Fuzzy \((r, s)\)-\(S_1\)-pre-semicontinuous mappings

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Abstract

In this paper, we introduce the notion of fuzzy \((r, s)\)-\(S_1\)-pre-semicontinuous mappings on intuitionistic fuzzy topological spaces in Šostak’s sense, which is a generalization of \(S_1\)-pre-semicontinuous mappings by Shi-Zhong Bai. The relationship between fuzzy \((r, s)\)-pre-semicontinuous mapping and fuzzy \((r, s)\)-\(S_1\)-pre-semicontinuous mapping is discussed. The characterizations for the fuzzy \((r, s)\)-\(S_1\)-pre-semicontinuous mappings are obtained.

Key Words: intuitionistic fuzzy topological space, \((r, s)\)-\(S_1\)-pre-semicontinuous

1. Introduction

The concept of fuzzy set was introduced by Zadeh [1]. Chang [2] defined fuzzy topological spaces. These spaces and its generalizations are later studied by several authors, one of which, developed by Šostak [3], used the idea of degree of openness. This type of generalization of fuzzy topological spaces was later rephrased by Chattopadhyay and his colleagues [4], and by Ramadan [5]. Shi-Zhong Bai [6] introduced the concept of fuzzy \(S_1\)-pre-semicontinuous mappings on Chang’s fuzzy topological spaces. As a generalization of fuzzy sets, the concept of intuitionistic fuzzy sets was introduced by Atanassov [7]. Recently, Çoker and his colleagues [8, 9] introduced intuitionistic fuzzy topological spaces using intuitionistic fuzzy sets. Using the idea of degree of openness and degree of nonopenness, Çoker and Demirci [10] defined intuitionistic fuzzy topological spaces in Šostak’s sense as a generalization of smooth topological spaces and intuitionistic fuzzy topological spaces. In the previous works, we also studied the structure of the category of intuitionistic fuzzy topological spaces, and investigated properties of fuzzy strongly \((r, s)\)-precontinuous mappings in these spaces [11, 12].

In this paper, we introduce the notion of fuzzy \((r, s)\)-\(S_1\)-pre-semicontinuous mappings on intuitionistic fuzzy topological spaces in Šostak’s sense, which is a generalization of \(S_1\)-pre-semicontinuous mappings by Shi-Zhong Bai. The relationship between fuzzy \((r, s)\)-pre-semicontinuous mapping and fuzzy \((r, s)\)-\(S_1\)-pre-semicontinuous mapping is discussed. The characterizations for the fuzzy \((r, s)\)-\(S_1\)-pre-semicontinuous mappings are obtained.

2. Preliminaries

We will denote the unit interval \([0, 1]\) of the real line by \(I\). A member \(\mu\) of \(I^X\) is called a fuzzy set in \(X\). For any \(\mu \in I^X\), \(\mu^c\) denotes the complement \(1 - \mu\). By \(\overline{0}\) and \(\overline{1}\) we denote constant mappings on \(X\) with value 0 and 1, respectively. All other notations are standard notations of fuzzy set theory.

Let \(X\) be a nonempty set. An intuitionistic fuzzy set \(A\) is an ordered pair \(A = (\mu_A, \gamma_A)\) where the functions \(\mu_A : X \to I\) and \(\gamma_A : X \to I\) denote the degree of membership and the degree of nonmembership, respectively and \(\mu_A + \gamma_A \leq 1\). Obviously every fuzzy set \(\mu\) in \(X\) is an intuitionistic fuzzy set of the form \((\mu, 1 - \mu)\).

Definition 2.1 [7]). Let \(A = (\mu_A, \gamma_A)\) and \(B = (\mu_B, \gamma_B)\) be intuitionistic fuzzy sets in \(X\). Then

\[(1)\ A \subseteq B \text{ if and only if } \mu_A \leq \mu_B \text{ and } \gamma_A \geq \gamma_B.\]
\[(2)\ A = B \text{ iff } A \subseteq B \text{ and } B \subseteq A.\]
\[(3)\ A^c = (\gamma_A, \mu_A).\]
\[(4)\ A \cap B = (\mu_A \land \mu_B, \gamma_A \lor \gamma_B).\]
\[(5)\ A \cup B = (\mu_A \lor \mu_B, \gamma_A \land \gamma_B).\]
\[(6)\ \overline{0} = (\overline{0}, \overline{1}) \text{ and } \overline{1} = (\overline{1}, \overline{0}).\]

A smooth topology on \(X\) is a mapping \(T : I^X \to I\) which satisfies the following properties:

\[(1)\ T(\overline{0}) = T(\overline{1}) = 1.\]
(2) \( T(\mu_1 \land \mu_2) \geq T(\mu_1) \land T(\mu_2) \).
(3) \( T(\vee \mu_i) \geq \bigwedge T(\mu_i) \).

The pair \((X, T)\) is called a smooth topological space.

An intuitionistic fuzzy topology on \(X\) is a family \(T\) of intuitionistic fuzzy sets in \(X\) which satisfies the following properties:

1. If \( A_1, A_2 \in T \), then \( A_1 \cap A_2 \in T \).
2. If \( A_i \in T \) for each \( i \), then \( \bigcup A_i \in T \).

The pair \((X, T)\) is called an intuitionistic fuzzy topological space.

Let \( I(X) \) be the family of all intuitionistic fuzzy sets in \( X \) and let \( I \otimes I \) be the set of the pair \((r, s)\) such that \( r, s \in I \) and \( r + s \leq 1 \).

**Definition 2.2 ([10]).** Let \( X \) be a nonempty set. An intuitionistic fuzzy topology in Šostak’s sense (SoIFT for short) \( T = (T_1, T_2) \) on \( X \) is a mapping \( T: I(X) \rightarrow I \otimes I \) which satisfies the following properties:

1. \( T_1(\emptyset) = T_1(\{1\}) = 1 \) and \( T_2(\emptyset) = T_2(\{1\}) = 0 \).
2. \( T_1(A \cap B) \geq T_1(A) \cap T_1(B) \) and \( T_2(A \cap B) \leq T_2(A) \vee T_2(B) \).
3. \( T_1(A \cup B) \geq T_1(A) \cup T_1(B) \) and \( T_2(A \cup B) \leq \bigvee T_2(A) \).

The \((X, T) = (X, T_1, T_2)\) is said to be an intuitionistic fuzzy topological space in Šostak’s sense (SoIFTs for short). Also, we call \( T_1(A) \) a gradation of openness of \( A \) and \( T_2(A) \) a gradation of nonopenness of \( A \).

**Definition 2.3 ([13]).** Let \( A \) be an intuitionistic fuzzy set in SoIFTs \((X, T_1, T_2)\) and \((r, s) \in I \otimes I\). Then \( A \) is said to be

1. fuzzy \((r, s)\)-open if \( T_1(A) \geq r \) and \( T_2(A) \leq s \).
2. fuzzy \((r, s)\)-closed if \( T_1(A^c) \geq r \) and \( T_2(A^c) \leq s \).

**Definition 2.4 ([13]).** Let \((X, T_1, T_2)\) be a SoIFTs. For each \((r, s) \in I \otimes I\) and for each \( A \in I(X)\), the fuzzy \((r, s)\)-interior is defined by

\[
\text{int}(A, r, s) = \bigcup \{ B \in I(X) \mid B \subseteq A, \text{ B is fuzzy \((r, s)\)-open} \}
\]

and the fuzzy \((r, s)\)-closure is defined by

\[
\text{cl}(A, r, s) = \bigcap \{ B \in I(X) \mid A \subseteq B, \text{ B is fuzzy \((r, s)\)-closed} \}
\]

**Lemma 2.5 ([13]).** For an intuitionistic fuzzy set \( A \) in a SoIFTs \((X, T_1, T_2)\) and \((r, s) \in I \otimes I\),

1. \( \text{int}(A, r, s)^c = \text{cl}(A^c, r, s) \).
2. \( \text{cl}(A, r, s)^c = \text{int}(A^c, r, s) \).

Let \((X, T_1, T_2)\) be an intuitionistic fuzzy topological space in Šostak’s sense. Then it is easy to see that for each \((r, s) \in I \otimes I\), the family \( T_{(r,s)} \) defined by

\[
T_{(r,s)} = \{ A \in I(X) \mid T_1(A) \geq r \text{ and } T_2(A) \leq s \}
\]

is an intuitionistic fuzzy topology on \(X\).

Let \((X, T)\) be an intuitionistic fuzzy topological space and \((r, s) \in I \otimes I\). Then the map \( T^{(r,s)}: I(X) \rightarrow I \otimes I \) defined by

\[
T^{(r,s)}(A) = \begin{cases} (1, 0) & \text{if } A = \emptyset, \{1\}, \\ (r, s) & \text{if } A \in T - \{0,1\}, \\ (0, 1) & \text{otherwise} \end{cases}
\]

becomes an intuitionistic fuzzy topology in Šostak’s sense on \(X\).

Let \( \alpha, \beta \in [0,1] \) with \( \alpha + \beta \leq 1 \). An intuitionistic fuzzy point \( x_{(\alpha,\beta)} \) in \( X \) is an intuitionistic fuzzy set in \( X \) defined by

\[
x_{(\alpha,\beta)}(y) = \begin{cases} (\alpha, \beta) & \text{if } y = x, \\ (0, 1) & \text{if } y \neq x. \end{cases}
\]

In this case, \( x \) is called the support of \( x_{(\alpha,\beta)} \), \( \alpha \) the value of \( x_{(\alpha,\beta)} \), and \( \beta \) the nonvalue of \( x_{(\alpha,\beta)} \). An intuitionistic fuzzy point \( x_{(\alpha,\beta)} \) is said to belong to an intuitionistic fuzzy set \( A = (\mu_A, \gamma_A) \) in \( X \), denoted by \( x_{(\alpha,\beta)} \in A \), if \( \mu_A(x) \geq \alpha \) and \( \gamma_A(x) \leq \beta \). An intuitionistic fuzzy set \( A \) in \( X \) is the union of all intuitionistic fuzzy points which belong to \( A \).

**Definition 2.6 ([13]).** Let \( A \) be an intuitionistic fuzzy set in a SoIFTs \((X, T_1, T_2)\) and \((r, s) \in I \otimes I\). Then \( A \) is said to be

1. fuzzy \((r, s)\)-semipopen if there is a fuzzy \((r, s)\)-open set \( B \) in \( X \) such that \( B \subseteq A \subseteq cl(B, r, s) \).
2. fuzzy \((r, s)\)-semiclosed if there is a fuzzy \((r, s)\)-closed set \( B \) in \( X \) such that \( int(B, r, s) \subseteq A \subseteq B \).

**Theorem 2.7 ([13]).** Let \( A \) be an intuitionistic fuzzy set in a SoIFTs \((X, T_1, T_2)\) and \((r, s) \in I \otimes I\). Then the following statements are equivalent:

1. \( A \) is a fuzzy \((r, s)\)-semipopen set.
2. \( A^c \) is a fuzzy \((r, s)\)-semiclosed set.
3. \( cl(int(A, r, s), r, s) \supseteq A \).
4. \( int(cl(A^c, r, s), r, s) \subseteq A^c \).

**Definition 2.8 ([14, 15]).** Let \( A \) be an intuitionistic fuzzy set in a SoIFTs \((X, T_1, T_2)\) and \((r, s) \in I \otimes I\). Then \( A \) is said to be
Definition 2.11 ([14, 15]). Let \((X, T_1, T_2)\) be a SoIFTS. For each \((r, s) \in I \otimes I\) and for each \(A \in I(X)\),

1. the fuzzy strongly \((r, s)\)-semiinterior is defined by
   \[
   \text{ssint}(A, r, s) = \bigcup \{ B \in I(X) \mid B \subseteq A, B \text{ is fuzzy strongly } (r, s)\text{-semiopen} \},
   \]
   \(A\) is fuzzy strongly \((r, s)\)-semiopen set.
2. the fuzzy strongly \((r, s)\)-semiclosure is defined by
   \[
   \text{sscl}(A, r, s) = \bigcap \{ B \in I(X) \mid A \subseteq B, B \text{ is fuzzy strongly } (r, s)\text{-semiclosed} \}.
   \]

Definition 2.12 ([6]). Let \(f : (X_1, \delta_1) \rightarrow (X_2, \delta_2)\) be a mapping from a fuzzy space \(X_1\) to another fuzzy space \(X_2\), \(f\) is called a fuzzy \(S_1\)-pre-continuous mapping if \(f^{-1}(B)\) is a fuzzy pre-semiopen set of \(X_1\) for each fuzzy strongly semiopen set \(B\) of \(X_2\).

3. Fuzzy \((r, s)\)-\(S_1\)-pre-continuous mappings

Now, we define the notion of fuzzy \((r, s)\)-\(S_1\)-pre-continuous mappings, and then we investigate some of their characterestic properties.

Definition 3.1. Let \(f : (X, T_1, T_2) \rightarrow (Y, U_1, U_2)\) be a mapping from a SoIFTS \(X\) to a SoIFTS \(Y\) and \((r, s) \in I \otimes I\). Then \(f\) is called a fuzzy \((r, s)\)-\(S_1\)-pre-continuous mapping if \(f^{-1}(B)\) is fuzzy \((r, s)\)-pre-semiopen in \(X\) for each fuzzy strongly \((r, s)\)-semiopen set \(B\) in \(Y\).

Definition 3.2. Let \(f : (X, T_1, T_2) \rightarrow (Y, U_1, U_2)\) be a mapping from a SoIFTS \(X\) to a SoIFTS \(Y\) and \((r, s) \in I \otimes I\). Then \(f\) is said to be fuzzy \((r, s)\)-\(S_1\)-pre-continuous at an intuitionistic fuzzy point \(x_{(\alpha, \beta)}\) in \(X\) if for each fuzzy strongly \((r, s)\)-semiopen set \(B\) in \(Y\) with \(f(x_{(\alpha, \beta)}) \in B\), there is a fuzzy \((r, s)\)-pre-semiopen set \(A\) in \(X\) such that \(x_{(\alpha, \beta)} \in A\) and \(f(A) \subseteq B\).

Theorem 2.33. Let \(f : (X, T_1, T_2) \rightarrow (Y, U_1, U_2)\) be a mapping from a SoIFTS \(X\) to a SoIFTS \(Y\) \((r, s) \in I \otimes I\). Then \(f\) is fuzzy \((r, s)\)-\(S_1\)-pre-continuous if and only if \(f\) is fuzzy \((r, s)\)-\(S_1\)-pre-continuous for each intuitionistic fuzzy point \(x_{(\alpha, \beta)}\) in \(X\).

Remark 3.4. It is clear that every fuzzy \((r, s)\)-\(S_1\)-pre-continuous mapping is fuzzy \((r, s)\)-pre-continuous. However, the following example shows that the converse need not be true.

Example 3.5. Let \(X = \{a, y, z\}\) and let \(A_1, A_2, A_3\) be intuitionistic fuzzy sets in \(X\) defined as
\[
A_1(x) = (0.3, 0.6), A_1(y) = (0.2, 0.6), A_1(z) = (0.4, 0.5);
\]
$A_2(x) = (0.4, 0.5), A_2(y) = (0.2, 0.6), A_2(z) = (0.5, 0.5);$ 
and 
$A_3(x) = (0.2, 0.7), A_3(y) = (0.6, 0.2), A_3(z) = (0.4, 0.5).$

Define $T : I(X) \rightarrow I \otimes I$ and $U : I(X) \rightarrow I \otimes I$ by

$$T(A) = (T_1(A), T_2(A)) = \begin{cases} 
(1, 0) & \text{if } A = A_0, \\
(1/2, 1/2) & \text{if } A = A_1, A_2, \\
(0, 1) & \text{otherwise};
\end{cases}$$

and

$$U(A) = (U_1(A), U_2(A)) = \begin{cases} 
(1, 0) & \text{if } A = A_0, \\
(1/2, 1/2) & \text{if } A = A_3, \\
(0, 1) & \text{otherwise}.
\end{cases}$$

Then $T$ and $U$ are SoIFTSs on $X$. Consider a mapping $f : (X, T) \rightarrow (X, U)$ defined by $f(x) = x, f(y) = y,$ and $f(z) = z$. Note that

$$A_i^c \subseteq \text{int}(\text{cl}(A_i^c, 1/2, 1, 1/2, 1/2, 1/2, 3/2))$$

$$= \text{int}(\text{cl}(A_3, 1/2, 1, 1/2, 1/2, 3/2))$$

$$= \text{int}(1/2, 1, 1/2, 3/2) = 1.$$

Hence $A_i^c$ is fuzzy strongly $(1/2, 1, 1/2)$-semiopen in $(X, U)$. Since

$$f^{-1}(A_3) = A_3 \subseteq \text{sint}(\text{cl}(A_3, 1/2, 1, 1/2, 1/2, 3/2))$$

$$= \text{sint}(A_2^c, 1/2, 1, 3/2) = A_2^c,$$

$f$ is fuzzy $(1/2, 1, 1/2)$-pre-semicontinuous. But $f$ is not fuzzy $(1/2, 1, 1/2)$-$S_1$-pre-semicontinuous, because

$$f^{-1}(A_i^c) = A_i^c \nsubseteq \text{sint}(\text{cl}(A_i^c, 1/2, 1, 1/2, 1/2, 3/2))$$

$$= \text{sint}(A_i^c, 1/2, 1, 3/2) = A_i^c.$$ 

Theorem 3.6. Let $f : (X, T_1, T_2) \rightarrow (Y, U_1, U_2)$ be a mapping from a SoIFTS $X$ to a SoIFTS $Y$ and $(r, s) \in I \otimes I$. Then the following statements are equivalent:

1. $f$ is fuzzy $(r, s)$-$S_1$-pre-semicontinuous.
2. $f^{-1}(B)$ is fuzzy $(r, s)$-pre-semiclosed in $X$ for each fuzzy strongly $(r, s)$-semiclosed set $B$ in $Y$.
3. For each intuitionistic fuzzy set $A$ in $X$,

$$f(\text{scl}(\text{int}(A, r, s), r, s)) \subseteq \text{sscl}(f(A), r, s).$$

4. For each intuitionistic fuzzy set $B$ in $Y$,

$$\text{scl}(\text{int}(f^{-1}(B), r, s), r, s)) \subseteq f^{-1}(\text{sscl}(B, r, s)).$$

(5) For each intuitionistic fuzzy set $B$ in $Y$,

$$f^{-1}(\text{ssint}(B, r, s)) \subseteq \text{sint}(f^{-1}(B), r, s), r, s).$$

Proof. (1) $\Leftrightarrow$ (2) Trivial.

(2) $\Rightarrow$ (3) Let $A \in I(X)$, then $f(A) \in I(Y)$. Since $\text{sscl}(f(A), r, s)$ is fuzzy strongly $(r, s)$-semiclosed in $Y$, $f^{-1}(\text{sscl}(f(A), r, s))$ is fuzzy $(r, s)$-pre-semiclosed in $X$ from (2). Hence

$$\text{scl}(\text{int}(A, r, s), r, s) \nsubseteq \text{sscl}(f(A), r, s)$$

$$\subseteq \text{scl}(\text{sscl}(f(A), r, s), r, s) \subseteq f^{-1}(\text{sscl}(f(A), r, s)).$$

Thus we have $f(\text{sscl}(A, r, s), r, s) \subseteq \text{sscl}(f(A), r, s)$.

(3) $\Rightarrow$ (4) Let $B \in I(Y)$, then $f^{-1}(B) \in I(X)$. By (3), we have

$$f(\text{sscl}(f^{-1}(B), r, s), r, s)) \subseteq \text{sscl}(f^{-1}(B), r, s)) \subseteq \text{sscl}(B, r, s).$$

Thus $\text{scl}(f^{-1}(B), r, s), r, s) \subseteq f^{-1}(\text{sscl}(B, r, s))$.

(4) $\Rightarrow$ (5) Let $B \in I(Y)$. By (4), we obtain

$$\text{scl}(f^{-1}(B), r, s), r, s) \subseteq f^{-1}(\text{sscl}(B, r, s)).$$

Hence we have

$$f^{-1}(\text{ssint}(B, r, s)) \subseteq \text{sint}(f^{-1}(B), r, s), r, s).$$

(5) $\Rightarrow$ (1) Let $B$ be a fuzzy $(r, s)$-pre-semiclosed set in $Y$. Then by (5), we have

$$f^{-1}(B) = f^{-1}(\text{ssint}(B, r, s)) \subseteq \text{scl}(f^{-1}(B), r, s), r, s).$$

Thus $f^{-1}(B)$ is fuzzy $(r, s)$-pre-semiclosed in $X$. Hence $f$ is fuzzy $(r, s)$-$S_1$-pre-semicontinuous.

Theorem 3.7. Let $f : (X, T_1, T_2) \rightarrow (Y, U_1, U_2)$ be a bijective mapping from a SoIFTS $X$ to a SoIFTS $Y$ and $(r, s) \in I \otimes I$. Then $f$ is fuzzy $(r, s)$-$S_1$-pre-semicontinuous if and only if $\text{ssmint}(f(A), r, s) \subseteq f(\text{cl}(\text{int}(A, r, s), r, s))$ for each intuitionistic fuzzy set $A$ in $X$.

Proof. Suppose that $f$ is fuzzy $(r, s)$-$S_1$-pre-semicontinuous. Let $A \in I(X)$, then $f(A) \in I(Y)$. Since $\text{ssmint}(f(A), r, s)$ is fuzzy $(r, s)$-pre-semiclosed in $Y$, $f^{-1}(\text{ssmint}(f(A), r, s))$ is fuzzy $(r, s)$-pre-semiclosed in $X$. Hence we have

$$\text{ssmint}(f(A), r, s)$$

$$= f^{-1}(\text{ssmint}(f(A), r, s))$$

$$\subseteq f(\text{ssmint}(f^{-1}(f(A)), r, s), r, s))$$

$$\subseteq f(\text{ssmint}(f^{-1}(f(A)), r, s), r, s)) = f(\text{ssmint}(A, r, s), r, s)).$$

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Conversely, let \( B \) be a fuzzy strongly \((r, s)\)-semiopen set in \( Y \). Then \( f^{-1}(B) \in \mathcal{I}(X) \). By hypothesis, we have

\[
B = \text{ssint}(B, r, s) = \text{ssint}(f(f^{-1}(B)), r, s) \subseteq f(\text{cl}(f^{-1}(B), r, s), r, s)).
\]

Thus \( f^{-1}(B) \subseteq \text{sin}(\text{cl}(f^{-1}(B), r, s), r, s)) \). Hence \( f^{-1}(B) \) is fuzzy \((r, s)\)-pre-semiopen in \( X \). Therefore \( f \) is fuzzy \((r, s)\)-\(S_1\)-pre-continuous. \( \square \)

**Theorem 3.8.** Let \( f : (X, T) \to (Y, U) \) and \( g : (Y, U) \to (Z, S) \) be mappings. If \( f \) is fuzzy \((r, s)\)-\(S_1\)-pre-continuous and \( g \) is fuzzy strongly \((r, s)\)-semiconnuous then \( g \circ f \) is fuzzy \((r, s)\)-pre-continuous.

**Proof.** Let \( B \) be a fuzzy \((r, s)\)-open set in \( Y \). Since \( g \) is fuzzy strongly \((r, s)\)-semiconnuous, \( g^{-1}(B) \) is fuzzy strongly \((r, s)\)-semiconnuous in \( Y \). Because \( f \) is fuzzy \((r, s)\)-\(S_1\)-pre-continuous, \( f^{-1}(g^{-1}(B)) = (g \circ f)^{-1}(B) \) is fuzzy \((r, s)\)-pre-semiopen in \( X \). Thus \( g \circ f \) is fuzzy \((r, s)\)-pre-continuous. \( \square \)

**References**


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