Ordinary Smooth Topological Spaces

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Abstract

In this paper, we introduce the concept of ordinary smooth topology on a set X by considering the gradation of openness of ordinary subsets of X. And we obtain the result [Corollary 2.13]: An ordinary smooth topology is fully determined its decomposition in classical topologies. Also we introduce the notion of ordinary smooth [resp. strong and weak] continuity and study some its properties. Also we introduce the concepts of a base and a subbase in an ordinary smooth topological space and study their properties. Finally, we investigate some properties of an ordinary smooth subspace.

Key words: ordinary smooth (co)topological space, r-level and strong r-level, ordinary smooth [resp. weak and strong] continuity, ordinary smooth open [resp. closed] mapping, ordinary smooth subspace, ordinary smooth base [resp. subbase].

1. Introduction and Preliminaries

Chang [1] introduce the concept of fuzzy topology on a set X by axiomatizing a collection of fuzzy sets in X. After that, Pu and Liu [7] and Lowen [5] advanced it. However, they did not consider the gradation of openness [resp. closedness] of fuzzy sets in X.

In 1992, Hazra et al.[4] have attempted to introduce a concept of gradation of openness of fuzzy sets in X by a mapping $\tau: I^X \rightarrow I$ satisfying the following axioms:

(i) $\tau(0) = \tau(1) = 1$,
(ii) $\tau(A_i) > 0$, $i = 1, 2$, implies $\tau(A_1 \cap A_2) > 0$,
(iii) $\tau(A_\alpha) > 0$, $\alpha \in \Gamma$, implies $\tau(\bigcup_{\alpha \in \Gamma} A_\alpha) > 0$.

On the other hand, Chattopadhyay et al.[2] modified the notion of gradation of openness of fuzzy sets in X by a mapping $\tau: I^X \rightarrow I$ satisfying the following axioms:

(i) $\tau(0) = \tau(1) = 1$,
(ii) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B), \forall A, B \in I^X$,
(iii) $\tau(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} \tau(A_\alpha), \forall \{A_\alpha\}_{\alpha \in \Gamma} \subset I^X$.

After then, some work has been done in this field by Ramadan [8], Chattopadhyay and Samanta [3], and Peeters [6]. In particular, Ying [9] introduced the concept of the topology considering the degree of openness of an ordinary subset of a set and studied some of its properties.

In this paper, we introduce the concept of ordinary smooth topology on a set X by considering the gradation of openness of ordinary subsets of X. And we obtain the result [Corollary 2.13]: An ordinary smooth topology is fully determined its decomposition in classical topologies. Also we introduce the notion of ordinary smooth [resp. strong and weak] continuity and study some its properties. Finally, we investigate some properties of an ordinary smooth subspace.

Throughout this paper, let $I = [0, 1]$ be the unit interval, let $I^X$ denote the set of all fuzzy sets in a set X, and we will write $I_0 = (0, 1]$ and $I_1 = [0, 1)$.

2. Definitions and general properties

Let $2 = \{0, 1\}$ and let $2^X$ denote the set of all ordinary subsets of X.

Definition 2.1. Let X be a nonempty set. Then a mapping $\tau: 2^X \rightarrow I$ is called an ordinary smooth topology (in short, ost) on X or a gradation of openness of ordinary subsets of X if $\tau$ satisfies the following axioms:

(OST$_1$) $\tau(\emptyset) = \tau(X) = 1$.
(OST$_2$) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B), \forall A, B \in 2^X$.
(OST$_3$) $\tau(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} \tau(A_\alpha), \forall \{A_\alpha\}_{\alpha \in \Gamma} \subset 2^X$.

The pair $(X, \tau)$ is called an ordinary smooth topological space (in short, osts). We will denote the set of all osts on X as OST(X).

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2000 Mathematics Subject Classification. 54A40.
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Then we can easily see that \( \tau \) satisfies the axioms in Definition 2.1 as a fuzzyfying topology [resp. fuzzy topology and bifuzzy topology] on \( X \).

**Example 2.3.** (a) Let \( X = \{a, b, c\} \). Then \( 2^X = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\} \).

We define the mapping \( \tau : 2^X \rightarrow I \) as follows:

\[
\tau(\emptyset) = \tau(X) = 1, \quad \tau(\{a\}) = 0.7, \quad \tau(\{b\}) = 0.4, \quad \tau(\{c\}) = 0.5,
\]

\[
\tau(\{a, b\}) = 0.6, \quad \tau(\{a, c\}) = 0.3, \quad \tau(\{b, c\}) = 0.8.
\]

Then we can easily see that \( \tau \in \text{OST}(X) \). In this case, \( \tau \) will be called the ordinary smooth discrete topology on \( X \).

(b) Let \( X \) be a nonempty set. We define the mapping \( \tau_0 : 2^X \rightarrow I \) as follows: For each \( A \in 2^X \),

\[
\tau_0(A) = \begin{cases} 
1, & \text{if } A = \emptyset \text{ or } A = X, \\
0, & \text{otherwise}.
\end{cases}
\]

Then we can easily see that \( \tau_0 \in \text{OST}(X) \). In this case, \( \tau_0 \) will be called the ordinary smooth indiscrete topology on \( X \).

(c) Let \( X \) be a nonempty set. We define the mapping \( \tau_X : 2^X \rightarrow I \) as follows: For each \( A \in 2^X \),

\[
\tau_X(A) = 1.
\]

Then clearly \( \tau_X \in \text{OST}(X) \). In this case, \( \tau_X \) will be called the ordinary smooth discrete topology on \( X \).

(d) Let \( X \) be a set and let \( r \in I_1 \) be fixed. We define the mapping \( \tau : 2^X \rightarrow I \) as follows: For each \( A \in 2^X \),

\[
\tau(A) = \begin{cases} 
1, & \text{if } A = \emptyset \text{ or } A^c \text{ is finite,} \\
r, & \text{otherwise}
\end{cases}
\]

Then it can be easily seen that \( \tau \in \text{OST}(X) \). In this case, \( \tau \) will be called the \( r \)-ordinary smooth finite complement topology on \( X \) and will be denoted by \( \text{OSCo}(X) \). \( \text{OSCo}(X) \) is of interest only when \( X \) is an infinite set because if \( X \) is finite, \( \text{OSCo}(X) \) coincides with \( \tau_X \) defined in (c).

(e) Let \( X \) be a set and let \( r \in I_1 \) be fixed. We define the mapping \( \tau : 2^X \rightarrow I \) as follows: For each \( A \in 2^X \),

\[
\tau(A) = \begin{cases} 
1, & \text{if } A = \emptyset \text{ or } A^c \text{ is countable,} \\
r, & \text{otherwise}
\end{cases}
\]

Then it can be easily seen that \( \tau \in \text{OST}(X) \). In this case, \( \tau \) will be called the \( r \)-ordinary smooth countable complement topology on \( X \) and will be denoted by \( \text{OSCo}(X) \).

**Remark 2.2.** Ying [9] called the mapping \( \tau : 2^X \rightarrow I \) [resp. \( \tau : I^X \rightarrow 2 \) and \( \tau : I^X \rightarrow I \)] satisfying the axioms in Definition 2.1 as a fuzzyfying topology [resp. fuzzy topology and bifuzzy topology] on \( X \).

**Example 2.3.** (a) Let \( X = \{a, b, c\} \). Then \( 2^X = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\} \).

We define the mapping \( \tau : 2^X \rightarrow I \) as follows:

\[
\tau(\emptyset) = \tau(X) = 1, \quad \tau(\{a\}) = 0.7, \quad \tau(\{b\}) = 0.4, \quad \tau(\{c\}) = 0.5,
\]

\[
\tau(\{a, b\}) = 0.6, \quad \tau(\{a, c\}) = 0.3, \quad \tau(\{b, c\}) = 0.8.
\]

Then we can easily see that \( \tau \in \text{OST}(X) \). In this case, \( \tau \) will be called the ordinary smooth discrete topology on \( X \).

(b) Let \( X \) be a nonempty set. We define the mapping \( \tau_0 : 2^X \rightarrow I \) as follows: For each \( A \in 2^X \),

\[
\tau_0(A) = \begin{cases} 
1, & \text{if } A = \emptyset \text{ or } A = X, \\
0, & \text{otherwise}
\end{cases}
\]

Then we can easily see that \( \tau_0 \in \text{OST}(X) \). In this case, \( \tau_0 \) will be called the ordinary smooth indiscrete topology on \( X \).

(c) Let \( X \) be a nonempty set. We define the mapping \( \tau_X : 2^X \rightarrow I \) as follows: For each \( A \in 2^X \),

\[
\tau_X(A) = 1.
\]

Then clearly \( \tau_X \in \text{OST}(X) \). In this case, \( \tau_X \) will be called the ordinary smooth discrete topology on \( X \).

(d) Let \( X \) be a set and let \( r \in I_1 \) be fixed. We define the mapping \( \tau : 2^X \rightarrow I \) as follows: For each \( A \in 2^X \),

\[
\tau(A) = \begin{cases} 
1, & \text{if } A = \emptyset \text{ or } A^c \text{ is finite,} \\
r, & \text{otherwise}
\end{cases}
\]

Then it can be easily seen that \( \tau \in \text{OST}(X) \). In this case, \( \tau \) will be called the \( r \)-ordinary smooth finite complement topology on \( X \) and will be denoted by \( \text{OSCo}(X) \). \( \text{OSCo}(X) \) is of interest only when \( X \) is an infinite set because if \( X \) is finite, \( \text{OSCo}(X) \) coincides with \( \tau_X \) defined in (c).

(e) Let \( X \) be a set and let \( r \in I_1 \) be fixed. We define the mapping \( \tau : 2^X \rightarrow I \) as follows: For each \( A \in 2^X \),

\[
\tau(A) = \begin{cases} 
1, & \text{if } A = \emptyset \text{ or } A^c \text{ is countable,} \\
r, & \text{otherwise}
\end{cases}
\]

Then it can be easily seen that \( \tau \in \text{OST}(X) \). In this case, \( \tau \) will be called the \( r \)-ordinary smooth countable complement topology on \( X \) and will be denoted by \( \text{OSCo}(X) \).
(c) From (b), it is also clear that \( \{ [\tau]^*_r : r \in I \} \) is a descending family of classical topologies on \( X \).

Let \( r \in I_1 \). Then \( [\tau]^*_r \supseteq \bigcup_{s > r} [\tau]^*_s \). Assume that \( A \not\subseteq [\tau]^*_r \). Then \( \tau(A) \leq r \). Thus \( \exists s \in I_1 \) such that \( \tau(A) \leq r < s \). So \( A \not\subseteq [\tau]^*_s \) for some \( r < s \), i.e., \( A \not\subseteq \bigcup_{s > r} [\tau]^*_s \).

Hence \( \bigcup_{s > r} [\tau]^*_s \subset [\tau]^*_r \). Therefore \( [\tau]^*_r = \bigcup_{s > r} [\tau]^*_s \). This completes the proof.

Proposition 2.11. Let \( X \) be a nonempty set and let \( \{ T_r : r \in I \} \) be a nonempty descending family of classical topologies on \( X \) such that \( T_0 \) is the classical discrete topology.

(a) We define the mapping \( \tau : 2^X \to I \) as follows: For each \( A \subseteq 2^X \),

\[
\tau(A) = \bigvee \{ r \in I : A \in T_r \}.
\]

Then \( \tau \in \text{OST}(X) \).

(b) For each \( r \in I_0 \), if \( T_r = \bigcap_{s \geq r} T_s \), then \( [\tau]^*_r = T_r \).

(b) For each \( r \in I_1 \), if \( T_r = \bigcup_{s > r} T_s \), then \( [\tau]^*_r = T_r \).

In this case, \( \tau \) is called the ordinary smooth topology generated by \( \{ T_r : r \in I \} \).

Proof. (a) From the definition of \( \tau \), it is clear that \( \tau(\emptyset) = \tau(X) = 1 \).

Thus \( \tau \) satisfies the axiom (OST1).

For any \( A_i \subseteq 2^X \), let \( \tau(A_i) = k_i \), \( i = 1, 2 \). Suppose \( k_i = 0 \) for some \( i \). Then clearly

\[
\tau(A_1 \cap A_2) \geq \tau(A_1) \cap \tau(A_2).
\]

Thus, without loss of generality, suppose \( k_i > 0 \) for \( i = 1, 2 \).

Then \( \exists r \in I_0 \) such that \( k_i - \epsilon < r_i < k_i \) and \( A_i \in T_{r_i} \), \( i = 1, 2 \).

Let \( r = r_1 \wedge r_2 \) and let \( k = k_1 \wedge k_2 \). Since \( \{ T_r : r \in I_0 \} \) is a descending family and \( A_i \in T_{r_i}, A_1, A_2 \in T_r \). Thus \( A_1 \cap A_2 \in T_r \). So, by the definition of \( \tau \),

\[
\tau(A_1 \cap A_2) \geq r > k - \epsilon.
\]

Since \( \epsilon > 0 \) is arbitrary, it follows that \( \tau(A_1 \cap A_2) \geq k = k_1 \wedge k_2 = \tau(A_1) \cap \tau(A_2) \).

Hence \( \tau \) satisfies the axiom (OST2).

Now let \( \{ A_\alpha \}_{\alpha \in \Gamma} \subset 2^X \), let \( \tau(A_\alpha) = l_\alpha \) for each \( \alpha \in \Gamma \) and let \( l = \bigwedge_{\alpha \in \Gamma} l_\alpha \). Suppose \( l = 0 \). Then clearly

\[
\tau(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq l = \bigwedge_{\alpha \in \Gamma} \tau(A_\alpha).
\]

Suppose \( l > 0 \) and let \( \epsilon > 0 \). Then \( 0 < l - \epsilon < l \) for each \( \alpha \in \Gamma \). Since \( A_\alpha \subseteq T_{l_\alpha} \) for each \( \alpha \in \Gamma \) and \( \{ T_r : r \in I_0 \} \) is a descending family, \( A_\alpha \in T_{l_\alpha - \epsilon} \) for each \( \alpha \in \Gamma \). Since \( T_{l_\alpha - \epsilon} \) is a classical topology on \( X \), \( \bigcup_{\alpha \in \Gamma} A_\alpha \in T_{l_\alpha - \epsilon} \). Thus, by the definition of \( \tau \),

\[
\tau(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq l - \epsilon.
\]

Since \( \epsilon > 0 \) is arbitrary,

\[
\tau(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq l = \bigwedge_{\alpha \in \Gamma} \tau(A_\alpha).
\]

So \( \tau \) satisfies the axiom (OST3). Hence \( \tau \in \text{OST}(X) \).

(b) Suppose \( T_r = \bigcap_{s \geq r} T_s \) for each \( r \in I_0 \) and let \( A \subseteq T_0 \). Then clearly \( \tau(A) \geq r \). Thus \( A \in \tau_r \). So \( \tau_r \subseteq \tau_r \) for each \( r \in I_0 \). Let \( A \subseteq \tau_r \). Then \( \tau(A) \geq r \). Thus, by the definition of \( \tau \),

\[
\tau(A) = \bigvee_{A \subseteq T_s} s \geq r.
\]

Let \( \epsilon > 0 \). Then \( \exists k \in I_0 \) such that \( s - \epsilon < k \) and \( A \subseteq T_k \).

Thus \( r - \epsilon \leq s - \epsilon < k \) and \( A \subseteq T_k \).

So \( A \subseteq T_{r-\epsilon} \). Since \( \epsilon > 0 \) is arbitrary, by the hypothesis, \( A \subseteq T_{r-\epsilon} \). Hence \( \tau_r \subseteq \tau_{r-\epsilon} \). Therefore \( \tau_r = T_r \) for each \( r \in I_0 \).

(b) By the similar arguments of the proof of (b), we can prove that \( [\tau]^*_r \in T_r \), for each \( r \in I_1 \). This completes the proof.

Since every mapping \( t : 2^X \to I \) is greater than or equal to 0 on all elements on which it is defined, note that indeed an extra requirement here is that \( T_0 \) is the classical discrete topology \( 2^X \). Thus from now on we take this supplementary condition for granted.

The following is the immediate result of Propositions 2.5 and 2.6.

Corollary 2.12. Let \( X \) be a nonempty set, let \( \tau \in \text{OST}(X) \) and let \( \{ [\tau]^*_r : r \in I \} \) be the family of all \( r \)-level classical topologies with respect to \( \tau \). We define the mapping \( \tau_1 : 2^X \to I \) as follows: For each \( A \subseteq 2^X \),

\[
\tau_1(A) = \bigvee \{ r \in I : A \subseteq [\tau]^*_r \}.
\]

Then \( \tau_1 = \tau \).

The fact that an ordinary smooth topological space is fully determined by its decomposition in classical topologies is restated in the following result.

Corollary 2.13. Let \( X \) be a nonempty set and let \( \tau_1, \tau_2 \in \text{OST}(X) \). Then \( \tau_1 = \tau_2 \) if and only if \( [\tau_1]^*_r = [\tau_2]^*_r \), for each \( r \in I \), or alternatively, if and only if \( [\tau_1]^*_r = [\tau_2]^*_r \), for each \( r \in I \).

Remark 2.14. In a similar way, we study the levels of an ordinary smooth cotopology \( C \) on a nonempty set \( X \): For each \( r \in I \),

\[
[C]^*_r = \{ A \subseteq 2^X : C(A) \geq r \}
\]

and

\[
[C]^*_r = \{ A \subseteq 2^X : C(A) > r \}.
\]

Definition 2.15. Let \( X \) be a nonempty set, let \( T \) be a classical topology and let \( \tau \in \text{OST}(X) \). Then \( \tau \) is said to be compatible with \( T \) if \( T = S(\tau) \), where \( S(\tau) = \{ A \subseteq 2^X : \tau(A) > 0 \} \).
Example 2.16. (a) Let $\tau_0$ be the ordinary smooth indiscrete topology on a nonempty set $X$ and let $l$ be the classical indiscrete topology on $X$. Then clearly
\[ S(\tau_0) = \{ A \in 2^X : \tau_0(A) > 0 \} = \{ \emptyset, X \} = l. \]
Thus $\tau_0$ is compatible with $l$.
(b) Let $\tau_X$ be the ordinary smooth discrete topology on a nonempty set $X$ and let $\mathcal{D}$ be the classical discrete topology on $X$. Then
\[ S(\tau_X) = \{ A \in 2^X : \tau_X(A) > 0 \} = 2^X = \mathcal{D}. \]
Thus $\tau_X$ is compatible with $\mathcal{D}$.
(c) Let $X$ be a nonempty set and let $r \in (0, 1)$ be fixed. We define the mapping $\tau : 2^X \rightarrow I$ as follows: For each $A \in 2^X$,
\[ \tau(A) = \begin{cases} 1, & \text{if } A = \emptyset \text{ or } A = X, \\ r, & \text{if } A \in T \setminus \{ \emptyset, X \}, \\ 0, & \text{otherwise}. \end{cases} \]
Then clearly $\tau \in \text{OST}(X)$ and $\tau$ is compatible with $D$. \( \square \)

From the following result, every classical topology can be considered as an ordinary smooth topology.

Proposition 2.17. Let $T$ be a classical topology on a nonempty set $X$ and let $r \in (0, 1)$. Then $\exists! T^r \in \text{OST}(X)$ such that $T^r$ is compatible with $T$. Moreover $(T^r)_r = T$. In this case, $T^r$ is called an $r$-th ordinary smooth topology on $X$ and $(X, T^r)$ is called an $r$-th ordinary smooth topological space.

Proof. Let $r \in (0, 1)$ be fixed and we define the mapping $T^r : 2^X \rightarrow I$ as follows: For each $A \in 2^X$,
\[ T^r(A) = \begin{cases} 1, & \text{if } A = \emptyset \text{ or } A = X, \\ r, & \text{if } A \in T \setminus \{ \emptyset, X \}, \\ 0, & \text{otherwise}. \end{cases} \]
Then we can easily see that $T^r \in \text{OST}(X)$ (and $(T^r)_r = T$).
On the other hand, by the definition of $T^r$,
\[ S(T^r) = \{ A \in 2^X : T^r(A) > 0 \} = T. \]
So $T^r$ is compatible with $T$. \( \square \)

Proposition 2.18. Let $T$ be a classical topology on a nonempty set $X$ and let $C(T)$ be the set of all ordinary smooth topologies on $X$ compatible with $T$. Then there is a one-to-one correspondence between $C(T)$ and the set $\tilde{I}_0^T$, where $\tilde{T} = T \setminus \{ \emptyset, X \}$.

Proof. We define two mappings $F : C(T) \rightarrow \tilde{I}_0^T$ and $G : \tilde{I}_0^T \rightarrow C(T)$ as follows, respectively:
\[ [F(\tau)](A) = f_\tau(A) = \tau(A), \quad \forall \tau \in C(T), \forall A \in \tilde{T} \]
and
\[ [G(f)](A) = \tau_f(A) = \begin{cases} 1, & \text{if } A = \emptyset \text{ or } A = X, \\ f(A), & \text{if } A \in \tilde{T}, \\ 0, & \text{otherwise}, \forall f \in \tilde{I}_0^T, \forall A \in 2^X. \end{cases} \]
Then, by the definition of $F$, it is clear that $F(\tau) = f_\tau \in I_0^T, \forall \tau \in C(T)$. Thus $F$ is well-defined. Also, by the definition of $G$, we can easily see that $G(f) = \tau_f \in \text{OST}(X)$ such that $\tau_f$ is compatible with $T, \forall f \in \tilde{I}_0^T$. So $G$ is well-defined.
Now let $\tau \in C(T)$. Then
\[ (G \circ F)(\tau) = G(F(\tau)) = G(f_\tau) = \tau_f. \]
Thus, for each $A \in 2^X$,
\[ \tau_f(A) = \begin{cases} 1 = \tau(A), & \text{if } A = \emptyset \text{ or } A = X, \\ f_\tau(A) = \tau(A), & \text{if } A \in \tilde{T}, \\ 0, & \text{otherwise}. \end{cases} \]
So $\tau_f = \tau$. Hence $G \circ F = id_{C(T)}$.
Similarly, it can be proved that $(F \circ G)(f) = f, \forall f \in \tilde{I}_0^T$. Thus $F \circ G = id_{\tilde{I}_0^T}$. This completes the proof. \( \square \)

3. Ordinary smooth continuous mappings

It is well-known that for any classical topological spaces $(X, T_1)$ and $(Y, T_2)$ a mapping $f : (X, T_1) \rightarrow (Y, T_2)$ is continuous if and only if $f^{-1}(A) \in T_1$ for each $A \in T_2$.

Definition 3.1. Let $(X, \tau_1)$ and $(Y, \tau_2)$ be ordinary smooth topological spaces. Then a mapping $f : X \rightarrow Y$ is said to be:

(i) [10] ordinary smooth continuous if $\tau_2(A) \leq \tau_1(f^{-1}(A)), \forall A \in 2^Y$.

(ii) ordinary smooth weakly continuous if $\tau_2(A) > 0 \Rightarrow \tau_1(f^{-1}(A)) > 0, \forall A \in 2^Y$.

(iii) ordinary smooth strongly continuous if $\tau_2(A) = \tau_1(f^{-1}(A)) > 0, \forall A \in 2^Y$.

In this manner, we obtain an obvious generalization of the known concept of classical continuity. It is clear that ordinary smooth strong continuity $\Rightarrow$ ordinary smooth continuity $\Rightarrow$ ordinary smooth weak continuity. However, the converse is not necessarily true.

Example 3.2. (a) Let $X = \{ a, b, c, d \}$, let $A = \{ b, d \}$ and let $B = \{ a, c \}$. For each $i = 1, 2$, we define a mapping $\tau_i : 2^X \rightarrow I$ as follows: For each $C \in 2^A$,
\[ \tau_i(\emptyset) = \tau_i(X) = 1, \]
\[ \tau_i(C) = \begin{cases} 1, & \text{if } C = A \text{ or } C = B, \\ 0, & \text{otherwise}. \end{cases} \]
Thus $\tau_1, \tau_2 \in \text{OST}(X)$. Consider the identity mapping $id : (X, \tau_2) \rightarrow (X, \tau_1)$. Then we can easily
see that id is ordinary smooth weakly continuous, but it is not ordinary smooth continuous.  

(b) Let O be the set of all odd number in N and let  

\[ \mathcal{A}_n = \{1, 3, \cdots, 2n-1\} \]  

for each \( n \in N \). For each \( i = 1, 2 \), we define a mapping \( \tau_i : 2^N \to I \) as follows: For each \( A \in 2^N, \)  

\[ \tau_i(A) = \begin{cases}  
\frac{1}{i}, & \text{if } A = O, \\
\frac{1}{i}, & \text{if } A = \mathcal{A}_n, \\
1, & \text{otherwise}. 
\end{cases} \]  

Then clearly \( \tau_1, \tau_2 \in \text{OST}(X) \). Consider the identity mappings \( id : (X, \tau_2) \to (X, \tau_1) \) and \( id : (X, \tau_1) \to (X, \tau_2) \). Then we can easily see that \( id : (X, \tau_2) \to (X, \tau_1) \) is ordinary smooth weakly continuous, but not ordinary smooth continuous and \( id : (X, \tau_1) \to (X, \tau_2) \) is ordinary smooth continuous, but not ordinary smooth strongly continuous. \( \square \)  

The following is the immediate result of Theorem 2.6 and Definition 3.1.  

**Theorem 3.3.** Let \( (X, \tau_1) \) and \( (Y, \tau_2) \) be two osts’s. Then  

(a) \( \tau \) is ordinary smooth continuous if and only if  

\[ C_{\tau_2}(A) \subseteq C_{\tau_1}(f^{-1}(A)), \forall A \in 2^Y. \]  

(b) \( \tau \) is ordinary smooth weakly continuous if and only if  

\[ C_{\tau_2}(A) = 0 \Rightarrow C_{\tau_1}(f^{-1}(A)) = 0, \forall A \in 2^Y. \]  

(c) \( \tau \) is ordinary smooth strongly continuous if and only if  

\[ C_{\tau_2}(A) = C_{\tau_1}(f^{-1}(A)), \forall A \in 2^Y. \]  

The following are the immediate results of Definition 3.1.  

**Proposition 3.4.** (See Lemma 2.1 in [10]) Let \( (X, \tau_1), (Y, \tau_2) \) and \((Z, \tau_3)\) be osts’s. If \( f : X \to Y \) and \( g : Y \to Z \) are ordinary smooth continuous, then so is  

\[ g \circ f. \]  

**Proposition 3.5.** Let \((X, \tau)\) be an osts. Then the identity mapping \( id : X \to X \) is ordinary smooth continuous.  

**Theorem 3.6.** Let \((X, \tau)\) and \((Y, \tau')\) be two osts’s and let \( f : X \to Y \) be a mapping. Then \( f \) is ordinary smooth continuous if and only if \( (X, [\tau]_r) \to (Y, [\tau']_r) \) is classical continuous for each \( r \in I_0 \).  

**Proof.** \((\Rightarrow):\) Suppose \( f \) is ordinary smooth continuous and let \( r \in I_0 \). Then  

\[ f^{-1}([\tau']_r) \subseteq [\tau]_r. \]  

Thus  

\[ f^{-1}(A) \subseteq \tau_r(A). \]  

So \( f : (X, [\tau]_r) \to (Y, [\tau']_r) \) is classical continuous.  

\((\Leftarrow):\) Suppose the necessary condition holds and let \( A \in 2^Y \).  

If \( \tau(A) = 0 \), then clearly \( \tau(A) \leq \tau(f^{-1}(A)) \).  

If \( \tau(A) = r \), then \( A \in [\tau]_r \). Thus, by the hypothesis, \( f^{-1}(A) \subseteq [\tau]_r \). So \( \tau(A) = r \leq \tau(f^{-1}(A)) \).  

Hence f : (X, \tau) \to (Y, \tau') is ordinary smooth continuous. This completes the proof. \( \square \)  

**Theorem 3.7.** Let \((X, T_1)\) and \((Y, T_2)\) be two classical topological spaces and let \( f : X \to Y \) be a mapping. Then \( f : (X, T_1) \to (Y, T_2) \) is classical continuous if and only if \( f : (X, T_1^r) \to (Y, T_2^r) \) is ordinary smooth continuous for each \( r \in I_0 \).  

**Proof.** \((\Rightarrow):\) Suppose \( f : (X, T_1) \to (Y, T_2) \) is classical continuous and let \( A \in 2^Y \). Then we have the following possibilities:  

(i) \( A = \emptyset \) or \( Y \),  

(ii) \( A \in T_2 \),  

(iii) \( A \notin T_2 \).  

In case (i), \( f^{-1}(\emptyset) = \emptyset \) and \( f^{-1}(Y) = Y \). By Proposition 2.16, \( T_1^r \in \text{OST}(X) \) and \( T_2^r \in \text{OST}(Y) \) for each \( r \in I_0 \). Thus  

\[ T_1^r(f^{-1}(A)) = 1 \geq T_2^r(A). \]  

In case (ii), \( T_2^r(A) = r \), by Proposition 2.16. Since \( f : (X, T_1) \to (Y, T_2) \) is classical continuous and \( A \in T_2 \), \( f^{-1}(A) \in T_1 \). Thus  

\[ T_1^r(f^{-1}(A)) = r. \]  

So \( T_2^r(A) \leq T_1^r(f^{-1}(A)) \).  

In case (iii), \( T_2^r(A) = 0 \), by Proposition 2.16. Thus  

\[ 0 = T_2^r(A) \leq T_1^r(f^{-1}(A)) \].  

Hence \( f : (X, T_1^r) \to (Y, T_2^r) \) is ordinary smooth continuous for each \( r \in I_0 \).  

\((\Leftarrow):\) Suppose the necessary condition holds. Then it follows from Proposition 2.16 and Theorem 3.6. \( \square \)  

**Theorem 3.8.** Let \((X, \tau)\) be an osts and let \( f : X \to Y \) be a mapping. Let \( T'_r : r \in I_0 \) be a descending family of classical topologies on \( Y \) and let \( \tau' \) be the osts on \( Y \) generated by this family. For each \( r \in I_0 \), let \( B_r \) be a base and \( s_r \) be a subbase for \( T'_r \).  

Then  

(a) \( f : (X, \tau) \to (Y, \tau') \) is ordinary smooth continuous if and only if \( r \leq \tau(f^{-1}(A)), \forall A \in T'_r, \forall r \in I_0 \).  

(b) \( f : (X, \tau) \to (Y, \tau') \) is ordinary smooth continuous if and only if \( r \leq \tau(f^{-1}(A)), \forall A \in B_r, \forall r \in I_0 \).  

(c) \( f : (X, \tau) \to (Y, \tau') \) is ordinary smooth continuous if and only if \( r \leq \tau(f^{-1}(A)), \forall A \in s_r, \forall r \in I_0 \).  

**Proof.** \((\Rightarrow):\) Suppose \( f : (X, \tau) \to (Y, \tau') \) is ordinary smooth continuous. Let \( r \in I_0 \) and let \( A \in T'_r \). Then  

\[ r \leq \tau'(A) \leq \tau(f^{-1}(A)). \]  

\((\Leftarrow):\) Suppose the necessary condition holds. Let \( A \in 2^Y \) and let \( \tau'(A) = r > 0 \). Then clearly \( A \in T'_r \). Thus  

\[ \tau'(A) = r \leq \tau(f^{-1}(A)). \]  

Arguing as above and using the definition of base and subbase for a classical topology, we have (b) and (c). \( \square \)  

**Definition 3.9.** [10] Let \( \tau_1 \in \text{OST}(X), \mathcal{C}_1 \in \text{OSTC}(X), \mathcal{C}_2 \in \text{OSTC}(Y) \) and \( \mathcal{C}_3 \in \text{OSTC}(Y) \). Then a mapping \( f : X \to Y \) is said to be:  

(i) ordinary smooth open if \( \tau_1(A) \leq \tau_2(f(A)), \forall A \in 2^X \),  

(ii) ordinary smooth closed if \( \mathcal{C}_1(A) \leq \mathcal{C}_3(f(A)), \forall A \in 2^X \).
Definition 3.10. [10] Let $\tau_1 \in \text{OST}(X)$ and let $\tau_2 \in \text{OST}(Y)$. Then a mapping $f : X \to Y$ is called an ordinary smooth homeomorphism if $f$ is bijective, and $f$ and $f^{-1}$ are ordinary smooth continuous.

The following is the immediate result of Definitions 3.1, 3.9 and Theorem 3.3 (a).

Theorem 3.11. Let $(X, \tau_1)$ and $(Y, \tau_2)$ be two osts’s and let $f : X \to Y$ be a mapping. Then the following are equivalent:

(a) $f$ is an ordinary smooth homeomorphism.

(b) $f$ is ordinary smooth open and ordinary smooth continuous.

(c) $f$ is ordinary smooth closed and ordinary smooth continuous.

The following is the immediate result of Proposition 2.11 and Definitions 3.1 and 3.9.

Proposition 3.12. Let $X$ and $Y$ be two sets, let $\{T_r : r \in I_0\}$ and $\{T'_r : r \in I_0\}$ be descending families of ordinary topologies on $X$ and $Y$, respectively. Let $\tau$ and $\tau'$ be osts’s on $X$ and $Y$, respectively generated by the families $\{T_r : r \in I_0\}$ and $\{T'_r : r \in I_0\}$, and let $f : X \to Y$ be a mapping. For each $r \in I_0$, if $f : (X, T_r) \to (Y, T'_r)$ is classical continuous [resp. classical open and classical closed], then $f : (X, \tau) \to (Y, \tau')$ is ordinary smooth continuous [resp. ordinary smooth open and ordinary smooth closed].

4. Bases for an ordinary smooth topology

Definition 4.1. [9] Let $(X, \tau)$ be an osts and let $x \in X$. Then $N_x$ is called the ordinary smooth neighborhood system (in short, osns) of $x$ if $N_x : 2^X \to I$ is the mapping defined as follows: For each $A \in 2^X$,

$$N_x(A) = \bigvee_{x \in B \subseteq A} \tau(B).$$

Result 4.A. [9, Lemma 3.1] Let $(X, \tau)$ be an osts and let $x \in X$. Then

$$\tau(A) = \bigwedge_{x \in A} \bigvee_{x \in B \subseteq A} \tau(B), \forall A \in 2^X.$$

Definition 4.2. Let $(X, \tau)$ be an ordinary smooth topological spaces and let $\mathfrak{B} : 2^X \to I$ be a mapping such that $\mathfrak{B} \leq \tau$. Then $\mathfrak{B}$ is called an ordinary smooth base for $\tau$ if for each $A \in 2^X$,

$$\tau(A) = \bigvee_{\mathfrak{B}(B_\alpha) \subseteq 2^X} \bigwedge_{\alpha \in \Gamma} \mathfrak{B}(B_\alpha).$$

Example 4.3. (a) Let $X$ be a set and let $\mathfrak{B} : 2^X \to I$ be the mapping defined by $\mathfrak{B}(\{x\}) = 1$ for each $x \in X$. Then $\mathfrak{B}$ is an ordinary smooth base for the ordinary smooth discrete topology $\tau_X$ on $X$.

(b) Let $X = \{a, b, c\}$, let $r \in I_1$ be fixed and let $\mathfrak{B} : 2^X \to I$ be the mapping defined as follows: For each $A \in 2^X$,

$$\mathfrak{B}(A) = \begin{cases} 1, & A = \{a, b\} \text{ or } \{b, c\} \text{ or } X; \\ r, & \text{otherwise.} \end{cases}$$

Then $\mathfrak{B}$ is not an ordinary smooth base for an ordinary smooth topology on $X$.

Assume that $\mathfrak{B}$ is an ordinary smooth base for an ordinary smooth topology $\tau$ on $X$. Then clearly $\mathfrak{B} \leq \tau$. Moreover, $\tau(\{a, b\}) = \tau(\{b, c\}) = 1$. Thus

$$\tau(\{b\}) = \bigvee_{\mathfrak{B}(A) \subseteq 2^X} \{x\} = \bigwedge_{A \in \Gamma} \mathfrak{B}(A)$$

This is a contradiction. Hence $\mathfrak{B}$ is not an ordinary smooth base for an ordinary smooth topology on $X$.

Theorem 4.4. Let $(X, \tau)$ be an ordinary smooth topological space and let $\mathfrak{B} : 2^X \to I$ be a mapping such that $\mathfrak{B} \leq \tau$. Then $\mathfrak{B}$ is an ordinary smooth base for $\tau$ if and only if $N_x(A) \leq \bigvee_{x \in B \subseteq A} \mathfrak{B}(B)$, for each $x \in X$ and each $A \in 2^X$.

Proof. ($\Rightarrow$) Suppose $\mathfrak{B}$ is an ordinary smooth base for $\tau$. Let $x \in X$ and let $A \in 2^X$. Then

$$N_x(A) = \bigvee_{x \in B \subseteq A} \tau(B) \quad \text{[By Definition 4.1]}$$

$$= \bigvee_{x \in B \subseteq A(\mathfrak{B}_\alpha) \subseteq 2^X} \bigwedge_{\alpha \in \Gamma} \mathfrak{B}(B_\alpha).$$

[By Definition 4.2]
Hence, by (4.1) and (4.2),

$$\tau(A) = \bigvee_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X} \bigwedge_{\alpha \in \Gamma} \mathfrak{B}(B_\alpha)$$

Thus

$$N_x(A) \leq \bigvee_{x \in B \subset A} \mathfrak{B}(B).$$

($\Leftarrow$): Suppose the necessary condition holds. Let $A \in 2^X$. Suppose $A = \bigcup_{\alpha \in \Gamma} B_\alpha$ and $\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X$.

Then

$$\tau(A) \geq \bigwedge_{\alpha \in \Gamma} \mathfrak{B}(B_\alpha) \quad (4.1)$$

On the other hand,

$$\tau(A) = \bigwedge_{x \in X} \bigvee_{x \in B \subset A} \tau(B) \quad [\text{By Result 4.1A}]$$

$$= \bigwedge_{x \in X} N_x(A) \quad [\text{By Definition 4.1}]$$

$$= \bigwedge_{x \in X} \bigvee_{x \in B \subset A} \mathfrak{B}(B) \quad [\text{By the hypothesis}]$$

$$= \bigvee_{f \in \prod_{x \in A} \mathfrak{B}_x} \bigwedge_{x \in A} \mathfrak{B}(f(x)),$$

where $\mathfrak{B}_x = \{B \in 2^X : x \in B \subset A\}$. Moreover,

$$A = \bigcup_{x \in A} f(x) \quad \text{for each } f \in \prod_{x \in A} \mathfrak{B}_x.$$ Thus

$$\tau(A) \leq \bigvee_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X} \bigwedge_{\alpha \in \Gamma} \mathfrak{B}(B_\alpha) \quad (4.2)$$

Hence, by (4.1) and (4.2),

$\mathfrak{B}$ is an ordinary smooth base for $\tau$.

The following is the restatement of Theorem 4.3.

**Theorem 4.5.** Let $\mathfrak{B} : 2^X \to I$ be a mapping. Then $\mathfrak{B}$ is an ordinary smooth base for some ordinary smooth topology $\tau$ on $X$ if and only if it satisfies the following conditions:

(a) $\bigvee_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X} \bigwedge_{\alpha \in \Gamma} \mathfrak{B}(B_\alpha) = 1$

(b) For any $A_1, A_2 \in 2^X$ and each $x \in A_1 \cap A_2$, $\mathfrak{B}(A_1) \cup \mathfrak{B}(A_2) \subseteq \bigvee_{x \in A \cap A_2} \mathfrak{B}(A)$

In fact, $\tau : 2^X \to I$ is the mapping defined as follows: For each $A \in 2^X$,

$$\tau(A) = \begin{cases} 1, & \text{if } A = \emptyset; \\ \bigvee_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X} \bigwedge_{\alpha \in \Gamma} \mathfrak{B}(B_\alpha), & \text{otherwise.} \end{cases}$$

In this case, $\tau$ is called the ordinary smooth topology on $X$ generated by $\mathfrak{B}$.

**Proof.** Since the proof is similar to that of Theorem 4.2 in [9], we omit it.

**Example 4.6.** (a) Let $X = \{a, b, c\}$ and let $r \in I_1$ be fixed. We define the mapping $\mathfrak{B} : 2^X \to I$ as follows: For each $A \in 2^X$,

$$\mathfrak{B}(A) = \begin{cases} 1, & \text{if } A = \{b\} \text{ or } \{a, b\} \text{ or } \{b, c\}; \\ r, & \text{otherwise.} \end{cases}$$

Then we can easily see that $\mathfrak{B}$ satisfies the conditions (a) and (b) in Theorem 4.3. Thus $\mathfrak{B}$ is an ordinary smooth base for an ordinary smooth topology $\tau$ on $X$. In fact, $\tau : 2^X \to I$ be the mapping defined as follows: For each $A \in 2^X$,

$$\tau(A) = \begin{cases} 1, & \text{if } A = \emptyset, \{b\}, \{a, b\}, \{b, c\}, X; \\ r, & \text{otherwise.} \end{cases}$$

(b) Let $r \in I_1$ be fixed. We define the mapping $\mathfrak{B} : 2^R \to I$ as follows: For each $A \in 2^R$,

$$\mathfrak{B}(A) = \begin{cases} 1, & \text{if } A = (a, b); \\ r, & \text{otherwise.} \end{cases}$$
Then it can be easily seen that $\mathfrak{B}$ satisfies the conditions (a) and (b) in Theorem 2.3. Thus $\mathfrak{B}$ is an ordinary smooth base for an ordinary smooth topology $\mathcal{U}_t$ on $\mathbb{R}$. In this case, $\mathcal{U}_t$ will be called the r-ordinary smooth usual topology.

(c) Let $r \in I_1$ be fixed. We define the mapping $\mathfrak{B} : 2^\mathbb{R} \to I$ as follows: For each $A \in 2^\mathbb{R}$, $$\mathfrak{B}(A) = \begin{cases} 1, & \text{if } A = [a, b]; \\ r, & \text{otherwise.} \end{cases}$$

Then we can see that $\mathfrak{B}$ satisfies the conditions (a) and (b) in Theorem 4.5. Thus $\mathfrak{B}$ is an ordinary smooth base for an ordinary smooth topology $\mathcal{U}_t$ on $\mathbb{R}$. Furthermore, $\mathcal{U}_t \not\subseteq \mathcal{U}_l$.

In this case, $\mathcal{U}_t$ will be called the r-ordinary smooth usual topology on $\mathbb{R}$. □

**Definition 4.7.** Let $\tau_1, \tau_2 \in \text{OST}(X)$, and let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be ordinary smooth bases for $\tau_1$ and $\tau_2$, respectively. Then $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are equivalent if $\tau_1 = \tau_2$.

**Theorem 4.8.** Let $\tau_1, \tau_2 \in \text{OST}(X)$, and let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be ordinary smooth bases for $\tau_1$ and $\tau_2$, respectively. Then $\tau_2$ is finer than $\tau_1$, i.e., $\tau_1 \leq \tau_2$ if and only if for each $x \in X$ and each $B \in 2^X$, if $x \in B$, then $\mathfrak{B}_1(B) \leq \bigvee_{x \in B \cap B'} \mathfrak{B}_2(B').$

**Proof.** ($\Rightarrow$) : Suppose $\tau_1 \leq \tau_2$. For each $x \in X$, let $B \in 2^X$ such that $x \in B$. Then

$$\mathfrak{B}_1(B) \leq \tau_1(B) \leq \tau_2(B) \leq \bigvee_{x \in B \cap B'} \mathfrak{B}_2(B').$$

Since $x \in B$, if $B = \bigcup_{\alpha \in \Gamma} B_{\alpha}$, then there exists $\alpha_0 \in \Gamma$ such that $x \in B_{\alpha_0}$. Thus

$$\bigwedge_{\alpha \in \Gamma} \mathfrak{B}_2(B_{\alpha}) \leq \mathfrak{B}_2(B_{\alpha_0}) \leq \bigvee_{x \in B \cap B'} \mathfrak{B}_2(B').$$

So

$$\mathfrak{B}_2(B) \leq \bigvee_{x \in B \cap B'} \mathfrak{B}_2(B').$$

($\Leftarrow$) : Suppose the necessary condition holds. Let $A \in 2^X$ and let $\mathcal{N}_{\tau_1}$ be the ordinary smooth neighborhood system of $x \in X$ w.r.t. $\tau_1$. Then

$$\tau_1(A) = \bigwedge_{x \in A} \mathcal{N}_{\tau_1}(A) \leq \bigwedge_{x \in A} \bigwedge_{x \in B \subseteq A} \mathfrak{B}_2(B) \leq \bigwedge_{x \in A} \bigwedge_{x \in B \subseteq A} \mathfrak{B}_2(B) \leq \bigwedge_{x \in A} \bigwedge_{x \in B \subseteq A} \mathfrak{B}_2(B) \leq \bigwedge_{x \in A} \bigwedge_{x \in B \subseteq A} \mathfrak{B}_2(B).$$

Thus $\tau_1 \leq \tau_2$. This completes the proof. □

The following is the immediate result of Definition 4.5 and Theorem 4.6.

**Corollary 4.9.** Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two ordinary smooth bases for ordinary smooth topologies on a set $X$, respectively. Then $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are equivalent if and only if

(a) For each $B_1 \subseteq 2^X$ and each $x \in B_1$, $\mathfrak{B}_1(B_1) \leq \bigvee_{x \in B_1 \subseteq B_2} \mathfrak{B}_2(B_2).$

(b) For each $B_2 \subseteq 2^X$ and each $x \in B_2$, $\mathfrak{B}_2(B_2) \leq \bigvee_{x \in B_2 \subseteq B_1} \mathfrak{B}_1(B_1).$

It is clear that the ordinary smooth topology itself forms an ordinary smooth base. Then every ordinary smooth topology has an ordinary smooth base. The following provides a condition for one to check to see if a mapping $\mathfrak{B} : 2^X \to I$ such that $\mathfrak{B} \leq \tau$ is an ordinary smooth base for $\tau$, where $\tau \in \text{OST}(X)$.

**Proposition 4.10.** Let $(X, \tau)$ be an ordinary smooth topological space, let $\mathfrak{B} : 2^X \to I$ a mapping such that $\mathfrak{B} \leq \tau$, and for each $x \in X$ and each $A \in 2^X$ with $x \in A$, let

$$\tau(A) \leq \bigvee_{x \in B \subseteq A} \mathfrak{B}(B).$$

Then $\mathfrak{B}$ is an ordinary smooth base for $\tau$. □
Proof.

\[
\bigvee_{\{B_\alpha\}_{\alpha \in \Gamma} \subseteq 2^X, X = \bigcup_{\alpha \in \Gamma} B_\alpha, B_\alpha \in \mathfrak{B}(B_\alpha) \}
\leq \bigvee_{\{B_\alpha\}_{\alpha \in \Gamma} \subseteq 2^X, X = \bigcup_{\alpha \in \Gamma} B_\alpha, B_\alpha \in \mathfrak{B}(B_\alpha) \}
\leq \bigvee_{\{B_\alpha\}_{\alpha \in \Gamma} \subseteq 2^X, X = \bigcup_{\alpha \in \Gamma} B_\alpha, B_\alpha \in \mathfrak{B}(B_\alpha) \}
\]

[By the axiom (OST3)]

\[
\tau(X) = \bigwedge_{x \in A, x \in B \subset X} \tau(B) \text{ [By Result 4.A]}
\]

\[
\leq \bigwedge_{x \in A, x \in B \subset X} \bigwedge_{x \in C \subset B} \mathfrak{B}(C) \text{ [By the hypothesis]}
\]

\[
= \bigwedge_{x \in C \subset X} \mathfrak{B}(C)
\]

\[
= \bigwedge_{\{B_\alpha\}_{\alpha \in \Gamma} \subseteq 2^X, X = \bigcup_{\alpha \in \Gamma} B_\alpha, B_\alpha \in \mathfrak{B}(B_\alpha) \}
\]

Then

\[
\tau(X) = \bigvee_{\{B_\alpha\}_{\alpha \in \Gamma} \subseteq 2^X, X = \bigcup_{\alpha \in \Gamma} B_\alpha, B_\alpha \in \mathfrak{B}(B_\alpha) \}
\]

Since \(\tau \in \text{OST}(X)\), \(\tau(X) = 1\). Thus

\[
\bigvee_{\{B_\alpha\}_{\alpha \in \Gamma} \subseteq 2^X, X = \bigcup_{\alpha \in \Gamma} B_\alpha, B_\alpha \in \mathfrak{B}(B_\alpha) \} = 1.
\]

So the condition (a) of Theorem 4.5 holds.

Now let \(A_1, A_2 \in 2^X\) and let \(x \in A_1 \cap A_2\). Then

\[
\mathfrak{B}(A_1) \wedge \mathfrak{B}(A_2) \leq \tau(A_1) \wedge \tau(A_2) \text{ [Since } \mathfrak{B} \leq \tau]\]

\[
\leq \bigwedge_{x \in A, x \in A_1 \cap A_2} \mathfrak{B}(A) \text{ [By the axiom (OST2)]}
\]

Thus the condition (b) of Theorem 4.5 holds. So, by Theorem 4.5, \(\mathfrak{B}\) is an ordinary smooth base for \(\tau\). This completes the proof.

**Definition 4.11.** Let \((X, \tau)\) be an ordinary smooth topological space, let \(\varphi : 2^X \rightarrow I\) a mapping. Then \(\varphi\) is called an ordinary smooth subbase for \(\tau\) if \(\varphi^{-1}\) is an ordinary smooth base for \(\tau\), where \(\varphi^{-1} : 2^X \rightarrow I\) is the mapping defined as follows: For each \(A \in 2^X\),

\[
\varphi^{-1}(A) = \bigvee_{\{B_\alpha\}_{\alpha \in \Gamma} \subseteq 2^X, A = \bigcap_{\alpha \in \Gamma} \mathfrak{B}(B_\alpha), B_\alpha \in \mathfrak{B}(B_\alpha) \}
\]

with \(\Box\) standing for “a finite subset of”.

**Example 4.12.** Let \(r \in I_1\) be fixed. We define the mapping \(\varphi : 2^R \rightarrow I\) as follows: For each \(A \in 2^R\),

\[
\varphi(A) = \begin{cases} 1, & \text{if } A = (a, \infty) \text{ or } -\infty, b \text{ or } (a, b); \\ r, & \text{otherwise}. \end{cases}
\]

where \(a, b \in \mathbb{R}\) such that \(a < b\). Then we can easily see that \(\varphi\) is an ordinary smooth subbase for the \(r\)-ordinary smooth usual topology \(\mathcal{U}_r\) on \(\mathbb{R}\).

**Result 4.B.** [9, Theorem 4.3] Let \(\varphi : 2^X \rightarrow I\) a mapping. Then \(\varphi\) is an ordinary smooth subbase for some ordinary smooth topology \(\tau\) on \(X\) if and only if

\[
\bigvee_{\{B_\alpha\}_{\alpha \in \Gamma} \subseteq 2^X, X = \bigcup_{\alpha \in \Gamma} B_\alpha, B_\alpha \in \mathfrak{B}(B_\alpha) \} \wedge \varphi(B_\alpha) = 1.
\]

In this case, \(\tau\) is called the ordinary smooth topology generated by \(\varphi\).

**Example 4.13.** Let \(X = \{a, b, c, d, e\}\) and let \(r \in I_1\) be fixed. We define the mapping \(\varphi : 2^X \rightarrow I\) as follows: For each \(A \in 2^X\),

\[
\varphi(A) = \begin{cases} 1, & \text{if } A \in \{\{a\}, \{a, b, c\}, \{b, c, d\}, \{c, e\}\}; \\ r, & \text{otherwise}. \end{cases}
\]

Then

\[
X = \{a\} \cup \{b, c, d\} \cup \{c, e\}
\]

and

\[
\varphi(\{a\}) \wedge \varphi(\{b, c, d\}) \wedge \varphi(\{c, e\}) = 1
\]

Thus

\[
\bigvee_{\{B_\alpha\}_{\alpha \in \Gamma} \subseteq 2^X, X = \bigcup_{\alpha \in \Gamma} B_\alpha, B_\alpha \in \mathfrak{B}(B_\alpha) \} \wedge \varphi(B_\alpha) = 1.
\]

So, by the result 4.B, \(\varphi\) is an ordinary smooth subbase for some ordinary smooth topology \(\tau\) on \(X\).
The following is the immediate result of Corollary 4.9 and Result 4.A.

**Proposition 4.14.** Let \( \varphi_1, \varphi_2 : 2^X \to I \) be two mappings such that

\[
\bigvee_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X, \ X = \bigcup_{\alpha \in \Gamma} B_\alpha \} \varphi_1(B_\alpha) = 1.
\]

and

\[
\bigvee_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X, \ X = \bigcup_{\alpha \in \Gamma} B_\alpha \} \varphi_2(B_\alpha) = 1.
\]

Suppose the two conditions holds:

(a) For each \( S_1 \in 2^X \) and each \( x \in S_1, \varphi_1(S_1) \leq \bigvee_{x \in S_1 \subset S_1} \varphi_2(S_2). \)

(b) For each \( S_2 \in 2^X \) and each \( x \in S_2, \varphi_2(S_2) \leq \bigvee_{x \in S_2 \subset S_2} \varphi_2(S_1). \)

Then \( \varphi_1 \) and \( \varphi_2 \) are ordinary smooth subbases for the some ordinary smooth topology on \( X. \)

5. Ordinary smooth subspace

**Proposition 5.1.** Let \( (X, \tau) \) be an osts and let \( A \subset X. \) We define a mapping \( \tau_A : 2^A \to I \) as follows : For each \( B \in 2^A, \)

\[
\tau_A(B) = \bigvee \{ \tau(C) : C \in 2^X \text{ and } C \cap A = B \}.
\]

Then \( \tau_A \in \text{OST}(A) \) and \( \tau(B) \leq \tau_A(B). \) In this case, \( (A, \tau_A) \) is called an ordinary smooth subspace of \( (X, \tau) \) and \( \tau_A \) is called the induced ordinary smooth topology on \( A \) by \( \tau. \)

**Proof.** (OST1) It is clear that \( \tau_A(\emptyset) = \tau_A(A) = 1. \)

(OST2) Let \( B_1, B_2 \in 2^A. \) Then

\[
\tau_A(B_1) \land \tau_A(B_2) = (\bigvee \{ \tau(C_1) : C_1 \in 2^X \text{ and } C_1 \cap A = B_1 \}) \land (\bigvee \{ \tau(C_2) : C_2 \in 2^X \text{ and } C_2 \cap A = B_2 \})
\]

\[
\leq \bigvee \{ \tau(C_1) \land \tau(C_2) : C_1, C_2 \in 2^X \text{ and } (C_1 \cap C_2) \cap A = B_1 \cap B_2 \}
\]

\[
= \tau_A(B_1 \cap B_2).
\]

(OST3) Let \( \{B_\alpha\}_{\alpha \in \Gamma} \subset 2^A. \) Then

\[
\tau_A(\bigcup_{\alpha \in \Gamma} B_\alpha) = \bigvee \{ \tau(C_\alpha) : C_\alpha \in 2^X \text{ and } C_\alpha \cap A = B_\alpha \}, \forall \alpha \in \Gamma.
\]

Thus

\[
\bigwedge_{\alpha \in \Gamma} \tau_A(B_\alpha) = \bigvee \{ \bigwedge_{\alpha \in \Gamma} \tau(C_\alpha) : C_\alpha \in 2^X \text{ and } \bigwedge_{\alpha \in \Gamma} C_\alpha \cap A = B_\alpha \}.
\]

Hence \( \tau_A \in \text{OST}(A). \)

Now let \( B \in 2^A. \) Then

\[
\tau_A(B) = \bigvee \{ \tau(C) : C \in 2^X \text{ and } C \cap A = B \}
\]

\[
\leq \tau(B). [\text{Since } B \subset A, B \cap A = B]
\]

This completes the proof. \( \square \)

**Proposition 5.2.** Let \( (X, \tau) \) be an osts, let \( (Y, \tau_Y) \) be an ordinary smooth subspace of \( (X, \tau) \) and let \( A \in 2^Y. \) Then

(a) \( C_{\tau_Y}(A) = \bigvee \{ C_{\tau_Y}(B) : B \in 2^X \text{ and } B \cap Y = A \}. \)

(b) If \( Z \subset Y \subset X \) then \( \tau_Z = (\tau_Y)_C. \)

**Proof.** (a)

\[
\mathfrak{F}_{\tau_Y}(A) = \tau_Y(Y - A)
\]

\[
= \bigvee \{ \tau(B) : B \in 2^X \text{ and } B \cap Y = Y - A \}
\]

\[
= \bigvee \{ \tau(B) : B^c \in 2^X \text{ and } B^c \cap Y = Y - A \}
\]

\[
= \bigvee \{ \mathfrak{F}_{\tau}(B^c) : B^c \in 2^X \text{ and } B^c \cap Y = Y - A \}
\]

\[
= \bigvee \{ \mathfrak{F}_{\tau}(C) : C \in 2^X \text{ and } C \cap Y = Y - A \}.
\]

(b) Let \( A \in 2^Z. \) Then

\[
(\tau_Y)_Z(A)
\]

\[
= \bigvee \{ \tau_Y(B) : B \in 2^X \text{ and } B \cap Z = A \}
\]

\[
= \bigvee \{ \tau(Y) : C \in 2^X \text{ and } C \cap Y = B \} : B \in 2^Y \text{ and } B \cap Z = A \}
\]

\[
= \bigvee \{ \tau(C) : C \in 2^X \text{ and } C \cap Z = A \}
\]

\[
= \tau_Z(A).
\]

\( \square \)

**Proposition 5.3.** (See Lemma 2.2 in [10]) Let \( (X, \tau) \) and \( (Y, \tau') \) be two osts’s, let \( f : X \to Y \) be ordinary smooth continuous and let \( A \subset X. \) Then the restriction \( f |_A : (A, \tau_A) \to (Y, \tau') \) is also ordinary smooth continuous.
Proof. Let $B \in 2^Y$. Then
\begin{align*}
\tau_A((f |_A)^{-1}(B)) & = \bigvee \{\tau(C) : C \in 2^X \text{ and } C \cap A = (f |_A)^{-1}(B)\} \\
& \geq \tau(f^{-1}(B)) \\
& \geq \tau(B).
\end{align*}
So $f |_A$ is ordinary smooth continuous. \qed

References


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