, \exists x \in X$ such that $f(x, y) = 1$,
(iv) injective if $f(x_1, y_1) \land f(x_2, y_2) \land E_Y(y_1, y_2) \leq E_X(x_1, x_2), \forall x_1, x_2 \in X, \forall y_1, y_2 \in Y$,
(v) bijective if it is surjective and injective,
(vi) strong bijective if it is strong surjective and injective.

**Example 3.3.** (a) Let $X, Y, E_X, E_Y$ and $f$ be same as in Example 3.2. Then $f : X \rightarrow Y$ is not strong but strong surjective. Moreover, it can be easily seen that $f$ is not injective.

(b) Let $X, Y, E_X$ and $E_Y$ be same as in Example 3.2. Define the fuzzy relation $g$ on $X \times Y$ as follows:

<table>
<thead>
<tr>
<th>$g$</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>$\tau$</td>
<td>1</td>
<td>0.7</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>1</td>
<td>0</td>
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</tbody>
</table>

Then we can easily see that $g$ is strong and strong surjective w.r.t. $E_X$ and $E_Y$. But $g$ is not injective.

(c) Let $X, Y, E_X$ and $E_Y$ be same as in Example 3.2. Define the fuzzy relation $h$ on $Y \times X$ as follows:

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\gamma$</th>
<th>$\tau$</th>
<th>$\varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0.5</td>
<td>0.4</td>
<td>1</td>
</tr>
<tr>
<td>$b$</td>
<td>1</td>
<td>0.7</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $h(a, \gamma) \land h(b, \tau) \land E_Y(a, b) = 0.5 \leq 0.3 = E_X(\gamma, \tau)$. Thus $h$ is not a fuzzy mapping w.r.t. $E_Y$ and $E_X$.

(d) Let $X, Y, E_X$ and $E_Y$ be same as in Example 3.2. Define the fuzzy relation $k$ on $Y \times X$ as follows:

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\gamma$</th>
<th>$\tau$</th>
<th>$\varepsilon$</th>
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<tbody>
<tr>
<td>$a$</td>
<td>1</td>
<td>0.3</td>
<td>0.2</td>
</tr>
<tr>
<td>$b$</td>
<td>1</td>
<td>0</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Then we can easily show that $k : Y \rightarrow X$ is strong and injective w.r.t. $E_Y$ and $E_X$.

**Definition 3.4[1].** The identify fuzzy mapping on $X$, denoted by $I_X$, is the fuzzy relation on $X \times X$ defined by:

$$I_X(x, y) = \begin{cases} 
1 & \text{if } x = y, \\
0 & \text{otherwise, for any } x, y \in X.
\end{cases}$$

**Remark 3.4.** (a) $I_X : X \rightarrow X$ is strong bijective w.r.t. any fuzzy equality on $X$. Moreover, $I_X$ itself is a fuzzy equality on $X$.

(b) An (ordinary) mapping $f : X \rightarrow Y$ is a fuzzy mapping w.r.t. $E_X = I_X \in E(X)$ and $E_Y = I_Y \in E(Y)$.

(c) Let $f : X \rightarrow Y$ be an (ordinary) mapping. If $f$ is injective [resp. surjective and bijective], then $f$ is injective [resp. strong surjective and strong bijective] w.r.t. $I_X \in E_X$ and $I_Y \in E(Y)$.

**Result 3.A[1, Proposition 2.1].** Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be fuzzy mappings w.r.t. $E_X \in E(X)$, $E_Y \in E(Y)$ and $E_Z \in E(Z)$. Then sup-min composition $g \circ f$ is a fuzzy mapping $g \circ f : X \rightarrow Z$ w.r.t. $E_X$ and $E_Z$.

**Proposition 3.5.** Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be fuzzy mappings w.r.t. $E_X \in E(X)$, $E_Y \in E(Y)$ and $E_Z \in E(Z)$. If $f$ and $g$ are strong [resp. injective, surjective, strong surjective, bijective and strong bijective], then so is $g \circ f$.

**Proof.** (i) Suppose $f$ and $g$ are strong and let $x \in X$. Since $f$ is strong, $\exists y_0 \in Y$ such that $f(x, y_0) = 1$.

Since $g$ is strong, $\exists z_0 \in Z$ such that $g(y_0, z_0) = 1$.

Thus

$$g \circ f(x, z_0) = \bigvee_{y \in Y} [f(x, y) \land g(y, z_0)] \geq f(x, y_0) \land g(y_0, z_0) = 1.$$

So $g \circ f$ is strong.

(ii) Suppose $f$ and $g$ are surjective and let $z \in Z$. Since $g$ is surjective, $\exists y_0 \in Y$ such that $g(y_0, z) > 0$.

Since $f$ is surjective, $\exists x_0 \in X$ such that $f(x_0, y_0) > 0$.

Thus

$$g \circ f(x_0, z) = \bigvee_{y \in Y} [f(x_0, y) \land g(y, z)] \geq f(x_0, y_0) \land g(y_0, z) > 0.$$

So $g \circ f$ is surjective.

(iii) Suppose $f$ and $g$ are strong surjective and let $z \in Z$. Since $g$ is strong surjective, $\exists y_0 \in Y$ such that...
Then so is $g$. Since $f$ is strong surjective, there exists $x_0 \in X$ such that $f(x_0, y_0) = 1$. Thus
\[(g \circ f)(x_0, z) = \bigvee_{y \in Y} [f(x, y) \wedge g(y, z)] \geq f(x_0, y_0) \wedge g(y_0, z) = 1.\]

So $g \circ f$ is strong surjective.

(iv) Suppose $f$ and $g$ are injective. Let $x_1, x_2 \in X$, let $y_1, y_2 \in Y$ and let $z_1, z_2 \in Z$.

Since $f$ is injective,
\[f(x_1, y_1) \wedge f(x_2, y_2) \leq E_Y(y_1, y_2) \leq E_X(x_1, x_2).\]

Since $g$ is injective,
\[g(y_1, z_1) \wedge g(y_2, z_2) \wedge E_Z(z_1, z_2) \leq E_Y(y_1, y_2).\]

By (3.1) and (3.2),
\[(f(x_1, y_1) \wedge g(y_1, z_1)) \wedge (f(x_2, y_2) \wedge g(y_2, z_2)) \wedge E_Z(z_1, z_2) \leq E_X(x_1, x_2).
\]

Thus,
\[\bigvee_{y \in Y} [f(x_1, y) \wedge g(y_1, z_1)] \wedge [f(x_2, y) \wedge g(y_2, z_2)] \wedge E_Z(z_1, z_2) \leq E_X(x_1, x_2).
\]

So $(g \circ f)(x_1, z_1) \wedge (g \circ f)(x_2, z_2) \wedge E_Z(z_1, z_2) \leq E_X(x_1, x_2)$. Hence $g \circ f$ is injective.

The remainders are obvious by (i), (ii), (iii) and (iv).

**Proposition 3.6.** Let $f : X \to Y$ and $g : Y \to Z$ be fuzzy mappings w.r.t. $E_X \in E(X)$, $E_Y \in E(Y)$ and $E_Z \in E(Z)$.

(a) If $g \circ f$ is strong, then so is $f$.

(b) If $g \circ f$ is surjective[resp. strong surjective], then so is $g$.

**Proof.** (a) Suppose $g \circ f$ is strong and let $x \in X$. Then there exists $z_0 \in Z$ such that $(g \circ f)(x, z_0) = 1$. Thus
\[(g \circ f)(x, z_0) = \bigvee_{y \in Y} [f(x, y) \wedge g(y, z_0)] = 1.\]

So $\exists y_0 \in Y$ such that $f(x, y_0) \wedge g(y_0, z_0) = 1$. In particular, $f(x, y_0) = 1$. Hence $f$ is strong.

(b) Suppose $g \circ f$ is surjective and let $z \in Z$. Then
\[\exists z_0 \in Z \text{ such that } (g \circ f)(x, z_0) > 0.
\]

Thus
\[\bigvee_{y \in Y} [f(x, y) \wedge g(y, z)] > 0.
\]

So $\exists y_0 \in Y$ such that $f(x, y_0) \wedge g(y_0, z) > 0$. In particular, $g(y_0, z) > 0$. Hence $g$ is surjective.

Now suppose $g \circ f$ is strong and let $x \in Z$. Then $\exists x_0 \in X$ such that $(g \circ f)(x_0, z) > 0$. Thus
\[\bigvee_{y \in Y} [f(x_0, y) \wedge g(y, z)] = 1.
\]

So $\exists y_0 \in Y$ such that $f(x_0, y_0) \wedge g(y_0, z) = 1$. In particular, $g(y_0, z) = 1$. Hence $g$ is surjective.

Let $f : X \to Y$ and $g : Y \to Z$ be two ordinary mappings. Then it is well-known that if $g \circ f : X \to Z$ is injective, then so is $f$. However, in case which $f$ and $g$ are fuzzy mappings, the above statement does not hold.

**Example 3.6.** Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$ and $Z = \{z_1, z_2\}$, let $E_X : X \times X \to I$ be the mapping defined by $E_X(x_i, x_j) = 1$ for $i = 1, 2, 3$, $E_X(x_1, x_2) = E_X(x_2, x_1) = E_X(x_3, x_1) = 0$ and $E_X(x_2, x_3) = E_X(x_3, x_2) = 0.5$. Then clearly $E_X \in E(X)$.

Now let $f : X \times Y \to I$ and $g : Y \times Z \to I$ be the mappings defined as follows, respectively:
\[f(x_1, y_1) = f(x_2, y_2) = 1, f(x_3, y_2) = 0.8\]
and
\[g(y_1, z_1) = 1, g(y_2, z_2) = 0.2.
\]

Then we can easily see that $f$ is a fuzzy mapping w.r.t. $E_X$ and $I_Y$, and $g$ is a fuzzy mapping w.r.t. $I_Y$ and $I_Z$. Furthermore, we can see that $g \circ f : X \to Z$ is a fuzzy injective mapping w.r.t. $E_X$ and $I_Z$. But $f$ is not injective.

**Definition 3.7.** Let $f : X \to Y$ be a fuzzy mapping w.r.t. $E_X \in E(X)$ and $E_Y \in E(Y)$. Then $f$ is said to be *invertible* if the fuzzy relation $f^{-1}$ on $X \times X$ is a fuzzy mapping $f^{-1} : Y \to X$ w.r.t. $E_Y$ and $E_X$.

**Lemma 3.8.** Let $f : X \to Y$ be a fuzzy mapping w.r.t. $E_X \in E(X)$ and $E_Y \in E(Y)$. If $f$ is invertible, then $f$ is bijective.

**Proof.** Suppose $f$ is invertible and let $y \in Y$. Since $f^{-1} : Y \to X$ is a fuzzy mapping w.r.t. $E_Y$ and $E_X$, $\exists x_0 \in X$ such that $f^{-1}(y, x_0) > 0$. Thus $f(x_0, y) > 0$. Hence $f$ is surjective. Now let $x_1, x_2 \in X$ and let $y_1, y_2 \in Y$. Since $f^{-1} : Y \to X$ is a fuzzy mapping, $f^{-1}(y_1, x_1) \wedge f^{-1}(y_2, x_2) \wedge E_Y(y_1, y_2) \leq E_X(x_1, x_2)$. Thus $f(x_1, y_1) \wedge f(x_2, y_2) \wedge E_Y(y_1, y_2) \leq E_X(x_1, x_2)$. Hence $f$ is injective. Hence $f$ is bijective.

**Lemma 3.9.** Let $f : X \to Y$ be a bijective fuzzy mapping w.r.t. $E_X \in E(X)$ and $E_Y \in E(Y)$. Then the fuzzy relation $f^{-1}$ on $Y \times X$ is a bijective fuzzy mapping $f^{-1} : Y \to X$ w.r.t. $E_Y$ and $E_X$.

**Proof.** Let $y \in Y$. Since $f$ is surjective, $\exists x_0 \in X$ such that $f(x_0, y) > 0$. Thus $f^{-1}(y, x_0) > 0$. Hence $f^{-1}$ satisfies the condition (1.1). Now let $y_1, y_2 \in Y$ and let $x_1, x_2 \in X$. Since $f$ is injective, $f(x_1, y_1) \wedge f(x_2, y_2) \wedge E_Y(y_1, y_2) \leq E_X(x_1, x_2)$. Then $f^{-1}(y_1, x_1) \wedge f^{-1}(y_2, x_2) \wedge E_Y(y_1, y_2) \leq E_X(x_1, x_2)$. Hence $f$ satisfies the condition (2.2). So $f^{-1} : Y \to X$ is a fuzzy mapping w.r.t. $E_Y$ and $E_X$. Let $x \in X$. Since $f$ is a fuzzy mapping, $\exists y_0 \in Y$ such that $f(x, y_0) \wedge g(y_0, z) = 1$. Hence $g$ is surjective.
such that \( f(x, y_0) > 0 \). Then \( f^{-1}(y_0, x) > 0 \). Thus \( f^{-1} \) is surjective. Now let \( y_1, y_2 \in Y \) and let \( x_1, x_2 \in X \). Since \( f \) is a fuzzy mapping, \( f(x_1, y_1) \land f(x_2, y_2) \land E \leq E_Y(y_1, y_2) \). Then \( f^{-1}(y_1, x_1) \land f^{-1}(y_2, x_2) \land E \leq E_Y(y_1, y_2) \). Thus \( f^{-1} \) is injective. So \( f^{-1} \) is bijective. This competes the proof.

The following shows us that \( f : X \to Y \) is strong surjective but \( f^{-1} : Y \to X \) is not strong surjective. Thus \( f \) is strong bijective but \( f^{-1} \) is not strong bijective.

Example 3.9. Let \( X, Y \), and \( E_X \) be the same as in Example 3.6. We define the mapping \( f : X \times Y \to I \) as follows:
\[
f(x_1, y_1) = f(x_2, y_2) = 1 \quad \text{and} \quad f(x_3, y_2) = 0.5.
\]
Then we can easily check that \( f \) is strong surjective but \( f^{-1} \) is not strong surjective. Moreover, \( f \) is injective. So \( f \) is strong bijective but \( f^{-1} \) is not strong bijective.

The following is the immediate result of Lemmas 3.8 and 3.9.

Theorem 3.10. [1, Proposition 2.2]. Let \( f : X \to Y \) be a fuzzy mapping w.r.t. \( E_X \in E(X) \) and \( E_Y \in E(Y) \). Then \( f \) is invertible if and only if \( f \) is bijective.

Result 3.B[1, Proposition 2.3]. If \( f : X \to Y \) is strong and injective w.r.t. \( E_X = I_X \in E(X) \) and \( E_Y \in E(Y) \), then \( f \circ f^{-1} = I_X \).

Lemma 3.11. Let \( f : X \to Y \) be a fuzzy mapping w.r.t. \( E_X \in E(X) \) and \( E_Y \in E(Y) \). If \( f \) is strong surjective and \( E_Y = I_Y \), then \( f \circ f^{-1} = I_Y \).

Proof. Let \( y, y' \in Y \). Then
\[
(f \circ f^{-1})(y, y') = \bigvee_{x \in X} [f^{-1}(y, x) \land f(x, y')]
\]
\[
= \bigvee_{x \in X} [f(x, y) \land f(x, y')]
\]
\[
= \bigvee_{x \in X} [f(x, y) \land f(x, y') \land E \geq E_X(x, x) \land E_Y(y, y')]
\]
\[
\leq E_Y(y, y') \quad \text{[\( f \) is a fuzzy mapping]}
\]
\[
= I_Y(y, y').
\]

Thus \( f \circ f^{-1} \subseteq I_Y \). Now let \( y, y' \in Y \). Then clearly \( I_Y(y, y') = 1 \) or \( I_Y(y, y') = 0 \). If \( I_Y(y, y') = 0 \), then clearly \( I_Y(y, y') \leq (f \circ f^{-1})(y, y') \). Suppose \( I_Y(y, y') = 1 \), i.e., \( y = y' \). Since \( f \) is strong surjective, for \( y \in Y \), \( \exists x \in X \) such that \( f(x_0, y) = 1 \). Thus
\[
(f \circ f^{-1})(y, y') = (f \circ f^{-1})(y, y')
\]
\[
= \bigvee_{x \in X} [f^{-1}(y, x) \land f(x, y')]
\]
\[
= \bigvee_{x \in X} [f(x, y) \land f(x, y')]
\]
\[
= \bigvee_{x \in X} [f(x, y)] = 1.
\]

So, in either cases, \( I_Y \subseteq f \circ f^{-1} \). Hence \( f \circ f^{-1} = I_Y \).

The following is the immediate result of Result 3.B and Lemma 3.11.

Theorem 3.12. Let \( f : X \to Y \) be a fuzzy mapping w.r.t. \( E_X \in E(X) \) and \( E_Y \in E(Y) \). If \( f \) is strong and strong bijective, \( E_X = I_X \) and \( E_Y = I_Y \), then \( f^{-1} \circ f = I_X \) and \( f \circ f^{-1} = I_Y \).

Result 3.C[1, Proposition 2.4]. Let \( f : X \to Y \) and \( g : Y \to Z \) be bijective w.r.t. \( E_X \in E(X) \) and \( E_Y \in E(Y) \) and \( E_Z \in E(Z) \). Then \( (g \circ f)^{-1} = f^{-1} \circ g^{-1} \) and the fuzzy relation \( (g \circ f)^{-1} \) is a fuzzy mapping \( (g \circ f)^{-1} : Z \to X \) w.r.t. \( E_Z \) and \( E_X \).

Definition 3.13. [1]. Let \( f : X \to Y \) be a fuzzy mapping, let \( A \in I^X \) and let \( B \in I^Y \). Then:
(i) The image of \( A \) under \( f \), denoted by \( f(A) \), is a fuzzy set in \( Y \) defined as follows:
\[
f(A)(y) = \bigvee_{x \in X} [A(x) \land f(x, y)], \forall y \in Y.
\]
(ii) The preimage of \( B \) under \( f \), denoted by \( f^{-1}(B) \), is a fuzzy set in \( X \) defined as follows:
\[
f^{-1}(B)(x) = \bigvee_{y \in Y} [B(y) \land f(x, y)], \forall x \in X.
\]

Remark 3.13. (a) If \( f : X \to Y \) is an (ordinary) mapping, then it is clear that Definition 3.13 is identical with Definition 2.1
(b) If \( f : X \to Y \) is strong surjective, then \( f(A)(y) = \bigvee_{x \in X} A(x), \forall y \in Y \).
(c) If \( f : X \to Y \) is strong, then \( f^{-1}(B)(x) = \bigvee_{y \in Y} B(y), \forall x \in X \).

The following is the immediate result of Definition 3.13.

Proposition 3.14. Let \( f : X \to Y \) and \( g : Y \to Z \) be fuzzy mappings w.r.t. \( E_X \in E(X) \), \( E_Y \in E(Y) \) and \( E_Z \in E(Z) \), let \( A \in I^X \) and let \( B \in I^Y \). Then:
(a) \( (g \circ f)(A) = g(f(A)) \).
(b) \( (g \circ f)(B) = f^{-1}(g^{-1}(B)) \).

Result 3.D[1, Proposition 2.5]. Let \( f : X \to Y \) be a fuzzy mapping w.r.t. \( E_X \in E(X) \) and \( E_Y \in E(Y) \), let \( A \in I^X \) and let \( B \in I^Y \).
(a) If \( f \) is strong, then \( A \subseteq f^{-1}(f(A)) \).
(b) If \( A = I_X \) and \( f \) is injective, then \( f^{-1}(f(A)) \subseteq A \).
(c) If \( f \) is strong surjective, then \( B \subseteq f(f^{-1}(B)) \).
(d) If $E_Y = I_Y$, then $f(f^{-1}(B)) \subseteq B$.

The following is the immediate result of Theorem 2.5 in [6] and Definition 3.13.

**Proposition 3.15.** Let $f : X \rightarrow Y$ be a fuzzy mapping w.r.t. $E_X \in E(X)$ and $E_Y \in E(Y)$.

(a) Define the (ordinary) relation $\overline{f}$ from $I^X$ to $I^Y$ as follows: $\overline{f}(A) = f(A), \forall A \in I^X$. Then $\overline{f} : I^X \rightarrow I^Y$ is an (ordinary) mapping.

(b) Define the (ordinary) relation $\overline{f}$ from $I^Y$ to $I^X$ as follows: $\overline{f}(B) = f^{-1}(B), \forall B \in I^Y$. Then $\overline{f} : I^Y \rightarrow I^X$ is an ordinary mapping.

The followings are the immediate results of Result 3.D and Proposition 3.14.

**Corollary 3.15.** Let $f : X \rightarrow Y$ be strong surjective w.r.t. $E_X \in E(X)$ and $E_Y = I_y \in E(Y)$. Then $\overline{f} \circ \overline{f} = \overline{f}$

**Proof.** Let $A \in I^X$. Since $f$ is strong surjective, by Result 3.D(c), $f(A) \subseteq f^{-1}(f(A))$. Since $E_Y = I_y$, by Result 3.D(d), $f^{-1}(f(A)) \subseteq f(A)$. So $f^{-1}(f(A)) = f(A)$. Hence $\overline{f} \circ \overline{f} = \overline{f}$.

**Proposition 3.16.** Let $f : X \rightarrow Y$ be a fuzzy mapping w.r.t. $E_X \in E(X)$ and $E_Y \in E(Y)$.

(a) If $f$ is strong, injective and $E_X = I_X$, then $\overline{f} \circ \overline{f}$ is bijective. Hence $\overline{f}$ is injective and $\overline{f}$ is surjective.

(b) If $f$ is strong surjective and $E_Y = I_y$, then $\overline{f} \circ \overline{f}$ is bijective. Hence $\overline{f}$ is surjective and $\overline{f}$ is injective.

(c) If $f$ is strong, strong bijective, $E_X = I_X$ and $E_Y = I_y$, then $\overline{f}$ and $\overline{f}$ are bijective.

**Proof.** (a) Clearly $\overline{f} \circ \overline{f} : I^X \rightarrow I^X$ is a mapping. Suppose $(\overline{f} \circ \overline{f})(A_1) = (\overline{f} \circ \overline{f})(A_2) \forall A_1, A_2 \in I^X$. Then $\overline{f}(f(A_1)) = \overline{f}(f(A_2))$. Thus, by the definitions of $\overline{f}$ and $\overline{f}$, $f^{-1}(f(A_1)) = f^{-1}(f(A_2))$. By Result 3.D, $A_1 = A_2$. So $\overline{f} \circ \overline{f}$ is injective. Let $A \in I^X$. Then clearly $f(A) \in I^Y$. Moreover, by Result 3.D, $f^{-1}(f(A)) = A$. Thus $\overline{f} \circ \overline{f}(A) = A$. So $\overline{f} \circ \overline{f}$ is surjective. Hence $\overline{f} \circ \overline{f}$ is bijective.

(b) Clearly $\overline{f} : I^Y \rightarrow I^X$ is a mapping. Suppose $(\overline{f} \circ \overline{f})(B_1) = (\overline{f} \circ \overline{f})(B_2), \forall B_1, B_2 \in I^Y$. Then $\overline{f}(\overline{f}(B_1)) = \overline{f}(\overline{f}(B_2))$, i.e., $f^{-1}(\overline{f}(B_1)) = f^{-1}(\overline{f}(B_2))$. By Result 3.D, $B_1 = B_2$. Thus $\overline{f} \circ \overline{f}$ is injective. Let $B \in I^Y$. Then clearly $f^{-1}(B) \in I^X$ and $f^{-1}(f^{-1}(B)) = B$. Thus $\overline{f} \circ \overline{f}(B) = B$. So $\overline{f} \circ \overline{f}$ is surjective. Hence $\overline{f} \circ \overline{f}$ is bijective.

(c) It is clear from (a) and (b).

**Result 3.E[1, Proposition 2.6].** Let $f : X \rightarrow Y$ be a fuzzy mapping w.r.t. $E_X \in E(X)$ and $E_Y \subseteq E(Y)$, let $A \subseteq I^X$ and let $B \subseteq I^Y$.

(a) If $E_X = I_X$ and $f$ is injective, then $f(A^c) \subseteq [f(A)]^c$.

(b) If $f$ is strong surjective, then $[f(A)]^c \subseteq f(A^c)$.

(c) If $f$ is strong, then $f^{-1}(B^c) \subseteq f^{-1}(B^c)$.

(d) If $E_Y = I_y$, then $f^{-1}(B^c) \subseteq f^{-1}(B^c)$.

**Result 3.F[1, Proposition 2.7].** Let $f : X \rightarrow Y$ be a fuzzy mapping w.r.t. $E_X \in E(X)$ and $E_Y \subseteq E(Y)$, let $\{A_\alpha\}_{\alpha \in \Gamma} \subseteq I^Y$ and let $\{B_\alpha\}_{\alpha \in \Gamma} \subseteq I^Y$.

(a) $f(\bigcup_{\alpha \in \Gamma} A_\alpha) = \bigcup_{\alpha \in \Gamma} f(A_\alpha)$.

(b) $f^{-1}(\bigcup_{\alpha \in \Gamma} B_\alpha) = \bigcup_{\alpha \in \Gamma} f^{-1}(B_\alpha)$.

(c) $f(\bigcap_{\alpha \in \Gamma} A_\alpha) \subseteq \bigcap_{\alpha \in \Gamma} f(A_\alpha)$.

(d) $f^{-1}(\bigcap_{\alpha \in \Gamma} B_\alpha) \subseteq \bigcap_{\alpha \in \Gamma} f^{-1}(B_\alpha)$.

(e) If $A_\alpha \subseteq A_\beta$ for $\alpha, \beta \in \Gamma$, then $f(A_\alpha) \subseteq f(A_\beta)$.

(f) If $B_\alpha \subseteq B_\beta$ for $\alpha, \beta \in \Gamma$, then $f^{-1}(B_\alpha) \subseteq f^{-1}(B_\beta)$.

(g) If $f$ is injective and $E_X = I_X$, then $\bigcap_{\alpha \in \Gamma} f(A_\alpha) \subseteq f(\bigcap_{\alpha \in \Gamma} A_\alpha)$.

(h) If $E_Y = I_y$, then $\bigcap_{\alpha \in \Gamma} f^{-1}(B_\alpha) \subseteq f^{-1}(\bigcap_{\alpha \in \Gamma} B_\alpha)$.

The following is the immediate result of Definition 3.1.

**Proposition 3.17.** Let $\{X_\alpha\}_{\alpha \in \Gamma}$ be a family of sets and let $X = \bigprod_{\alpha \in \Gamma} X_\alpha$ be the product of $\{X_\alpha\}_{\alpha \in \Gamma}$. If $E_{X_\alpha}$ is a fuzzy equality on $X_\alpha$ for each $\alpha \in \Gamma$, then $E_X = \bigprod_{\alpha \in \Gamma} E_{X_\alpha}$ is a fuzzy equality on $X$, where $E_X : X \times X \rightarrow I$ is the mapping defined as follows: $\forall (x_\alpha, y_\alpha) \in X$, $E_X((x_\alpha, y_\alpha)) = \bigcup_{\alpha \in \Gamma} E_{X_\alpha}(x_\alpha, y_\alpha)$.

The following is the immediate result of Definition 3.2 and Proposition 3.17.

**Proposition 3.18.** Let $X = \bigprod_{\alpha \in \Gamma} X_\alpha$ be the product of a family $\{X_\alpha\}_{\alpha \in \Gamma}$ of sets. For each $\alpha \in \Gamma$, we define the fuzzy relation $\pi_\alpha$ on $X \times X_\alpha$ as follows:

$$
\pi_\alpha((x_\alpha, x), x) = \begin{cases} 
1 & \text{if } x = x_\alpha, \\
0 & \text{if } x \neq x_\alpha, \forall (x_\alpha, y_\alpha) \in X, \forall x \in X_\alpha.
\end{cases}
$$
Then $\pi_\alpha : X \to \alpha$ is a fuzzy mapping w.r.t. $E_X = \prod_{\alpha \in \Gamma} E_{X_\alpha} \in E(X)$ and $E_X \in E(X_\alpha), \forall \alpha \in \Gamma$.

In this case, $\pi_\alpha$ is called the *fuzzy projection* of $X$ to $X_\alpha$.

From Proposition 3.18, it is clear that $\pi_\alpha$ is strong and strong surjective.

**Proposition 4.19.** Let $\pi_\alpha : X = \Pi_{\alpha \in \Gamma} X_\alpha \to \alpha$ be the fuzzy projection of $X$ to $X_\alpha$ and let $B_\alpha \in f^{X_\alpha}, \forall \alpha \in \Gamma$. Then

$$\bigcap_{\alpha \in \Gamma} \pi_\alpha^{-1}(B_\alpha) = \bigcap_{\alpha \in \Gamma} B_\alpha,$$

where $\bigcap_{\alpha \in \Gamma} B_\alpha$ is the fuzzy set in $X$ defined as follows:

$$\left(\bigcap_{\alpha \in \Gamma} B_\alpha\right)(x_\alpha) = \bigcap_{\alpha \in \Gamma} B_\alpha(x_\alpha), \forall (x_\alpha) \in X.$$

**Proof.** Let $(x_\alpha) \in X$. Then

$$\bigcap_{\alpha \in \Gamma} \pi_\alpha^{-1}(B_\alpha)(x_\alpha) = \bigcap_{\alpha \in \Gamma} \pi_\alpha^{-1}(B_\alpha)(x_\alpha) = \bigcap_{\alpha \in \Gamma} \bigcap_{x_\alpha \in X_\alpha} B_\alpha(x_\alpha)$$

$$= \bigcap_{\alpha \in \Gamma} \bigcap_{x_\alpha \in X_\alpha} B_\alpha(x_\alpha)$$

$$= \bigcap_{\alpha \in \Gamma} \bigcap_{x_\alpha \in X_\alpha} B_\alpha(x_\alpha)$$

$$= \bigcap_{\alpha \in \Gamma} \left(\bigcap_{x_\alpha \in X_\alpha} B_\alpha(x_\alpha)\right).$$

The following is the immediate result of Definition 3.2 and Proposition 3.17.

**Proposition 4.20.** Let $f : X \to Y$ be a fuzzy mapping w.r.t. $E_X \in E(X)$ and $E_Y \in E(Y)$. We define the fuzzy relation $g$ on $(X \times X) \times (Y \times X)$ as follows:

$$g((x, x'), (y, y')) = f(x, y) \wedge f(x', y'), \forall (x, x') \in X \times X, \forall (y, y') \in Y \times Y.$$

Then $g : X \times X \to Y \times Y$ is a fuzzy mapping w.r.t. $E_{X \times X} = E_X \times E_X \in E(X \times X)$ and $E_{Y \times Y} = E_Y \times E_Y \in E(Y \times Y)$. In this case, $g$ is called the *fuzzy product mapping* of $f$ and is denoted by $g = f \times f = f^2$.

4. Preimage and quotient of fuzzy equivalence relations.

**Proposition 4.1** Let $f : X \to Y$ be a strong fuzzy mapping w.r.t. $E_X \in E(X)$ and $E_Y \in E(Y)$, and let $R$ be a fuzzy equivalence relation on $Y$. Then $f^{-2}(R)$ is a fuzzy equivalence relation on $X$. In this case, $f^{-2}(R)$ is called the *preimage* of $R$ under $f$, where

$$f^{-2} = (f^2)^{-1} = (f \times f)^{-1}.$$

**Proof.** It is clear that $f^{-2}(R)$ is a fuzzy relation on $X$.

(i) Let $x \in X$. Then

$$f^{-2}(R)(x, x) = \bigvee_{(y, y') \in Y \times Y} \{R(y, y') \cap f^{-2}(x, x', x', y)\}$$

By Definition 3.13 and Notation $f^2 = f \times f$

$$= \bigvee_{(y, y') \in Y \times Y} \{R(y, y') \cap f(x, y) \cap f(x', y)\}$$

By Proposition 3.19

$$\geq R(y_0, y_0)$$

Since $f$ is strong, $\exists y_0 \in Y$ such that $f(x, y_0) = 1$. Thus $f^{-2}(R)$ is reflexive.

(ii) By the definition of $f^{-2}(R)$, it is clear that $f^{-2}(R)$ is symmetric.

(iii) Let $x, x'' \in X$. Then

$$f^{-2}(R) \circ f^{-2}(R)(x, x'')$$

$$= \bigvee_{x' \in X} \{R(y, y') \cap f^{-2}(x', x', x'')\}$$

By (ii)

$$\geq \bigvee_{(y', y'') \in Y \times Y} \{R(y', y'') \cap f(x, y) \wedge f(x', y') \wedge f(x', y'')\}$$

Thus $f^{-2}(R) \circ f^{-2}(R) \subseteq f^{-2}(R)$. So $f^{-2}(R)$ is transitive. Hence $f^{-2}(R)$ is fuzzy equivalence relation on $X$.

**Corollary 4.1** Let $f$ and $R$ be same as in Proposition 4.1. Then $f^{-2}(R) = f^{-1} \circ R \circ f$.

**Proof.** Let $a, b \in X$. Then

$$f^{-2}(R)(a, b) = \bigvee_{(c, d) \in Y \times Y} \{R(c, d) \cap f(x, y)\}$$

$$= \bigvee_{(c, d) \in Y \times Y} \{R(c, d) \cap f(a, y) \wedge f(b, y)\}$$

$$= \bigvee_{d \in Y} \{\bigvee_{(c, d) \in Y \times Y} \{R(c, d) \cap f(a, c) \cap f(b, d)\}\}$$

$$= \bigvee_{c \in Y} \{\bigvee_{d \in Y} \{R(c, d) \cap f(a, c) \cap f(b, d)\}\}$$

4. Preimage and quotient of fuzzy equivalence relations.
We define the fuzzy relation

\[ \text{Proof.} \]

\[ \text{In this case, } f \text{ is a fuzzy mapping w.r.t. } \pi. \]

Hence \( f^{-1}(R) = f^{-1} \circ R \circ f. \)

**Proposition 4.2** If \( R \) is a fuzzy equivalence relation on \( X \), then \( \exists \) the strong and strong surjective fuzzy mapping \( \pi : X \to X/R \) w.r.t. \( \text{E} \in \text{E}(X/R) \) where \( E_{X/R} : X/R \times X/R \to I \) is the fuzzy equality on \( X/R \) defined as follows: \( \forall a, b \in X, E_{X/R}(Ra, Rb) = R(a, b). \)

In this case, \( \pi \) is called the natural (or canonical) fuzzy mapping.

**Proof.** We define the fuzzy relation \( \pi : X \times X/R \to I \) as follows: \( \forall a, b \in X, \)

\[ \pi(a, b) = Rb(a) = R(b, a). \]

Then clearly \( \pi \) satisfies the condition (f.1). Let \( a_1, a_2, b_1, b_2 \in X. \) If \( a_1 \neq a_2, \) then clearly \( I_X(a_1, a_2) = 0. \)

Thus \[ \pi(a_1, Rb_1) \land \pi(a_2, Rb_2) \land I_X(a_1, a_2) \leq E_{X/R}(Rb_1, Rb_2). \]

Suppose \( a_1 = a_2. \) Then \( \pi(a_1, Rb_1) \land \pi(a_2, Rb_2) \land I_X(a_1, a_2) = R(b_1, b_2) \triangleq 1. \) \[ \text{Since } R \text{ is symmetric and } I_X(a_1, a_2) = 1. \]

Thus \( \pi \) satisfies the condition (f.2). So \( \pi : X \to X/R \) is a fuzzy mapping w.r.t. \( I_X \) and \( E_{X/R}. \) Moreover, it is clear that \( \pi \) is strong and strong surjective from the definition of \( \pi. \)

**Proposition 4.3** Let \( R \) and \( G \) be fuzzy equivalence relations on \( X \) such that \( R \subseteq G. \) We define the mapping \( G/R : X/R \times X/R \to I \) as follows:

\[ G/R(Ra, Rb) = G(a, b), \forall a, b \in X. \]

Then \( G/R \) is a fuzzy equivalence relation on \( X/R. \) In this case, \( G/R \) is called the fuzzy quotient of \( G \) by \( R. \)

**Proof.** It is clear that \( G/R \) is reflexive and symmetric. Let \( a, c \in X. \) Then \[ (G/R \circ G/R)(Ra, Re) \]

\[ = \bigvee_{b \in X} [G/R(Ra, Rb) \land G/R(Rb, Re)] \]

\[ = \bigvee_{b \in X} [G(a, b) \land G(b, c)] \]

\[ = (G \circ G)(a, c) \triangleq 1. \]

Thus \( G/R \) is a fuzzy equivalence relation on \( X/R. \)

The following is the immediate result of Proposition 4.3.

**Corollary 4.3** Let \( R, G \) and \( H \) be fuzzy equivalence relations on \( X \) such that \( R \subseteq G \subseteq H. \) Then \( G/R \subseteq H/R. \)

**Proposition 4.4** Let \( R, G \) and \( H \) be fuzzy equivalence relations on \( X \) such that \( R \subseteq G \subseteq H. \)

(a) \( R \subseteq G \subseteq H. \)

(b) If \( G \subseteq H \) is a fuzzy equivalence relation on \( X, \) then \( (G \circ H)/R \) is a fuzzy equivalence relation on \( X/R \) and \( G/R \circ H/R = (G \circ H)/R. \)

(c) \( G/R \circ H/R \) is a fuzzy equivalence relation on \( X/R. \)

**Proof.** (a) Let \( a, c \in X. \) Then \[ (G \circ H)(a, c) \]

\[ = \bigvee_{b \in X} [H(a, b) \land G(b, c)] \]

\[ = \bigvee_{b \in X} R(a, b) \land R(b, c) \]

\[ \geq R(a, c) \land R(c, c) = R(a, c). \]

Thus \( R \subseteq G \subseteq H. \)

(b) By the hypothesis and, (a) and Proposition 4.3, it is clear that \( (G \circ H)/R \) is a fuzzy equivalence relation on \( X/R. \) Let \( a, c \in X. \) Then \[ (G/R \circ H/R)(Ra, Rc) \]

\[ = \bigvee_{b \in X} [H(Ra, Rb) \land G(Rb, Rc)] \]

\[ = \bigvee_{b \in X} H(a, b) \land G(b, c) \]

\[ = R(a, c). \]

Thus \( G/R \circ H/R = (G \circ H)/R. \)

(c) It is obvious from (b).

**Proposition 4.5** Let \( R \) and \( G \) be fuzzy equivalence relations on \( X \) and \( Y, \) respectively. Let the fuzzy product of \( R \) and \( G, \) denoted by \( R \cdot G, \) be a fuzzy relation on \( (X \times Y) \times (X \times Y) \) defined as follows: \( \forall x_1, x_2 \in X, \forall y_1, y_2 \in Y, \)

\[ (R \cdot G)((x_1, y_1), (x_2, y_2)) = R(x_1, x_2) \land G(y_1, y_2). \]

Then \( R \cdot G \) is a fuzzy equivalence relation on \( X \times Y. \)

**Proof.** Let \( (a, b) \in X \times Y. \) Then \[ (R \cdot G)((a, b), (a, b)) \]

\[ = R(a, a) \land G(b, b). \]

\[ = 1. \]

Thus \( R \cdot G \) is reflexive. It is clear that \( R \cdot G \) is symmetric. Now let \( (a_1, b_1), (a_3, b_3) \in X \times Y. \) Then \[ [(R \cdot G) \circ (R \cdot G)]((a_1, b_1), (a_3, b_3)) \]
Proof. Let $\pi : X \rightarrow R$ be a strong fuzzy equivalence relation on $X$. If $\pi : X \rightarrow R$ is the natural fuzzy mapping w.r.t. $I_X \in E(X)$ and $E_{X/R} \in E(X/R)$, then $\pi = R_\pi$.

Thus $\pi$ is transitive. Hence $\pi$ is transitive.

The following is the immediate result of Propositions 4.2 and 5.1.

Corollary 5.1 Let $\pi : X \rightarrow R$ be a strong fuzzy equivalence relation on $X$. From Proposition 4.2, it is clear that $\pi$ is transitive and strong surjective. Let $a, b \in X$, Then $R_\pi(a, b) = \pi(a, R_\pi) \cap \pi(b, Rd) \cap E_{X/R}(Rc, Rd)]$

Proof. From Proposition 4.2, it is clear that $\pi$ is transitive and strong surjective. Let $a, b \in X$, Then $R_\pi(a, b) = \pi(a, R_\pi) \cap \pi(b, Rd) \cap E_{X/R}(Rc, Rd)]$

Thus $R_\pi \subset R$. On the other hand, $R_\pi(a, b) = \pi(a, R_\pi) \cap \pi(b, Rd) \cap E_{X/R}(Rc, Rd)]$

By the definitions of $\pi$ and $E_{X/R}$.

\[ \leq \bigvee_{(c, d) \in X \times Y} \left[ f(a, b) \wedge f(c, b') \wedge E_{Y}(b, b') \right] \]

\[ \leq R(a, c). \]  

Thus $R \subset R_\pi$. Hence $R = R_\pi$.

Remark 5.1 Corollary 5.1 is the generalization of Theorem 3.22 in [6] in fuzzy setting.

\[ \leq \bigvee_{(b, b') \in Y \times Y} \left[ f(a, b) \wedge f(c, b') \wedge E_{Y}(b, b') \right] \]

\[ \leq R(a, c). \]  

Thus $R \subset R_\pi$. Hence $R = R_\pi$.

Proposition 5.2 Let $f : X \rightarrow Y$ be a strong fuzzy mapping w.r.t. $I_X \in E(X)$ and $E_Y \in E(Y)$ and let $\text{ran } f = \{ y \in Y : \exists x \in X \text{ such that } f(x, y) > 0 \} \subset Y$. Let $R$ be the fuzzy equivalence relation determined by $f$. We define two fuzzy relations $s$ and $t$ on $X \times R \times \text{ran } f$ and $X \times Y$, respectively as follows:

\[ s(Ra, y) = f(a, y), \forall a \in X, \forall y \in \text{ran } f \]

and

\[ t(y, y') = \begin{cases} 1 & \text{if } y = y', \\ 0 & \text{if } y \neq y', \forall y \in \text{ran } f, \forall y' \in Y. \end{cases} \]
Then $s$ is strong and bijective, $t$ is strong and injective and $f = t \circ s \circ \pi$.

**Proof.** (i) From Proposition 4.2, it is clear that $\pi: X \to X/R$ is a strong and strong surjective fuzzy mapping w.r.t. $I_X$ and $E_{X/R} \in E(X/R)$.

(ii) It is easily seen that $s: X/R \to \text{ran } f$ is a fuzzy mapping w.r.t. $E_{X/R}$ and $E_Y$. Let $y \in \text{ran } f$. Then $\exists x \in X$ such that $f(x, y) > 0$. Thus $Rx \in X/R$ and $s(Rx, y) = f(x, y) > 0$. So $s$ is surjective. Now let $x_1, x_2 \in X$ and $y_1, y_2 \in \text{ran } f$. Then

$$E_{X/R}(Rx_1, Rx_2)
= R(x_1, x_2)
= \bigvee_{(c, d) \in Y \times Y} [f(x_1, c) \land f(x_2, d) \land E_Y(c, d)]$$

$R$ is the fuzzy equivalence relation determined by $f$.

Thus $s$ is injective. Since $f$ is strong, it is clear that $s$ is strong and bijective.

(iii) From the definition of $t$, it is clear that $t: \text{ran } f \to Y$ is strong and injective fuzzy mapping w.r.t. $E_Y$ and $E_{X/R}$.

(iv) Let $x \in X$ and let $y \in Y$. Then

$$t \circ s \circ \pi(x, y)
= [(t \circ s) \circ \pi](x, y)
= \bigvee_{Ra \in X/R} \big[\pi(x, Ra) \land (t \circ s)(Ra, y)\big]$$

$R$ is the fuzzy equivalence relation determined by $f$.

Thus $t \circ s \circ \pi = f$. This completes the proof.

**Proposition 5.3** Let $f: X \to Y$ be a strong fuzzy mapping w.r.t. $I_X \in E(X)$ and $E_Y \in E(Y)$. Let $R$ be the fuzzy equivalence relation on $X$ determined by $f$ and let $G$ be any fuzzy equivalence relation on $X$ such that $G \subset R$. We define the fuzzy relation $f/G$ on $X/G \times Y$ as follows:

$$[f/G](Gx, y) = f(x, y), \forall x \in X, \forall y \in Y.$$  

Then $f/G : X/G \to Y$ is a strong fuzzy mapping w.r.t. $E_{X/G} \in E(X/G)$ and $E_Y$. In this case, $f/G$ is called the fuzzy quotient of $f$ by $G$.

**Proof.** From the definition of $f/G$, it is clear that $f/G$ satisfies the condition (f.1). Let $Gx_1, Gx_2 \in X/G$ and let $y_1, y_2 \in Y$. Then

$$(f/G)(Gx_1, y_1) \land (f/G)(Gx_2, y_2) \land E_{X/G}(Gx_1, Gx_2)$$

$$= f(x_1, y_1) \land f(x_2, y_2) \land G(x_1, x_2)$$

$$\leq f(x_1, y_1) \land f(x_2, y_2) \land R(x_1, x_2) \quad [\text{Since } G \subset R]$$

$$f(x_1, y_1) \land f(x_2, y_2) \land (\bigvee_{(c, d) \in Y \times Y} f(x_1, c) \land f(x_2, d) \land E_Y(c, d)).$$

Then $f/G$ satisfies the condition (f.2). Since $f$ is strong, it is clear that $f/G$ is strong. Hence $f/G : X/G \to Y$ is strong w.r.t. $E_{X/G}$ and $E_Y$.

**Proposition 5.4** Let $f$, $R$, $G$ and $f/G$ be same as in Proposition 5.3. Then $R/G$ is the fuzzy equivalence relation on $X/G$ determined by $f/G$.

**Proof.** Let $R_{f/G}$ be the fuzzy equivalence relation on $X/G$ determined by $f/G$ and let $Ga, Gb \in X/G$. Then

$$R_{f/G}(Ga, Gb)$$

$$= \bigvee_{(c, d) \in Y \times Y} [(f/G)(Ga, c) \land f/G(Gb, d) \land E_Y(c, d)]$$

$$= \bigvee_{(c, d) \in Y \times Y} [f(a, c) \land f(b, d) \land E_Y(c, d)]$$

$$= R(a, b) \quad [\text{By Proposition 5.1}]$$

Thus $R_{f/G} = R/G$. So $R/G$ is the fuzzy equivalence relation on $X/G$ determined by $f/G$.
Remark 5.4 Proposition 5.4 is the generalization of Theorem 3.26 in [6] in fuzzy setting.

Proposition 5.5 Let $R$ and $G$ be fuzzy equivalence relations on $X$ such that $G \subseteq R$. Then $\exists$ a strong and strong bijective fuzzy mapping $h : (X/G)/(R/G) \rightarrow X/R$.

Proof. By Proposition 4.2, $\exists$ a strong and strong surjective fuzzy mapping $\pi : X \rightarrow X/R$ w.r.t. $I_X \in E(X)$ and $E_{X/R} \in E(X/R)$. By Corollary 5.1, it is clear that $R$ is the fuzzy equivalence relation on $X$ determined by $\pi$. Then, by Proposition 5.3, $\pi/G : X/G \rightarrow X/R$ is strong w.r.t. $E_{X/G} \in E(X/G)$ and $E_{X/R}$. Thus, by Proposition 5.4, $R/G$ is the fuzzy equivalence relation determined by $\pi$. Since $\pi$ is strong surjective, $\pi/G$ is strong and surjective. Hence, by Corollary 5.2, $\exists$ a strong and strong bijective fuzzy mapping $h : (X/G)/(R/G) \rightarrow X/R$. □

The following is the immediate result of Proposition 5.5.

Corollary 5.5 Let $R$ and $G$ be any fuzzy equivalence relations on $X$. Then:

(a) $\exists$ a bijective fuzzy mapping $g : X/(R \circ G) \rightarrow (X/R)/(R \circ G/R)$.

(b) $\exists$ a bijective fuzzy mapping $h : X/R \rightarrow (X/R \cap G)/(R/R \cap G)$.

Proposition 5.6 Let $f : X \rightarrow Y$ be a strong and strong surjective fuzzy mapping w.r.t. $I_X \in E(X)$ and $E_Y \in E(Y)$, and let $R$ be an fuzzy equivalence relation on $X$. Then $f^2(R)$ is a fuzzy equivalence relation on $Y$. In this case, $f^2(R)$ is called the image of $R$ under $f$.

Proof. Let $y \in Y$. Then

$\begin{align*}
\left[f^2(R)(y,y')\right] & = \bigvee_{(x,x') \in X} [R(x,x') \wedge f^2((x,x'),(y,y))] \\
& = \bigvee_{(x,x') \in X} [R(x,x') \wedge f(x,y) \wedge f(x',y')] \\
& \geq R(x_0,x_0) \\
& \because f(x_0,y) = 1
\end{align*}$

Thus $f^2(R)$ is reflexive. From the definition of $f^2(R)$, it is clear that $f^2(R)$ is symmetric. Now let $y, y' \in Y$.

Then

$\begin{align*}
[f^2(R) \circ f^2(R)](y,y') & = \bigvee_{y' \in Y} [f^2(R)(y,y') \wedge f^2(R)(y',y'')] \\
& \leq \bigvee_{y \in Y} [R(x_0,x') \wedge R(x_0,x'') \wedge f(x,y) \wedge f(x',y'')] \\
& \leq \bigvee_{y \in Y} [R(x_0,x') \wedge R(x_0,x'') \wedge f(x,y) \wedge f(x',y'')] \\
& = f^2(G)(x_0,x') \\
& \because f^2(G)(x_0,x') \wedge f^2((x,x'),(y,y'))
\end{align*}$

Thus $R \subseteq f^2(G)$.

(b) $\Rightarrow$ Suppose $H = f^2(G)$ and let $y, y' \in Y$. Then

$\begin{align*}
[f^2(H)(y,y')] & = \bigvee_{(x,x') \in X \times X} [H(x,x') \wedge f^2((x,x'),(y,y'))] \\
& = \bigvee_{(x,x') \in X \times X} [f^{-2}(G)(x,x') \wedge f(x,y) \wedge f(x',y')] \\
& = f^{-2}(G)(x_0,x_0) \\
& \because f^{-2}(G)(x_0,x_0) \wedge f^2((x,x'),(y,y'))
\end{align*}$

Thus $f(x_0,y) = 1.$

Theorem 5.7 Let $f : X \rightarrow Y$ be strong and strong surjective w.r.t. $I_X \in E(X)$ and $E_Y \in E(Y)$, let $R$ be the fuzzy equivalence relation on $X$ induced by $f$ and let $G$ be any fuzzy equivalence relation on $Y$. Then:

(a) $R \subseteq f^{-2}(G)$.

(b) $H = f^{-2}(G)$ if and only it $G = f^2(H)$.

Hence $\exists$ a bijection $h : FER(Y) \rightarrow FER_R(X)$, where $FER_R(X)$ denotes the set of all fuzzy equivalence relations on $X$ containing $R$.

Proof. (a) Let $x, x' \in X$. Then

$\begin{align*}
R(x,x') & = \bigvee_{(y,y') \in Y \times Y} [f(x,y) \wedge f(x',y') \wedge E_Y(y,y')] \\
& \leq \bigvee_{(y,y') \in Y \times Y} [f(x,y) \wedge f(x',y')] \\
& = \bigvee_{(y,y') \in Y \times Y} [G(y_0,y_0) \wedge f(x,y) \wedge f(x',y')] \\
& \leq \bigvee_{(y,y') \in Y \times Y} [G(y_0,y_0) \wedge f^2((x,x'),(y,y'))] \\
& = f^{-2}(G)(x,x')
\end{align*}$

Thus $R \subseteq f^{-2}(G)$.

(b) $\Rightarrow$ Suppose $H = f^{-2}(G)$ and let $y, y' \in Y$. Then

$\begin{align*}
[f^2(H)(y,y')] & = \bigvee_{(x,x') \in X \times X} [H(x,x') \wedge f^2((x,x'),(y,y'))] \\
& = \bigvee_{(x,x') \in X \times X} [f^{-2}(G)(x,x') \wedge f(x,y) \wedge f(x',y')] \\
& = f^{-2}(G)(x_0,x_0) \\
& \because f^{-2}(G)(x_0,x_0) \wedge f^2((x,x'),(y,y'))
\end{align*}$

Thus $f(x_0,y) = 1.$

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Thus \( f^2(H) = G \).

(\( \Leftarrow \)): Suppose \( f^2(H) = G \) and let \( x, x' \in X \). Then

\[
\begin{align*}
f^{-2}(G)(x, x') &= \bigvee_{(y, y') \in Y \times Y} [G(y, y') \land f^2((x, x'), (y, y'))] \\
&= \bigvee_{(y, y') \in Y \times Y} [f^2(H)(y, y') \land f(x, y) \land f(x', y')]
\end{align*}
\]

Thus \( f^{-2}(G) = H \).

Now we define \( h: \text{FER}(Y) \to \text{FER}_R(X) \) as follows:

\[
\begin{align*}
h(G) &= f^{-2}(G).\]
\]

Since \( f \) is strong, \( \exists y_0, y'_0 \in Y \) such that

\[
\begin{align*}
f(x, y_0) &= f(x', y'_0) = 1.
\end{align*}
\]

Thus \( f^{-2}(G) = H \).

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