Interval-valued Fuzzy Ideals and Bi-ideals of a Semigroup

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Abstract

We apply the concept of interval-valued fuzzy sets to theory of semigroups. We give some properties of interval-valued fuzzy ideals and interval-valued fuzzy bi-ideals, and characterize which is left [right] simple, left [right] duo and a semilattice of left [right] simple semigroups or another type of semigroups in terms of interval-valued fuzzy ideals and interval-valued fuzzy bi-ideals.

Key Words: interval-valued fuzzy set, interval-valued fuzzy semigroup, interval-valued fuzzy ideal, interval-valued fuzzy bi-ideal, interval-valued fuzzy duo.

1. Introduction

As a generalization of fuzzy sets introduced by Zadeh\(^14\), he\(^15\) introduced the concept of interval-valued fuzzy sets. After that time, Gorzalczany\(^4\) applied it to a method of inference in approximate reasoning, Biswass\(^\) to group theory and Montal and Samanta\(^12\) to topology. Recently, Hur et al.\(^5\) introduced the notion of interval-valued fuzzy relations and obtained some of its properties. Moreover, Choi et al.\(^3\) introduced the concept of interval-valued smooth topological spaces and studied it. Kang and Hur\(^6\) investigated interval-valued fuzzy subgroups and rings.

In this paper, we apply the notion of interval-valued fuzzy sets to theory of semigroups. We give some properties of interval-valued fuzzy ideals and interval-valued fuzzy bi-ideals, and characterize which is left [right] simple, left [right] duo and a semilattice of left [right] simple semigroups or another type of semigroups in terms of interval-valued fuzzy ideals and interval-valued fuzzy bi-ideals.

2. Preliminaries

We will list some concepts needed in the later sections.

Let \(D(I)\) be the set of all closed subintervals of the unit interval \(I = [0, 1]\). The elements of \(D(I)\) are generally denoted by capital letters \(M, N, \cdot \cdot \cdot\), and note that \(M = [M^L, M^U]\), where \(M^L\) and \(M^U\) are the lower and the upper end points respectively. Especially, we denoted \(0 = [0, 0]\), \(1 = [1, 1]\), and \(a = [a, a]\) for every \(a \in (0, 1)\).

We also note that

(i) \(\forall M, N \in D(I)\)
\[M = N \iff M^L = N^L, M^U = N^U\],

(ii) \(\forall M, N \in D(I)\)
\[M \leq N \iff M^L \leq N^L, M^U \leq N^U\].

For every \(M \in D(I)\), the complement of \(M\), denoted by \(M^c\), is defined by \(M^c = 1 - M = [1 - M^U, 1 - M^L]\) (See \(12\)).

Definition 2.1. \(^{[4, 12, 15]}\) A mapping \(A : X \rightarrow D(I)\) is called an interval-valued fuzzy set (in short, IVFS) in \(X\), denoted by \(A = [A^L, A^U]\), if \(A^L, A^U \in I^X\) such that \(A^L \leq A^U\), i.e., \(A^L(x) \leq A^U(x)\) for each \(x \in X\), where \(A^L(x)[resp. A^U(x)]\) is called the lower[resp. upper] end point of \(x\) to \(A\). For any \([a, b] \in D(I)\), the interval-valued fuzzy set \(A\) in \(X\) defined by \(A(x) = [A^L(x), A^U(x)] = [a, b]\) for each \(x \in X\) is denoted by \([a, b]\) and if \(a = b\), then the IVFS \([a, b]\) is denoted by simply \(a\). In particular, \(0\) and \(1\) denote the interval-valued fuzzy empty set and the interval-valued fuzzy whole set in \(X\), respectively.

We will denote the set of all IVFSs in \(X\) as \(D(I)^X\). It is clear that \(A = [A, A] \in D(I)^X\) for each \(A \in I^X\).

Definition 2.2. \(^{[12]}\) Let \(A, B \in D(I)^X\) and let \(\{A_\alpha\}_{\alpha \in \Gamma} \subset D(I)^X\). Then:

(i) \(A \subset B\) iff \(A^L \leq B^L\) and \(A^U \leq B^U\).

(ii) \(A = B\) iff \(A \subset B\) and \(B \subset A\).
(iii) \( A^c = [1 - A^U, 1 - A^L] \).

(iv) \( A \cup B = [A^L \lor B^L, A^U \lor B^U] \).

\[
\bigcup_{\alpha \in \Gamma} A_\alpha = \left[ \bigvee_{\alpha \in \Gamma} A^L_\alpha, \bigvee_{\alpha \in \Gamma} A^U_\alpha \right].
\]

(v) \( A \cap B = [A^L \land B^L, A^U \land B^U] \).

\[
\bigcap_{\alpha \in \Gamma} A_\alpha = \left[ \bigwedge_{\alpha \in \Gamma} A^L_\alpha, \bigwedge_{\alpha \in \Gamma} A^U_\alpha \right].
\]

**Definition 2.3.** [6] An interval-valued fuzzy set \( A \) in \( G \) is called an interval-valued fuzzy subgroupoid (in short, IVGP) in \( G \) if

\[
A^L(xy) \geq A^L(x) \land A^L(y),
\]

and

\[
A^U(xy) \geq A^U(x) \land A^U(y), \quad \forall x, y \in G.
\]

It is clear that \( \tilde{0}, \tilde{1} \in \text{IVGP}(G) \). We will denote the IVGPs in \( G \) as IVGP(G).

**Definition 2.4.** [6] Let \( A \) be an IVFS of a group \( G \) and \( \lambda, \mu \in D(I) \). Then the subgroup \( A^{[\lambda, \mu]} \) is called a \([\lambda, \mu]\)-level subset of \( A \).

### 3. Interval-valued fuzzy ideals and bi-ideals of a semigroup

Let \( S \) be a semigroup. By a subsemigroup of \( S \) we mean a non-empty subset \( A \) of \( S \) such that \( A^2 \subseteq A \) and by a left [resp. right] ideal of \( S \) we mean a non-empty subset \( A \) of \( S \) such that

\[
SA \subseteq A \quad \text{[resp. \( AS \subseteq A \)]}.
\]

By two-sided ideal or simply ideal we mean a subset \( A \) of \( S \) which is both a left and a right ideal of \( S \). A semigroup \( S \) is said to be left [resp. right] simple if \( S \) itself is the only left [resp. right] ideal of \( S \). \( S \) is said to be simple if it contains no proper ideal.

**Definition 3.1.** Let \( S \) be a semigroup and let \( A \in D(I)^S \). Then \( A \) is called an:

1. **interval-valued fuzzy subsemigroup** (in short, IVSG) of \( S \) if
   \[
   A^L(xy) \geq A^L(x) \land A^L(y),
   \]
   and
   \[
   A^U(xy) \geq A^U(x) \land A^U(y)
   \]
   for any \( x, y \in S \).

2. **interval-valued fuzzy left ideal** (in short, IVLI) of \( S \) if
   \[
   A^L(xy) \geq A^L(y), \quad \text{and} \quad A^U(xy) \geq A^U(y)
   \]
   for any \( x, y \in S \).

3. **interval-valued fuzzy right ideal** (in short, IVRI) of \( S \) if
   \[
   A^L(xy) \geq A^L(x), \quad \text{and} \quad A^U(xy) \geq A^U(x)
   \]
   for any \( x, y \in S \).

4. **interval-valued fuzzy (two-sided) ideal** (in short, IVI) of \( S \) if it is both an interval-valued fuzzy left and an interval-valued fuzzy right ideal of \( S \).

We will denote the set or all IVSGs [resp. IVLIs, IVRIs and IVIs] of \( S \) as IVSG(S) [resp. IVLI(S), IVRI(S) and IVI(S)].

It is clear that \( A \in \text{IVI}(S) \) if and only if

\[
A^L(xy) \geq A^L(x) \land A^L(y),
\]

and

\[
A^U(xy) \geq A^U(x) \land A^U(y)
\]

for any \( x, y \in S \), and if \( A \in \text{IVLI}(S) \), then \( A \in \text{IVSG}(S) \).

**Remark 3.2.** Let \( S \) be a semigroup.

(a) If \( A \) is a fuzzy subsemigroup of \( S \), then

\[
\]

(b) If \( A \in \text{IVSG}(S) \) [resp. \( \text{IVI}(S), \text{IVLI}(S) \) and \( \text{IVRI}(S) \)], then \( A^L \) and \( A^U \) are fuzzy subsemigroup [resp. ideal, left ideal and right ideal] of \( S \).

**Result 3.A.** [6, Proposition 3.7] Let \( A \) be a non-empty subset of a groupoid \( S \). \( A \) is a subgroupoid of \( S \) if and only if \( \langle A, A \rangle \in \text{IVGP}(S) \).

The following is the immediate result of Definition 3.1 and Result 3.A.

**Theorem 3.3.** Let \( A \) be a non-empty subset of a semigroup \( S \). Then \( A \) is a subsemigroup of \( S \) if and only if \( \langle A, A \rangle \in \text{IVSG}(S) \).

**Result 3.B.** [6, Proposition 6.6] Let \( R \) be a ring. Then \( A \) is an ideal [resp. a left ideal and a right ideal] of \( R \) if and only if \( \langle A, A \rangle \in \text{IVI}(R) \) [resp. \( \text{IVRI}(R) \) and \( \text{IVLI}(R) \)].

The following is the immediate result of Definition 3.1 and Result 3.B.

**Theorem 3.4.** Let \( A \) be a nonempty subset of a semigroup \( S \). Then \( A \) is an ideal [resp. a left ideal and a right ideal] of \( S \) if and only if \( \langle A, A \rangle \in \text{IVI}(S) \) [resp. \( \text{IVLI}(S) \) and \( \text{IVRI}(S) \)].

**Proposition 3.5.** Let \( S \) be a semigroup. If \( A \in \text{IVSG}(S) \) [resp. \( \text{IVLI}(S), \text{IVRI}(S) \) and \( \text{IVI}(S) \)], then \( A^{[\lambda, \mu]} \) is a subsemigroup [resp. ideal, left ideal and right ideal] of \( S \).
The following result is the converse of Proposition 3.5:

**Proposition 3.6.** Let $S$ be a semigroup and let $A \in D(I)^S$. If $A^{[\lambda, \mu]}$ is a subsemigroup [resp. ideal, left ideal and right ideal] of $S$ for each $[\lambda, \mu] \in D(I)$, then $A \in IVS(S)$ [resp. IVI(S), IVLI(S) and IVRI(S)].

**Proof.** Suppose $A^{[\lambda, \mu]}$ is a subsemigroup of $S$ for each $[\lambda, \mu] \in D(I)$. For any $x, y, z \in S$, let $A(x) = [\lambda_1, \mu_1]$ and $A(y) = [\lambda_2, \mu_2]$. Then $A^L(x) = [\lambda_1 \wedge \lambda_2, \mu_1 \wedge \mu_2]$ and $A^U(y) = [\lambda_2 \wedge \lambda_1, \mu_2 \wedge \mu_1]$. Thus $x, y \in A^{[\lambda_1 \wedge \lambda_2, \mu_1 \wedge \mu_2]}.$ Since $[\lambda_1 \wedge \lambda_2, \mu_1 \wedge \mu_2] \subseteq D(I)$, by the hypothesis, $x y \in A^{[\lambda_1 \wedge \lambda_2, \mu_1 \wedge \mu_2]}$. Then $A^L(xy) \geq [\lambda_1 \wedge \lambda_2, \mu_1 \wedge \mu_2] = A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq [\lambda_1 \wedge \lambda_2, \mu_1 \wedge \mu_2] = A^U(x) \wedge A^U(y).$ Hence $A \in IVS(S)$.

Now suppose $A^{[\lambda, \mu]}$ is a left ideal of $S$ for each $[\lambda, \mu] \in D(I)$. For each $y \in S$, let $A(y) = [\lambda, \mu]$. Then clearly $y \in A^{[\lambda, \mu]}.$ Let $x \in S$. Then, by the hypothesis, $x y \in A^{[\lambda, \mu]}$. Thus $A^L(xy) \geq [\lambda, \mu] = A^L(y)$ and $A^U(xy) \geq [\lambda, \mu] = A^U(y).$ Hence $A \in IVLI(S)$.

Also, we easily see the rest. This completes the proof.

A subsemigroup $A$ of a semigroup $S$ is called a bi-ideal of $S$ if $ASA \subseteq A$. We will denote the set of all bi-ideals of $S$ by $B(S)$.

**Definition 3.7.** Let $S$ be a semigroup and let $A \in IVS(G(S)).$ Then $A$ is called an interval-valued fuzzy bi-ideal (in short, IVBI) of $S$ if

$$A^L(xyz) \geq A^L(x) \wedge A^L(z),$$

and

$$A^U(xyz) \geq A^U(x) \wedge A^U(z)$$

for any $x, y, z \in S$.

We will denote the set of all IVBIs of $S$ as $IVBI(S)$. The following result shows that the concept of an IVBI in a semigroup is an extended one of a bi-ideal.

**Theorem 3.8.** Let $A$ be a non-empty subset of a semigroup $S$. Then $A$ is a bi-ideal of $S$ if and only if $[\chi_A, \chi_A] \in IVBI(S)$.

**Proof.** ($\Rightarrow$): Suppose $A \in B(S)$ and let $x, y, z \in S$.

Case (i): Suppose $x \in A$ and $z \in A$. Then $\chi_A(x) = \chi_A(z) = 1$. Since $A$ is a bi-ideal of $S$, $xyz \in ASA \subseteq A$. Thus $\chi_A(xyz) = 1 = \chi_A(x) \wedge \chi_A(z)$.

Case (ii): Suppose $x \notin A$ or $z \notin A$. Then $\chi_A(x) = 0$ or $\chi_A(z) = 0$. Thus $\chi_A(xyz) \geq 0 = \chi_A(x) \wedge \chi_A(z)$. So, in either cases, $\chi_A(xyz) \geq \chi_A(x) \wedge \chi_A(z)$. Moreover, by Theorem 3.2, $[\chi_A, \chi_A] \in IVS(S)$. Hence $[\chi_A, \chi_A] \in IVBI(S)$.

($\Leftarrow$): Suppose $[\chi_A, \chi_A] \in IVBI(S)$. Let $t \in ASA$. Then there exist $x, z \in A$ and $y \in S$ such that $t = xyz$. Since $x, z \in A$, $\chi_A(x) = \chi_A(z) = 1$. Since $[\chi_A, \chi_A] \in IVBI(S)$, $\chi_A(xyz) \geq \chi_A(x) \wedge \chi_A(z) = 1$. Then $\chi_A(xyz) = 1$. Thus $t = xyz \in A$. So $ASA \subseteq A$. Moreover, by Theorem 3.3, $A$ is a subsemigroup of $S$. Hence $A \in B(S)$.

**Theorem 3.9.** Let $S$ be a semigroup. Then $S$ is a group if and only if every IVBI of $S$ is a constant mapping.

**Proof.** ($\Rightarrow$): Suppose $S$ is a group with the identity $e$. Let $A \in IVBI(S)$, and let $a \in S$. Then

$$A^L(a) = A^L(eae) \geq A^L(e) \wedge A^L(e) = A^L(e) = A^L(1) = A^L((aa^{-1})(a^{-1}a)) = A^L(a) \wedge A^L(a) = A^L(a).$$

By the similar arguments, we have that $A^U(a) \geq A^U(a)$. Thus $A(a) = A(e)$. Hence $A$ is a constant mapping.

($\Leftarrow$): Suppose the necessary condition holds. Assume that $S$ is not a group. Then it follows from p.84 in [2] that $S$ contains a proper bi-ideal $A$ of $S$. Then there exists an $x \in S$ such that $x \notin A$. Let $y \in A$ with $y \neq x$. Since $A$ is a bi-ideal of $S$, by Theorem 3.8, $[\chi_A, \chi_A] \in IVBI(S)$. By the hypothesis, $[\chi_A, \chi_A]$ is a constant mapping. Thus $[\chi_A, \chi_A](x) = [\chi_A, \chi_A](y)$, i.e., $\chi_A(x) = \chi_A(y)$. Since $x \notin A$ and $y \in A$, $\chi_A(x) = 0 < \chi_A(y) = 1$, i.e., $[\chi_A, \chi_A](x) = [0, 0] \neq [1, 1] = [\chi_A, \chi_A](y)$. This is a contradiction. Hence $S$ is a group. This completes the proof.

**Proposition 3.10.** Every IVLI[resp. IVRI and IVI] of $S$ is an IVBI of $S$.

**Proof.** Suppose $A \in IVLI(S)$, and let $x, y, z \in S$. Then

$$A^L(xyz) = A^L((xyz)z) \geq A^L(z) \geq A^L(x) \wedge A^L(z)$$

and

$$A^U(xyz) = A^U((xyz)y) \geq A^U(z) \geq A^U(x) \wedge A^U(z).$$

So $A \in IVBI(S)$. Similarly, we can see that the other case holds.

**Theorem 3.11.** Let $S$ be a semigroup and let $A \in D(I)^S$. Then $A \in IVBI(S)$ if and only if $A^{[\lambda, \mu]} \in B(S)$ for each $[\lambda, \mu] \in D(I)$.

**Proof.** ($\Rightarrow$): Suppose $A \in IVBI(S)$, and let $[\lambda, \mu] \in D(I)$. Since $A \in IVSGS$, by Proposition 3.5, $A^{[\lambda, \mu]}$ is a subsemigroup of $S$. Let $t \in A^{[\lambda, \mu]}$. Then there exist $x, z \in A^{[\lambda, \mu]}$ and $y \in S$ such that $t = xyz$. Since $A \in IVBI(S)$, we have

$$A^L(t) \geq A^L(x) \wedge A^L(z) \geq \lambda,$$

and

$$A^U(t) \geq A^U(x) \wedge A^U(y) \geq \mu.$$
Thus $t \in A^{[\lambda, \mu]}$. So $A^{[\lambda, \mu]}SA^{[\lambda, \mu]} \subset A^{[\lambda, \mu]}$. Hence $A^{[\lambda, \mu]} \in \mathcal{B}(S)$.

($\Leftarrow$): Suppose the necessary condition holds. Since $A^{[\lambda, \mu]}$ is a subsemigroup of $S$, by Proposition 3.6, $A \in \mathcal{IVSG}(S)$. For any $x, z \in S$, let $A(x) = [\lambda_1, \mu_1]$ and $A(z) = [\lambda_2, \mu_2]$. Then, by the process of the proof of Proposition 3.6, $x, z \in A^{[\lambda_1, \lambda_2, \mu_1, \mu_2]}$. Let $y \in S$. Then, by the hypothesis, $xyz \in A^{[\lambda_1, \lambda_2, \mu_1, \mu_2]}$. Thus

$$A^L(xyz) \geq \lambda_1 \wedge \lambda_2 = A^L(x) \wedge A^L(z),$$

and

$$A^U(xyz) \geq \mu_1 \wedge \mu_2 \leq A^U(x) \wedge A^U(z).$$

Hence $A \in \mathcal{IVBI}(S)$. This completes the proof. \hfill $\Box$

### 4. Interval-valued fuzzy duos, ideals and bi-ideals of a regular semigroup

A semigroup $S$ is said to be regular of for each $a \in S$ there exists an $x \in S$ such that $a = axa$.

A semigroup $S$ is said to be left duo [resp. right duo] if every left [resp. right] ideal of $S$ is a two-sided ideal of $S$.

A semigroup $S$ is said to be duo if it is both left and right duo.

**Definition 4.1.** A semigroup $S$ is said to be:

1. interval-valued fuzzy left duo (in short, $\mathcal{IVLD}$) if every $\mathcal{IVLI}$ of $S$ is an $\mathcal{IVI}$ of $S$.
2. interval-valued fuzzy right duo (in short, $\mathcal{IVRD}$) if every $\mathcal{IVRI}$ of $S$ is an $\mathcal{IVI}$ of $S$.
3. interval-valued fuzzy duo (in short, $\mathcal{IVD}$) if it is both interval-valued fuzzy left and interval-valued fuzzy right duo.

**Theorem 4.2.** Let $S$ be a regular semigroup. Then $S$ is left duo if and only if $S$ is $\mathcal{IVLD}$.

**Proof.** ($\Rightarrow$): Suppose $S$ is left duo. Let $A \in \mathcal{IVLI}(S)$ and let $a, b \in S$. Then, by the process of the proof of Theorem 3.1 in [8], $ab \in (aSa)b \subset (Sa)S \subset Sa$. Thus there exists an $x \in S$ such that $ab = axa$. Since $A \in \mathcal{IVLI}(S)$, $A^L(ab) = A^L(xa) \geq A^L(a)$, and $A^U(ab) = A^U(xa) \geq A^U(a)$. Hence $A \in \mathcal{IVRI}(S)$.

($\Leftarrow$): Suppose $S$ is $\mathcal{IVLD}$, and let $A$ be any left ideal of $S$. Then, by Theorem 3.4, $[\chi_A, \chi_A] \in \mathcal{IVLI}(S)$. By the assumption, $[\chi_A, \chi_A] \in \mathcal{IVI}(S)$. Since $A \neq \emptyset$, by Theorem 3.4, $A$ is an ideal of $S$. Hence $S$ is left duo. This completes the proof. \hfill $\Box$

**Theorem 4.2’.** [The dual of Theorem 4.2] Let $S$ be a regular semigroup. Then $S$ is right duo if and only if $S$ is $\mathcal{IVRD}$.

The following is the immediate result of Theorem 4.2 and 4.2’.

**Theorem 4.3.** Let $S$ be a regular semigroup. Then $S$ is duo if and only if $S$ is $\mathcal{IVD}$.

**Theorem 4.4.** Let $S$ be a regular semigroup. Then every bi-ideal of $S$ is a right ideal of $S$ if and only if every $\mathcal{IVBI}$ of $S$ is an $\mathcal{IVRI}$ of $S$.

**Proof.** ($\Rightarrow$): Suppose every bi-ideal of $S$ is a right ideal of $S$. Let $A \in \mathcal{IVBI}(S)$ and let $a, b \in S$. Then, by the process of proof of Theorem 3.4 in [8], $ab \in (aSa)b \subset aSa$. Thus there exists an $x \in S$ such that $ab = axa$. Since $A \in \mathcal{IVBI}(S)$, we have $A^L(ab) = A^L(xa) \geq A^L(a) \wedge A^L(a) = A^L(a)$, and $A^U(ab) = A^U(xa) \geq A^U(a) \wedge A^U(a) = A^U(a)$.

Hence $A \in \mathcal{IVRI}(S)$.

($\Leftarrow$): Suppose that every $\mathcal{IVBI}$ of $S$ is an $\mathcal{IVRI}$ of $S$, and let $A$ be any bi-ideal of $S$. Then, by Theorem 3.8, $[\chi_A, \chi_A] \in \mathcal{IVBI}(S)$. By the assumption, $[\chi_A, \chi_A] \in \mathcal{IVRI}(S)$. Since $A \neq \emptyset$, by Theorem 3.4, $A$ is a right ideal of $S$. This completes the proof. \hfill $\Box$

**Result 4.4A.** [11, Theorem 3] Every bi-ideal of a regular left duo semigroup $S$ is a right ideal of $S$.

**Corollary 4.5.** Let $S$ be a regular duo semigroup. Then every $\mathcal{IVBI}$ of $S$ is an $\mathcal{IVRI}$ of $S$.

**Proof.** By Result 4.4A, every bi-ideal of $S$ is a right ideal of $S$. Hence, by Theorem 4.3, it follows that every $\mathcal{IVBI}$ of $S$ is an $\mathcal{IVRI}$ of $S$. \hfill $\Box$

**Theorem 4.4’.** [The dual of Theorem 4.4] Let $S$ be a regular semigroup. Then every bi-ideal of $S$ is a left ideal of $S$ if and only if every $\mathcal{IVBI}$ of $S$ is an $\mathcal{IVLI}$ of $S$.

The following is the immediate result of Theorem 4.4 and 4.4’.

**Theorem 4.6.** Let $S$ be a regular duo semigroup. Then every bi-ideal of $S$ is an ideal of $S$ if and only if every $\mathcal{IVBI}$ of $S$ is an $\mathcal{IVRI}$ of $S$.  

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A semigroup \( S \) is called a semilattice of groups [2] if it is the set-theoretical union of a set of mutually disjoint subgroups \( G_\alpha (\alpha \in \Gamma) \), i.e., \( S = \bigcup_{\alpha \in \Gamma} G_\alpha \) such that for any \( \alpha, \beta \in \Gamma \), \( G_\alpha G_\beta \subseteq G_\gamma \) and \( G_\beta G_\alpha \subseteq G_\gamma \), for some \( \gamma \in \Gamma \).

**Result 4.B.** [10, Theorem 4] Every bi-ideal of a semigroup \( S \) which is a semilattice of groups, is an ideal of \( S \).

The following is the immediate result of Result 4.B and Theorem 4.6.

**Corollary 4.7.** Let \( S \) be a semigroup which is a semilattice of groups. Then every IVBI of \( S \) is an IVI of \( S \).

We denote by \( L[a] \) [resp. \( J[a] \)] the principle left [resp. two-sided] ideal of a semigroup \( S \) generated by \( a \) in \( S \), i.e.,

\[
L[a] = \{ a \} \cup S a,
\]

and

\[
J[a] = \{ a \} \cup S a \cup a S \cup S a S.
\]

It is well-known [2, Lemma 2.13] that if \( S \) is a regular semigroup, then \( L[a] = S a \) for each \( a \in S \).

A semigroup \( S \) is said to be right zero [resp. left zero] if \( xy = y [\text{resp. } xy = x] \) for any \( x, y \in S \).

**Theorem 4.8.** Let \( S \) be a regular semigroup and let \( E_S \) the set of all idempotent elements of \( S \). Then \( E_S \) forms a left zero subsemigroup of \( S \) if and only if for each \( A \in \text{IVLI}(S) \), \( A(e) = A(f) \) for any \( e, f \in E_S \), where \( E_S \) denotes the set of all idempotent elements of \( S \).

**Proof.** (\( \Rightarrow \)): Suppose \( E_S \) forms a left zero subsemigroup of \( S \). Let \( e, f \in E_S \). Then, by the hypothesis, \( ef = e \) and \( fe = f \). Since \( A \in \text{IVLI}(S) \), we have

\[
A^L(e) = A^L(ef) \geq f^L = A^L(f) \geq A^L(e),
\]

and

\[
A^U(e) = A^U(ef) \geq f^U = A^U(f) \geq A^U(e).
\]

Hence \( A(e) = A(f) \).

(\( \Leftarrow \)): Suppose the necessary condition holds. Since \( S \) is regular, \( E_S \neq \emptyset \). Let \( e, f \in E_S \). Then, by Theorem 3.4, \( \chi_{[L]} \chi_{[J]} \in \text{IVLI}(S) \). Thus \( \chi_{[L]}(ef) = \chi_{[J]}(f) = 1 \). So \( e \in [L] \cap [J] \). Then there exists an \( x \in S \) such that \( e = xf \). Hence \( E_S \) is a left zero semigroup. This completes the proof.

**Corollary 4.9.** Let \( S \) be an idempotent semigroup. Then \( S \) is left zero if and only if for each \( A \in \text{IVLI}(S) \), \( A(e) = A(f) \) for any \( e, f \in S \).

**Theorem 4.8’ [The dual of Theorem 4.8]** Let \( S \) be a regular semi group. Then \( E_S \) forms a right zero subsemigroup of \( S \) if and only if for each \( A \in \text{IVRI}(S) \), \( A(e) = A(f) \) for any \( e, f \in E_S \).

**Corollary 4.9’ [The dual of Corollary 4.9]** Let \( S \) be a semigroup. Then \( S \) is right zero if and only if for each \( A \in \text{IVRI}(S) \), \( A(e) = A(f) \) for any \( e, f \in S \).

**Theorem 4.10.** Let \( S \) be a regular semigroup. Then \( S \) is a group if and only if for each \( A \in \text{IVBI}(S) \), \( A(e) = A(f) \) for any \( e, f \in E_S \).

**Proof.** (\( \Rightarrow \)): Suppose \( S \) is a group. Let \( A \in \text{IVBI}(S) \). Then, by Theorem 3.8, \( A \) is a constant mapping. Hence \( A(e) = A(f) \) for any \( e, f \in E_S \).

(\( \Leftarrow \)): Suppose the necessary condition holds. Let \( e, f \in E_S \). Let \( B[x] \) denote the principal bi-ideal of \( S \) generated by \( x \) in \( S \), i.e., \( B[x] = \{ x \} \cup \{ x^2 \} \cup x S x \) [2, p.84]. Moreover, if \( S \) is regular, then \( B[x] = x S x \) for each \( x \in S \). Then, by Theorem 3.8, \( \chi_{B[x]} \in \text{IVBI}(S) \). Since \( f \in B[f] \), \( \chi_{B[f]}(e) = \chi_{B[f]}(f) = 1 \). Then \( e \in B[f] = f S f \). Thus, by the process of the proof of Theorem 3.14 in [8], \( e = f \). Since \( S \) is regular, \( E_S \neq \emptyset \) and \( S \) contains exactly one idempotent. So it follows from [2, p.33(Ex. 4)] that \( S \) is a group. This completes the proof.

5. Intra-regular semigroups

A semigroup \( S \) is said to be intra-regular if for each \( a \in S \), there exist \( x, y \in S \) such that \( a = xa^2 y \). For characterization of such a semigroup, see [2, Theorem 4.4] and [13, II.4.5 Theorem].

**Theorem 5.1.** Let \( S \) be a semigroup. Then \( S \) is intra-regular if and only if for each \( A \in \text{IVI}(S) \), \( A(a) = A(a^2) \) for each \( a \in S \).

**Proof.** (\( \Rightarrow \)): Suppose \( S \) is intra-regular. Let \( A \in \text{IVI}(S) \), and let \( a \in S \). Then, by the hypothesis, there exist \( x, y \in S \) such that \( a = xa^2 y \). Since \( A \in \text{IVI}(S) \), we have

\[
A^L(a) = A^L(xa^2 y) \geq A^L(xa^2) \geq A^L(a^2) \geq A^L(a),
\]

and

\[
A^U(a) = A^U(xa^2 y) \geq A^U(xa^2) \geq A^U(a^2) \geq A^U(a).
\]

Hence \( A(a) = A(a^2) \) for each \( a \in S \).

(\( \Leftarrow \)): Suppose the necessary condition holds and let \( a \in S \). Then, by Theorem 3.4, \( \chi_{J[a^2]} \in \text{IVI}(S) \). Since \( a^2 \in J[a^2] \), \( \chi_{J[a^2]}(a) = \chi_{J[a^2]}(a^2) = 1 \). Thus \( a \in J[a^2] \). For we can easily see that \( S \) is intra-regular. This completes the proof.

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Proposition 5.2. Let $S$ be an intra-regular semigroup. Then for each $A \in \text{IVI}(S)$, $A(ab) = A(a^2)$ for each $a, b \in S$.

Proof. Let $A \in \text{IVI}(S)$, and let $a, b \in S$. Then, by Theorem 5.1, $A^L(ab) = A^L((ab)^2) \geq A^L(ab) = A^L((ba)^2) = A^L(ba(a)b) \geq A^L(ba)$. By the similar arguments, we have that $A^U(ab) \geq A^U(ab)$. Thus $A(ab) = A(ba)$. This completes the proof.

6. Completely regular semigroups

A semigroup $S$ is said to be completely regular if for each $a \in S$, there exists an $x \in S$ such that

$$a = axa \text{ and } ax = xa.$$ 

A semigroup is said to be left regular [resp. right regular] if for each $a \in S$, there exists an $x \in S$ such that

$$a = xa^2 \text{ [resp. } a = a^2x].$$

For characterizations of such a semigroup, see [2, Theorem 4.2]. It is well-known [2, Theorem 4.3] that $S$ is completely regular if and only if it is left and right regular.

Result 6A. [13, p. 105] Let $S$ be a semigroup. Then the followings are equivalent:

1. $S$ is completely regular.
2. $S$ is a union of groups.
3. $a \in a^2Sa^2$ for each $a \in S$.

Theorem 6.1. Let $S$ be a semigroup. Then $S$ is left regular if and only if, for each $A \in \text{IVLI}(S)$, $A(a) = A(a^2)$ for each $a \in S$.

Proof. ($\Rightarrow$): Suppose $S$ is left regular. Let $A \in \text{IVLI}(S)$, and let $a \in S$. Then, by the hypothesis, there exists an $x \in S$ such that $a = ax^2$. Since $A \in \text{IVLI}(S)$, $A^L(a) = A^L(a^2x^2) \geq A^L(a^2) \geq A^L(a) \text{ and } A^U(a) = A^U(a^2x^2) \geq A^U(a^2) \geq A^U(a)$. Hence $A(a) = A(a^2)$, for each $a \in S$.

($\Leftarrow$): Suppose the necessary condition holds. Let $a \in S$. Then, by Theorem 3.4, $(\chi_{L[a^2]}, \chi_{L[a^2]}) \in \text{IVLI}(S)$. Since $a^2 \in L[a^2]$, $(\chi_{L[a^2]}(a)) = \chi_{L[a^2]}(a^2) = 1$. Then $a \in L[a^2] = \{a^2\} \cup Sa^2$. Hence $S$ is left regular. This completes the proof.

Theorem 6.2. Let $S$ be a semigroup. Then the followings are equivalent:

1. $S$ is completely regular.
2. For each $A \in \text{IVBI}(S)$, $A(a) = A(a^2)$ for each $a \in S$.
3. For each $B \in \text{IVLI}(S)$ and each $C \in \text{IVRI}(S)$, $B(a) = B(a^2)$ and $C(a) = C(a^2)$ for each $a \in S$.

Proof. It is clear that (1) $\Leftrightarrow$ (3) by Theorem 6.1 and 6.1'. Thus it is sufficient to show that ($1 \Leftrightarrow 2$).

($1 \Rightarrow 2$): Suppose the condition (1) holds. Let $A \in \text{IVBI}(S)$, and let $a \in S$. Then, by Result 6A(3), there exists an $x \in S$ such that $a = a^2x^2a^2$. Since $A \in \text{IVBI}(S)$, $A^L(a) = A^L(a^2x^2a^2) \geq A^L(a^2) \wedge A^L(a^2) = A^L(a) \wedge A^L(a) = A^L(a)$. By the similar arguments, we have that $A^U(a) \geq A^U(a)$. Hence $A(a) = A(a^2)$.

($2 \Rightarrow 1$): Suppose the condition (2) holds. For each $x \in S$, let $B[x]$ denote the principal bi-ideal of $S$ generated by $x$, i.e., $B[x] = \{x\} \cup \{x^2\} \cup xSx$. Let $a \in S$. Then, by Theorem 3.8, $[\chi_{B[a^2]}, \chi_{B[a^2]}] \in \text{IVBI}(S)$. Since $a^2 \in B[a^2]$, $\chi_{B[a^2]}(a) = \chi_{B[a^2]}(a^2) = 1$. Thus $a \in B[a^2] = \{a^2\} \cup \{a^4\} \cup a^2Sa^2$. Hence $S$ is completely regular. This completes the proof.

Result 6B. [9, Theorem 1] Let $S$ be a semigroup. Then $S$ is a semilattice of groups if and only if $B[1(S)]$ is a semilattice under the multiplication of subsets.

Theorem 6.3. Let $S$ be a semigroup. Then $S$ is a semilattice of groups if and only if for each $A \in \text{IVBI}(S)$, $A(a) = A(a^2)$ and $A(ab) = A(ba)$ for any $a, b \in S$.

Proof. ($\Rightarrow$): Suppose $S$ is a semilattice of groups. Then $S$ is a union of groups. By Result 6A, $S$ is completely regular. Let $A \in \text{IVBI}(S)$, and let $a \in S$. Then, by Theorem 6.2, $A(a) = A(a^2)$. Now let $a, b \in S$. Then, by the process of the proof of Theorem 6 in [7], there exists an $x \in S$ such that $(a^3x = (ba)x^3(ba))$. Thus $A^L(ab) = A^L((ab)^3) = A^L((ba)x^3(ba)) \geq A^L(ba) \wedge A^L(ba) = A^L(ba)$. By the similar arguments, we have that $A^U(ab) \geq A^U(ab)$. Similarly, we can see that $A^L(ba) \geq A^L(ab)$ and $A^U(ba) \geq A^U(ab)$. So $A(ab) = A(ba)$. Hence the necessary conditions hold.

($\Leftarrow$): Suppose the necessary conditions hold. Then, by the first condition and Theorem 6.2, $S$ is completely regular. Thus it is easily shown that $A$ is idempotent for each $A \in B(S)$. Let $A, B \in B(S)$, and let $t \in BA$. Then there exist $a \in A$ and $b \in B$ such that $t = ab$. Moreover $B[t] = B[ab] \in B(S)$. By Theorem 3.8, $[\chi_{B[ab]}, \chi_{B[ab]}] \in \text{IVBI}(S)$. By the hypothesis, $[\chi_{B[ab]}, \chi_{B[ab]}](ab) = [\chi_{B[ab]}, \chi_{B[ab]}](ba)$. Since $ab \in B[ab]$, $\chi_{B[ab]}(ab) = \chi_{B[ab]}(ba) = 1$. Then $ba \in B[ab] = \{ab\} \cup \{abab\} \cup abSab$. It follows from the process of the proof of Theorem 6 in [7] that $BA = AB$.  

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So \((BI(S), \cdot)\) is a commutative idempotent semigroup. Hence, by Result 6.B, \(S\) is a semilattice of groups. This completes the proof. □

**Corollary 6.4.** Let \(S\) be an idempotent semigroup. Then \(S\) is commutative if and only if for each \(A \in \text{IVBI}(S)\), \(A(ab) = A(ba)\) for any \(a, b \in S\).

### 7. Semigroups that are semilattices of left [resp. right] simple semigroups

**Result 7.A.** [9, Theorem 7 and 18, Theorem] Let \(S\) be a semigroup. Then the followings are equivalent:

1. \(S\) is a semilattice of left simple semigroups.
2. \(S\) is left regular and \(AB = BA\) for any two left ideals \(A\) and \(B\) of \(S\).
3. \(S\) is left regular and every left ideal of it is an ideal of \(S\).

The following result can be proved in a similar way as in the proof of Theorem 4.2 and 4.2'.

**Theorem 7.1.** Let \(S\) be a left [resp. right] regular semigroup. Then \(S\) is left [resp. right] duo if and only if for each \(A \in \text{IVLD} [\text{resp. IVRD}]\).

The characterization of a semigroup that is a semilattice of left simple semigroups can be found in [13, Theorem II.4.9].

**Theorem 7.2.** Let \(S\) be a semigroup. Then \(S\) is a semilattice of left simple semigroups if and only if for each \(A \in \text{IVLI}(S)\), \(A(a) = A(a^2)\) and \(A(ab) = A(ba)\) for any \(a, b \in S\).

**Proof.** (⇒): Suppose \(S\) is a semilattice of left simple semigroups. Let \(A \in \text{IVLI}(S)\), and let \(a, b \in S\). Then, by Result 7.A, \(S\) is left regular. By Theorem 6.1, \(A(a) = A(a^2)\). By the hypothesis and Result 6.A, \(S\) is left duo. Thus, by Theorem 7.1, \(S\) is IVLD. Then \(A \in \text{IVLI}(S)\). Hence, by the process of the proof of Theorem 6.3 in [8], we have \(AB = BA\). Hence, by Result 6.A, \(S\) is a semilattice of left simple semigroups. This completes the proof. □

**Theorem 7.2'.** [The dual of Theorem 7.2] Let \(S\) be a semigroup. Then the \(S\) is a semilattice of right simple semigroups if and only if for each \(A \in \text{IVRI}(S)\), \(A(a) = A(a^2)\) and \(A(ab) = A(ba)\) for any \(a, b \in S\).

### 8. Left [resp. right] simple semigroups

**Definition 8.1.** A semigroup \(S\) is said to be interval-valued fuzzy left simple [resp. interval-valued fuzzy right simple] if every IVLI [resp. IVRI] of \(S\) is a constant mapping and is said to be interval-valued fuzzy simple if every IVI of \(S\) is a constant mapping.

**Theorem 8.2.** Let \(S\) be a semigroup. Then \(S\) is left simple if and only if \(S\) is interval-valued fuzzy left simple.

**Proof.** (⇒): Suppose \(S\) is left simple. Let \(A \in \text{IVLI}(S)\), and let \(a, b \in S\). Since \(S\) is left simple, from [2, p.6], there exist \(x, y \in S\) such that \(b = xa\) and \(a = yb\). Since \(A \in \text{IVLI}(S)\), \(A^L(a) = A^L(yb) \geq A^L(b) = A^L(xa) \geq A^L(a)\) and \(A^U(a) = A^U(yb) \geq A^U(b) = A^U(xa) \geq A^U(a)\). Thus, \(A(a) = A(b)\). So \(A\) is a constant mapping. Hence \(S\) is interval-valued fuzzy left simple.

(⇐): Suppose the necessary condition holds. Let \(A\) be any left ideal of \(S\). By Theorem 3.4, \([\chi_A, \chi_A] \in \text{IVLI}(S)\). By the hypothesis, \([\chi_A, \chi_A] = 1\) is a constant mapping. Since \(A \neq \emptyset\), \([\chi_A, \chi_A] = 1\). Then \(\chi_A(a) = 1\) for each \(a \in S\). Thus \(A = A\) for each \(a \in S\), i.e., \(S \subset A\). Hence \(S\) is left simple. This completes the proof. □

The following two results can be seen in a similar way as in the proof of Theorem 8.2.

**Theorem 8.2'.** [The dual of Theorem 8.2] Let \(S\) be a semigroup. Then \(S\) is right simple if and only if \(S\) is interval-valued fuzzy simple.

**Theorem 8.3.** Let \(S\) be a semigroup. Then \(S\) is simple if and only if \(S\) is interval-valued fuzzy simple.

It is well-known that a semigroup \(S\) is a group if and only if it is left and right simple. Thus from this and Theorem 8.2 and 8.2', we obtain the following result:

**Theorem 8.4.** Let \(S\) be a semigroup. Then \(S\) is a group if and only if \(S\) is both interval-valued fuzzy left and interval-valued fuzzy right simple.

**Proposition 8.5.** Let \(S\) be a left simple semigroup. Then every IVBI of \(S\) is an IVRI of \(S\).
Proof. Let $A \in \text{IVBI}(S)$, and let $a, b \in S$. Since $S$ is left simple, there exists an $x \in S$ such that $b = xa$. Since $A \in \text{IVBI}(S)$, $A^L(ab) = A^L(axa) \geq A^L(a) \wedge A^L(a) = A^L(a)$ and $A^U(ab) = A^U(axa) \geq A^U(a) \wedge A^U(a) = A^U(a)$. Hence $A \in \text{IVRI}(S)$. This completes the proof. □

Corollary 8.6. Let $S$ be a left simple semigroup. Then every bi-ideal of $S$ is a right ideal of $S$.

References


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