A Note on the Fuzzy Linear Maps over the Fuzzy Quotient Spaces

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Abstract

In this paper we study various properties of the fuzzy linear maps over the fuzzy quotient spaces. In particular we obtain some exact sequences of the fuzzy linear maps over the fuzzy quotient spaces.

Key Words: Fuzzy R-module, Fuzzy linear map(Fuzzy R-map), Fuzzy weak isomorphism, Quasi-monic, Epic, Fuzzy quotient spaces.

1. Introduction

Fuzzy modules were introduced by Negoita and Ralescu [1]. Katsaras and Liu [2], and Lowen [3] have developed the theory of fuzzy vector spaces.

Fu-Zhen Pan [4] investigated fuzzy vector spaces for the following purposes; to establish a fundamental frame of the theory of fuzzy vector space by virtue of homological algebra and modular theory, and to stretch it out to study general fuzzy modules.

In fact, fuzzy vector spaces are the simplest kind of fuzzy free modules. The theory of fuzzy modules has been a virgin field for a long time.

Recently, many authors presented the same research on fuzzy modules, properties of fuzzy finitely generated modules and fuzzy quotient modules, etc.

In particular, Fu-Zhen Pan [5] and Kim [6] investigated the properties of the sequence of fuzzy linear maps and studied the situations in connection with two exact sequences of fuzzy linear maps.

In this paper, we study various properties of fuzzy linear maps over fuzzy quotient spaces.

2. Preliminaries

In this section, we review some definitions and some results which will be used in the later sections. Throughout this paper, we assume that all modules are equipped with the same underlying commutative ring R.

Definition 2.1 ([5]). A R-module M together with a function χ from M into [0, 1] is called a fuzzy R-module if it satisfies the following conditions;

1. χ(a + b) ≥ min{χ(a), χ(b)}
2. χ(−a) = χ(a)
3. χ(0) = 1
4. χ(ra) ≥ χ(a),

for any a, b ∈ M and r ∈ R, denoted by (M, χ) or χM.

Definition 2.2 ([5]). Let χM, ηN be any two fuzzy R-modules, then f : χM −→ ηN is called a fuzzy linear map (or fuzzy R-map) if there exists a linear map f : M −→ N such that η(f(a)) ≥ χ(a) for all a ∈ M.

Remark 2.3. Let χM, ηN be any two fuzzy R-modules. Then (f : χM −→ ηN) is called a fuzzy strong linear map if there exists a linear map f : M −→ N such that η(f(a)) = χ(a) for all a ∈ M.

Definition 2.4 ([5]). Let f : χV −→ ηW be a fuzzy linear map. f is called a fuzzy weak isomorphism if f is an isomorphism, denoted χV ∼ 1WηW.

Definition 2.5 ([5]). For a fuzzy linear map f : χM −→ ηN, η1Mf is called the image of f denoted by η1Mf. Further, χM0, where M0 = \{m ∈ M | η(f(m)) = 1\} is called the Kernel of f denoted by χKerf.

Theorem 2.6 ([5]). Let f : χM −→ ηN be a fuzzy linear map, then χKerf is a fuzzy subspace of χM and η1Mf is a fuzzy subspace of χN.

Remark 2.7. For any fuzzy linear map f : χM −→ ηN, χKerf ≤ χKerf and η1Mf = η1Mf.
Definition 2.8 ([5]). A fuzzy linear map \( \tilde{f} : \chi_M \rightarrow \eta_N \) is called epic (or monic) iff \( f : M \rightarrow N \) is epic (or monic).

Definition 2.9 ([5]). A fuzzy linear map \( \tilde{f} : \chi_M \rightarrow \eta_N \) is called quasi-monic iff \( \chi_{Ker\tilde{f}} = \chi_{M'} \), where \( M' = \{ m \in M \mid \chi(m) = 1 \} \).

Remark 2.10. Obviously, when \( \chi_{Ker\tilde{f}} = \emptyset \), quasi-monic is just ordinary monic.

Definition 2.11 ([5]). Two fuzzy maps
\[
\chi_M \xrightarrow{i} \eta_N \xrightarrow{\tilde{g}} \rho_V \xrightarrow{j} 1
\]
are exact at \( \eta_N \) iff \( \eta_{im'} = \chi_{Ker\tilde{g}} \).

Remark 2.12. By the induction, from Definition 2-11, we can define an exact sequence of fuzzy linear maps.

Theorem 2.13 ([5]). A fuzzy \( R \)-module \( \chi_M \) is called a fuzzy singular \( R \)-module iff \( \chi(m) = 1 \) for all \( m \in M \), denoted by 1.

Theorem 2.14 ([5]). An exact sequence
\[
1 \xrightarrow{i} \chi_M \xrightarrow{\tilde{f}} \eta_N \xrightarrow{\tilde{g}} \rho_V \xrightarrow{j} 1
\]
where the two 1’s are the appropriate singular fuzzy \( R \)-modules and \( i, j \) are the fuzzy identity map and an epic map, respectively, is called a short exact sequence of fuzzy linear maps.

Theorem 2.15 ([5]). Given a short exact sequence of fuzzy linear maps,
\[
1 \xrightarrow{i} \chi_M \xrightarrow{\tilde{f}} \eta_N \xrightarrow{\tilde{g}} \rho_V \xrightarrow{j} 1
\]
1. \( Im\tilde{i} = Ker\tilde{f} = \chi_{M'} \),
2. \( Im\tilde{f} = Ker\tilde{g} \geq \eta_{N'} \),
3. \( \tilde{g} \) is epic,
where \( M' = \{ m \in M \mid \chi(m) = 1 \} \) and \( N' = \{ n \in N \mid \eta(n) = 1 \} \).

Proposition 2.16 ([3]). For any fuzzy linear map \( \tilde{f} : \chi_M \rightarrow \eta_N \),
\[
\chi_{M'} \subseteq \chi_{Ker\tilde{f}}
\]
where \( M' = \{ m \in M \mid \chi(m) = 1 \} \).

Definition 2.17 ([8]). By the coimage \( Coim h \) and the cokernel \( Coker h \) of a homomorphism \( h : X \rightarrow Y \), we mean the quotient modules
\[
Coim h = X/Ker h, Coker h = Y/Im h.
\]

3. Fuzzy linear maps over fuzzy quotient spaces

In this section, we study various properties of fuzzy linear maps over fuzzy quotient spaces.

Let \( \chi_V \) be an arbitrary fuzzy subspace of a fuzzy vector space \( \chi_V \). Then a map \( \tilde{\chi} : V/W \rightarrow [0, 1] \) given by
\[
\tilde{\chi}(u + W) = \begin{cases} 1 & \text{if } v \in W \\ \inf \{ \chi(v') \mid v' \in v + W \} & \text{if } v \notin W 
\end{cases}
\]
can determine the fuzzy quotient space \( (V/W, \tilde{\chi}) \), denoted \( V/W \) or \( \chi_{V/W} \) (See [5]).

Theorem 3.1. Consider the following diagram of fuzzy \( R \)-maps of \( R \)-modules:
\[
\begin{array}{ccc}
\chi_M & \xrightarrow{\tilde{f}} & \eta_N \\
\downarrow \tilde{a} & & \downarrow \tilde{b} \\
\mu_V & \xrightarrow{\tilde{f}_1} & \nu_P \\
\tilde{a} & & \tilde{b} \\
\tilde{v} & & \tilde{v}
\end{array}
\]
where the rows are exact and the squares are commutative. If \( \tilde{b} \) is quasi-monic, then the map
\[
\tilde{f}^* : \tilde{\chi}_{Coim(\tilde{a})} \rightarrow \tilde{\eta}_{Coim(\tilde{b})}
\]
defined by
\[
\tilde{f}^*(m + Ker\tilde{a}) = f(m) + Ker\tilde{b}
\]
is a fuzzy linear map.

Proof. First we prove that \( \tilde{f}^* \) is well-defined. Let \( m + Ker\tilde{a} = m' + Ker\tilde{a} \). Then
\[
m = m' + x(x \in Ker\tilde{a}).
\]
Thus
\[
f(m) + Ker\tilde{b} = f(m') + f(x) + Ker\tilde{b}.
\]
Since
\[
\nu\beta f(x) = \nu f_1\alpha(x) \geq \mu\alpha(x) = 1,
\]
we have
\[
\nu\beta f(x) = 1.
\]
So
\[
f(x) \in Ker\tilde{b}.
\]
Thus
\[
f(m) + Ker\tilde{b} = f(m') + Ker\tilde{b}.
\]
Thus $\tilde{f}^*$ is well-defined. On the other hand,
\[
\tilde{f}^*[m + \text{Ker} \tilde{a}] = \tilde{f}^* (m + m' + \text{Ker} \tilde{a}) = f(m + m') + \text{Ker} \tilde{\beta} = f(m) + f(m') + \text{Ker} \tilde{\beta} = (f(m) + \text{Ker} \tilde{\beta}) + (f(m') + \text{Ker} \tilde{\beta}) = \tilde{f}^*(m + \text{Ker} \tilde{a}) + \tilde{f}^*(m' + \text{Ker} \tilde{a}).
\]

For any $r \in R$,
\[
\tilde{f}^*(r(m + \text{Ker} \tilde{a})) = \tilde{f}^*(rm + \text{Ker} \tilde{a}) = f(rm) + \text{Ker} \tilde{\beta} = rf(m) + \text{Ker} \tilde{\beta} = r[f(m) + \text{Ker} \tilde{\beta}] = \tilde{r}\tilde{f}^*(m + \text{Ker} \tilde{a}).
\]

Thus $\tilde{f}^*$ is a linear map. To prove that $\tilde{f}^*$ is a fuzzy linear map, we must show that
\[
\tilde{\eta}\tilde{f}^*(m + \text{Ker} \tilde{a}) \geq \tilde{\chi}(m + \text{Ker} \tilde{a})
\]
for all $m + \text{Ker} \tilde{a} \in \tilde{\chi}_{\text{Coim} m}$. Let $m + \text{Ker} \tilde{a}$ be any element of $\tilde{\chi}_{\text{Coim} m}$. If $m \in \text{Ker} \tilde{a}$, then
\[
\tilde{\eta}\tilde{f}^*(m + \text{Ker} \tilde{a}) = \tilde{\eta}\tilde{f}^*(0) = \tilde{\eta}(0) = 1 \geq \tilde{\chi}(m + \text{Ker} \tilde{a}).
\]
Let $m \notin \text{Ker} \tilde{a}$. If $f(m) \in \text{Ker} \tilde{\beta}$, then
\[
\tilde{\eta}\tilde{f}^*(m + \text{Ker} \tilde{a}) = \tilde{\eta}(f(m) + \text{Ker} \tilde{\beta}) = 1 \geq \tilde{\chi}(m + \text{Ker} \tilde{a}).
\]
If $f(m) \notin \text{Ker} \tilde{\beta}$, then
\[
\tilde{\eta}\tilde{f}^*(m + \text{Ker} \tilde{a}) = \tilde{\eta}(f(m) + \text{Ker} \tilde{\beta}) = \inf\{\eta(y) \mid y \in f(m) + \text{Ker} \tilde{\beta}\} \geq \inf\{\min\{\eta f(m), \eta(y)\} \mid y \in \text{Ker} \tilde{\beta}\} \geq \inf\{\min\{\chi(m), \eta(y)\} \mid y \in \text{Ker} \tilde{\beta}\} = \tilde{\chi}(m + \text{Ker} \tilde{a}).
\]

\[\tilde{\eta}\tilde{f}^* : \tilde{\eta}_{\text{Coim} m} \to \tilde{\rho}_{\text{Coim} m}\]

where the rows are exact and the squares are commutative. If $\tilde{\beta}$ and $\tilde{\gamma}$ are quasi-monic, then the following sequence
\[
\tilde{\chi}_{\text{Coim} m} \xrightarrow{f^*} \tilde{\eta}_{\text{Coim} m} \xrightarrow{\tilde{\rho}} \tilde{\rho}_{\text{Coim} m}
\]
is exact.

\textbf{Proof.} By Theorem 3.1, we can construct the fuzzy linear map $\tilde{g}^* : \tilde{\eta}_{\text{Coim} m} \to \tilde{\rho}_{\text{Coim} m}$ defined by
\[
\tilde{g}^*(n + \text{Ker} \tilde{\beta}) = g(n) + \text{Ker} \tilde{\gamma}
\]
To prove that the given sequence in Theorem 3.2 is exact, we must show that
\[
\tilde{\eta}_{\text{Im} f^*} = \tilde{\eta}_{\text{Ker} \tilde{g}^*}.
\]
Let $n + \text{Ker} \tilde{\beta}$ be any element of $\text{Im} f^*$. Then there exists $m + \text{Ker} \tilde{a} \in \text{Coim} m$ such that
\[
\tilde{f}^*(m + \text{Ker} \tilde{a}) = n + \text{Ker} \tilde{\beta}.
\]
Thus $f(m) + \text{Ker} \tilde{\beta} = n + \text{Ker} \tilde{\beta}$ and thus $n = f(m) + b(b \in \text{Ker} \tilde{\beta})$. Since $\tilde{\beta}$ is quasi-monic, $b \in \{x \mid \eta(x) = 1\}$. So $\eta(b) = 1$ and so $pg(b) = 1$. If $n \in \text{Ker} \tilde{\beta}$, then
\[
\tilde{\rho}g^*(n + \text{Ker} \tilde{\beta}) = \tilde{\rho}g^*(0) = \tilde{\rho}(0) = 1.
\]
Let $n \notin \text{Ker} \tilde{\beta}$. If $g(n) \in \text{Ker} \tilde{\gamma}$, then
\[
\tilde{\rho}g^*(n + \text{Ker} \tilde{\beta}) = \tilde{\rho}(g(n) + \text{Ker} \tilde{\gamma}) = \tilde{\rho}(0) = 1.
\]
Let $g(n) \notin \text{Ker} \tilde{\gamma}$. If $z \in \text{Ker} \tilde{\gamma}$, then $\tilde{\epsilon}(z) = 1 = \rho(z)$, since $\tilde{\gamma}$ is quasi-monic. Also since $\text{Im} f = \text{Ker} \tilde{g}$ and $f(m) \in \text{Im} f, f(m) \in \text{Ker} \tilde{g}$. So $pg(m) = 1$. Thus
\[
\tilde{\rho}g^*(n + \text{Ker} \tilde{\beta}) = \tilde{\rho}(g(n) + \text{Ker} \tilde{\gamma}) = \inf\{\rho(x) \mid x \in g(n) + \text{Ker} \tilde{\gamma}\} = \inf\{\rho(g(n) + z) \mid z \in \text{Ker} \tilde{\gamma}\} \geq \inf\{\min\{\rho g(n), \rho(z)\} \mid z \in \text{Ker} \tilde{\gamma}\} = \rho g(n) = 1,
\]
since $\rho g(n) = \rho g(f(m) + b) = \rho(g(f(m)) + g(b)) \geq \min\{\rho g(f(m)), \rho(g(b))\} = 1$. In any case, we have $\tilde{\rho}g^* (n + \text{Ker} \tilde{\beta}) = 1$. Thus $n + \text{Ker} \tilde{\beta} \in \text{Ker} \tilde{g}^*$. Hence
\[
\tilde{\eta}_{\text{Im} f^*} \subseteq \tilde{\eta}_{\text{Ker} \tilde{g}^*}.
\]

\textbf{Theorem 3.2.} Consider the following diagram of fuzzy R-maps of R-modules:
\[\begin{array}{cccc}
\chi_M & \xrightarrow{\tilde{f}} & \eta_N & \xrightarrow{\tilde{\gamma}} & \rho_W \\
\mu_V & \xrightarrow{\tilde{f}_1} & \nu_P & \xrightarrow{\tilde{\beta}} & \tilde{\epsilon}_S
\end{array}\]
Conversely, if \( n + \text{Ker}\tilde{\beta} \in \tilde{n}_{\text{Ker}\gamma} \), then
\[
\tilde{\rho}\tilde{g}^*(n + \text{Ker}\tilde{\beta}) = \tilde{\rho}(g(n) + \text{Ker}\gamma) = 1
\]
Thus \( g(n) \in \text{Ker}\gamma \). For if \( g(n) \notin \text{Ker}\gamma \), then
\[
1 = \tilde{\rho}\tilde{g}^*(n + \text{Ker}\tilde{\beta}) = \tilde{\rho}(g(n) + \text{Ker}\gamma) = \inf\{\rho(g(n) + z) \mid z \in \text{Ker}\gamma\}.
\]
Thus \( \rho g(n) = 1 \). Since \( \mathcal{E}\gamma g(n) \geq \rho g(n) = 1 \), \( \mathcal{E}\gamma g(n) = 1 \) and thus \( g(n) \in \text{Ker}\tilde{\gamma} \). This is a contradiction. Thus \( g(n) \in \text{Ker}\tilde{\gamma} \). Since \( \tilde{\gamma} \) is quasi-monic, we have \( \rho g(n) = 1 \).
Thus \( n \in \text{Ker}\tilde{g} \). Since \( \text{Ker}\tilde{g} = \text{Im}\tilde{f} \), there exists \( m \in \chi_M \), such that \( f(m) = n \). Thus
\[
f^*(m + \text{Ker}\tilde{\alpha}) = f(m) + \text{Ker}\tilde{\beta} = n + \text{Ker}\tilde{\beta}.
\]
Thus \( n + \text{Ker}\tilde{\beta} \in \tilde{n}_{\text{Im}\tilde{f}} \). Hence
\[
\tilde{n}_{\text{Ker}\gamma^*} \subseteq \tilde{n}_{\text{Im}\tilde{f}}^*.
\]
This completes the proof. \( \square \)

**Theorem 3.3.** Consider the following diagram of fuzzy \( R \)-maps of \( R \)-modules:
\[
\begin{align*}
\chi_M & \xrightarrow{\tilde{f}} \eta_N & \tilde{\gamma} & \rho_{\mathcal{E}} \\
\mu_{\mathcal{V}} & \xrightarrow{\tilde{f}_1} & \nu_{\mathcal{P}} & \tilde{g} & \mathcal{E}\mathcal{S}
\end{align*}
\]
where the rows are exact and the squares are commutative. If \( \tilde{f} \) is epic, then the map
\[
\tilde{f}_1^* : \tilde{\mu}_{\text{Coker}(\tilde{f})} \rightarrow \tilde{\nu}_{\text{Coker}(\tilde{g})}
\]
defined by
\[
\tilde{f}_1^*(v + \text{Im}\tilde{\alpha}) = f_1(v) + \text{Im}\tilde{\beta}
\]
is a fuzzy linear map.

**Proof.** First we prove that \( \tilde{f}_1^* \) is well-defined. Let \( v + \text{Im}\tilde{\alpha} = v' + \text{Im}\tilde{\alpha} \). Then
\[
v = v' + x(\in \text{Im}\tilde{\alpha}).
\]
Thus
\[
f_1(v) + \text{Im}\tilde{\beta} = f_1(v') + x + \text{Im}\tilde{\beta} = f_1(v') + \text{Im}\tilde{\beta}.
\]
Since \( x \in \text{Im}\tilde{\alpha} \), there exists \( m \in M \) such that \( \tilde{\alpha}(m) = x \). Thus
\[
f_1(x) = f_1\alpha(m) = \beta f(m).
\]
So \( f_1(x) \in \text{Im}\tilde{\beta} \).

Thus \( f_1(v) + \text{Im}\tilde{\beta} = f_1(v') + \text{Im}\tilde{\beta} \). Thus \( \tilde{f}_1^* \) is well-defined. On the other hand,
\[
\tilde{f}_1^*[v + \text{Im}\tilde{\alpha}] = [v' + \text{Im}\tilde{\alpha}]
\]
\[
= \tilde{f}_1^*[v + v' + \text{Im}\tilde{\beta}]
\]
\[
= f_1(v + v') + \text{Im}\tilde{\beta}
\]
\[
= f_1(v) + f_1(v') + \text{Im}\tilde{\beta}
\]
\[
= (f_1(v) + \text{Im}\tilde{\beta}) + (f_1(v') + \text{Im}\tilde{\beta})
\]
\[
= f_1^*(v + \text{Im}\tilde{\alpha}) + f_1^*(v' + \text{Im}\tilde{\alpha}).
\]
For any \( r \in \mathcal{R} \),
\[
\tilde{f}_1^*(r(v + \text{Im}\tilde{\alpha})) = \tilde{f}_1^*(rv + \text{Im}\tilde{\beta})
\]
\[
= f_1(rv) + \text{Im}\tilde{\beta}
\]
\[
= rf_1(v) + \text{Im}\tilde{\beta}
\]
\[
= [f_1(v) + \text{Im}\tilde{\beta}]
\]
\[
= f_1^*(v + \text{Im}\tilde{\alpha}).
\]
Thus \( \tilde{f}_1^* \) is a linear map. To prove that \( \tilde{f}_1^* \) is a fuzzy linear map, we must show that
\[
\tilde{\nu}\tilde{f}_1^*(v + \text{Im}\tilde{\alpha}) \geq \tilde{\mu}(v + \text{Im}\tilde{\alpha})
\]
for all \( v + \text{Im}\tilde{\alpha} \in \text{Coker}(\tilde{f}) \). Let \( v + \text{Im}\tilde{\alpha} \) be any element of \( \text{Coker}(\tilde{f}) \). Then
\[
\tilde{\nu}\tilde{f}_1^*(v + \text{Im}\tilde{\alpha}) = \tilde{\nu}(f_1(v) + \text{Im}\tilde{\beta}).
\]
If \( f_1(v) \in \text{Im}\tilde{\beta} \), then
\[
\tilde{\nu}\tilde{f}_1^*(v + \text{Im}\tilde{\alpha}) = \tilde{\nu}(f_1(v) + \text{Im}\tilde{\beta}) = \tilde{\nu}(0) = 1 \geq \tilde{\mu}(v + \text{Im}\tilde{\alpha}).
\]
If \( f_1(v) \notin \text{Im}\tilde{\beta} \), then
\[
\tilde{\nu}\tilde{f}_1^*(v + \text{Im}\tilde{\alpha}) = \tilde{\nu}(f_1(v) + \text{Im}\tilde{\beta})
\]
\[
= \tilde{\nu}(0) = 1 \geq \tilde{\mu}(v + \text{Im}\tilde{\alpha}).
\]
Hence \( \tilde{f}_1^* \) is a fuzzy linear map. \( \square \)
Theorem 3.4. Consider the following diagram of fuzzy $R$-maps of $R$-modules:

$$
\begin{array}{cccc}
\chi_M & \longrightarrow & \eta_N & \longrightarrow \rho_W \\
\downarrow \bar{\alpha} & & \downarrow \bar{\beta} & \downarrow \gamma \\
\mu_V & \longrightarrow & \nu_P & \longrightarrow \xi_S
\end{array}
$$

where the rows are exact and the squares are commutative.

If $f$ and $g$ are epic, then the following sequence

$$
\bar{\mu}_{\text{Coker}(\bar{\alpha})} \xrightarrow{\bar{f}_1^*} \bar{\nu}_{\text{Coker}(\bar{\beta})} \xrightarrow{\bar{g}_1} \bar{\xi}_{\text{Coker}(\bar{\gamma})}
$$

is exact.

**Proof.** By Theorem 3.3, we can define the fuzzy linear map

$$
g_1^* : \bar{\nu}_{\text{Coker}(\bar{\beta})} \longrightarrow \bar{\xi}_{\text{Coker}(\bar{\gamma})}
$$

by

$$
g_1^*(n + \text{Im} \bar{\beta}) = g_1(n) + \text{Im} \bar{\gamma}
$$

To prove that the given sequence in Theorem 3.4 is exact, we must show that

$$\bar{\nu}_{\text{Im} \bar{f}_1^*} = \bar{\nu}_{\text{Ker} \bar{g}_1^*}.
$$

Let $p + \text{Im} \bar{\beta} \in \bar{\nu}_{\text{Ker} \bar{g}_1^*}$, then

$$
\bar{\xi} \bar{g}_1^*(p + \text{Im} \bar{\beta}) = \bar{\xi}(g_1(p) + \text{Im} \bar{\gamma}) = 1
$$

If $g_1(p) \in \text{Im} \bar{\gamma}$, then there exists $w \in W$ such that $\gamma(w) = g_1(p)$. Since $\bar{g}$ is epic, there exists $n \in N$ such that $g(n) = w$. Thus

$$
g_1(p) = \gamma g(n) = g_1 \beta(n).
$$

Thus $p - \beta(n) \in \text{Ker} \bar{g}_1$. Since $\text{Ker} \bar{g}_1 = \text{Im} \bar{f}_1$, there exists $v \in V$ such that $f_1(v) = p - \beta(n)$. Thus

$$
\bar{f}_1^*(v + \text{Im} \bar{\alpha}) = f_1(v) + \text{Im} \bar{\beta} = p - \beta(n) + \text{Im} \bar{\beta} = p + \text{Im} \bar{\beta}.
$$

Hence

$$p + \text{Im} \bar{\beta} \in \bar{\nu}_{\text{Im} \bar{f}_1^*}.
$$

If $g_1(p) \notin \text{Im} \bar{\gamma}$, then

$$
1 = \bar{\xi} \bar{g}_1^*(p + \text{Im} \bar{\beta}) = \bar{\xi}(g_1(p) + \text{Im} \bar{\gamma}) = \inf \{ \bar{\xi}(g_1(p) + z) \mid z \in \text{Im} \bar{\gamma} \} \leq \bar{\xi} g_1(p).
$$

Thus $\bar{\xi} g_1(p) = 1$. So $p \in \text{Ker} \bar{g}_1$. Since $\text{Ker} \bar{g}_1 = \text{Im} \bar{f}_1$, there exists $v \in V$ such that $f_1(v) = p$. Thus

$$
\bar{f}_1^*(v + \text{Im} \bar{\alpha}) = f_1(v) + \text{Im} \bar{\beta} = p + \text{Im} \bar{\beta}.
$$

Hence

$$p + \text{Im} \bar{\beta} \in \bar{\nu}_{\text{Im} \bar{f}_1^*}.
$$

and hence

$$\bar{\nu}_{\text{Im} \bar{f}_1^*} \subseteq \bar{\nu}_{\text{Ker} \bar{g}_1^*}.
$$

Conversely let $p + \text{Im} \bar{\beta}$ be any element of $\bar{\nu}_{\text{Im} \bar{f}_1^*}$. Then there exists $v + \text{Im} \bar{\alpha} \in \bar{\mu}_{\text{Coker}(\bar{\alpha})}$ such that

$$
\bar{f}_1^*(v + \text{Im} \bar{\alpha}) = f_1(v) + \text{Im} \bar{\beta} = p + \text{Im} \bar{\beta}.
$$

Thus $f_1(v) - p \in \text{Im} \bar{\beta}$. So there exists $n \in N$ such that $\beta(n) = f_1(v) - p$. If $g_1(p) \in \text{Im} \bar{\gamma}$, then

$$
\bar{\xi} \bar{g}_1^*(p + \text{Im} \bar{\beta}) = \bar{\xi}(g_1(p) + \text{Im} \bar{\gamma}) = \bar{\xi}(0) = 1.
$$

Let $g_1(p) \notin \text{Im} \bar{\gamma}$. Since $f_1(v) \in \text{Im} \bar{f}_1 = \text{Ker} \bar{g}_1$, $\bar{\xi} g_1 f_1(v) = 1$. Also Since $f_1 \alpha(m) \in \text{Im} \bar{f}_1 = \text{Ker} \bar{g}_1$, $\bar{\xi} g_1 f_1 \alpha(m) = 1$ for all $m \in M$. Thus we have

$$
\bar{\xi} \bar{g}_1^*(p + \text{Im} \bar{\beta}) = \bar{\xi}(g_1^*(f_1(v) - \beta(n) + \text{Im} \bar{\beta})) = \bar{\xi}(g_1^*(f_1(v) + \text{Im} \bar{\gamma})) = \inf \{ \bar{\xi}(g_1 f_1(v) + z) \mid z \in \text{Im} \bar{\gamma} \} \geq \inf \{ \min \{ \bar{\xi}(g_1 f_1(v) + \gamma(w)) \mid w \in W \} \} = \inf \{ \bar{\xi}(\gamma f(m)) \mid m \in M \} = \inf \{ \bar{\xi}(g_1 f_1 \alpha(m)) \mid m \in M \} = 1.
$$

Thus

$$\bar{\xi} \bar{g}_1^*(p + \text{Im} \bar{\beta}) = 1
$$

and hence

$$p + \text{Im} \bar{\beta} \in \bar{\nu}_{\text{Ker} \bar{g}_1^*}.
$$

Hence

$$\bar{\nu}_{\text{Im} \bar{f}_1^*} \subseteq \bar{\nu}_{\text{Ker} \bar{g}_1^*}.
$$

This completes the proof. \(\square\)

**Lemma 3.5.** Consider the following diagram of fuzzy $R$-maps of $R$-modules:

$$
\begin{array}{cccc}
\chi_M & \longrightarrow & \eta_N & \longrightarrow \rho_W \\
\downarrow \bar{\alpha} & & \downarrow \bar{\beta} & \downarrow \gamma \\
\mu_V & \longrightarrow & \nu_P & \longrightarrow \xi_S
\end{array}
$$

A Note on the Fuzzy Linear Maps over the Fuzzy Quotient Spaces

117
where the rows are exact and the squares are commutative. If \( \tilde{g} \) is epic, then for each \( w \in \rho_{\text{Ker}\tilde{g}} \), there exist \( n \in N \) and \( v \in V \) such that

\[
g(n) = w \quad \text{and} \quad f_1(v) = \beta(n).
\]

**Proof.** Let \( w \) be any element of \( \rho_{\text{Ker}\tilde{g}} \). Then \( \mathcal{E}\gamma(w) = 1 \). Since \( \tilde{g} \) is epic, there exists \( n \in N \) such that \( g(n) = w \). Thus

\[
\mathcal{E}\gamma_1\beta(n) = \mathcal{E}\gamma g(n) = \mathcal{E}\gamma(w) = 1.
\]

Thus

\[
\beta(n) \in \text{Ker} \tilde{g}_1.
\]

Since \( \text{Ker}\tilde{g}_1 = \text{Im}\tilde{f}_1 \), we have

\[
\beta(n) \in \text{Im}\tilde{f}_1.
\]

Hence there exists \( v \in V \) such that \( f_1(v) = \beta(n) \).

**Theorem 3.7.** Consider the following diagram of fuzzy R-maps of R-modules:

\[
\begin{array}{ccc}
\chi_M & \xrightarrow{\tilde{f}} & \eta_N \\
\downarrow\alpha & & \downarrow\beta \\
\mu_V & \xrightarrow{\tilde{f}_1} & \nu_P \\
\end{array}
\]

where the rows are exact and the squares are commutative. If \( \tilde{f} \) and \( \tilde{g} \) are epic, and if \( f_1 \) is monic, then the map

\[
h : \rho_{\text{Ker}\tilde{g}} \rightarrow \tilde{\mu}_{\text{Coker}\tilde{g}}
\]

defined by

\[
h(w) = v + \text{Im}\alpha,
\]

where \( g(n) = w, f_1(v) = \beta(n) \) are given by Lemma 3.5, is a zero fuzzy linear map and \( \tilde{h}(w) \) does not depend on the choice of \( n \) and \( v \).

**Proof.** Let \( w \) be any element of \( \rho_{\text{Ker}\tilde{g}} \) and let \( g(n) = w, g(n') = w, \beta(n) = f_1(v) \) and \( \beta(n') = f_1(v') \) (\( n, n' \in N \) and \( v, v' \in V \)). Then \( h(w) = v + \text{Im}\alpha \) and \( h(w) = v' + \text{Im}\alpha \), and also \( g(n - n') = 0 \) and so \( n - n' \in \text{Ker}\tilde{g} \). Since \( \text{Ker}\tilde{g} = \text{Im}\tilde{f} \), \( n - n' \in \text{Im}\tilde{f} \). Thus there exists \( m \in M \) such that \( f(m) = n - n' \). Thus

\[
f_1\alpha(m) = \beta f(m) = \beta(n - n') = f_1(v - v').
\]

Since \( f_1 \) is monic, \( \alpha(m) = v - v' \). Thus

\[
v + \text{Im}\alpha = v' + \alpha(m) + \text{Im}\alpha = v' + \text{Im}\alpha.
\]

Let \( w \) be any element of \( \rho_{\text{Ker}\tilde{g}} \) and let \( \tilde{h}(w) = v + \text{Im}\alpha \). Then there exist \( n \in N \) and \( v \in V \) such that \( g(n) = w \) and \( \beta(n) = f_1(v) \). Since \( f \) is epic, there exists \( m \in M \) such that \( f(m) = n \). Thus

\[
f_1\alpha(m) = \beta f(m) = \beta(n).
\]

Thus

\[
h(w) = \alpha(m) + \text{Im}\alpha = \text{Im}\alpha = 0
\]

by definition of \( \tilde{h} \). Hence \( \tilde{h} \) is a zero map.

**Theorem 3.8** ([7]). Consider the following diagram of fuzzy R-maps of R-modules:

\[
\begin{array}{ccc}
\chi_M & \xrightarrow{\tilde{f}} & \eta_N \\
\downarrow\alpha & & \downarrow\beta \\
\mu_V & \xrightarrow{\tilde{f}_1} & \nu_P \\
\end{array}
\]

\[
\begin{array}{ccc}
\rho_W & \xrightarrow{\tilde{g}} & \tilde{\mu}_S \\
\downarrow\gamma & & \downarrow\delta \\
\tilde{E}_S & & \tilde{E}_S
\end{array}
\]

118
Thus \( x \). Conversely let \( \tilde{\eta} \).

Since \( \tilde{\eta} \) is epic, there exists \( y \in \eta_N \) such that \( g(y) = x \).

Since \( \tilde{\eta} \) is epic, then the sequence of fuzzy \( R \)-maps,

\[
\chi_{Ker\tilde{\alpha}} \to \eta_{Ker\tilde{\beta}} \to \rho_{Ker\gamma}
\]

is exact at \( \eta_{Ker\tilde{\beta}} \).

From Theorem 3.4 and Theorem 3.8, we have the following theorem.

**Theorem 3.9.** Consider the following diagram of fuzzy \( R \)-modules:

\[
\begin{array}{ccc}
\chi_M & \xrightarrow{f} & \eta_N \\
\downarrow\tilde{a} \quad & \downarrow\tilde{b} \quad & \downarrow\tilde{c} \\
\mu_V & \xrightarrow{\nu} & \nu_p \\
\end{array}
\]

where the rows are exact and the squares are commutative.

If \( f_1 \) is monic and if \( g_1 \) is quasi-monic, then if \( \tilde{f} \) and \( \tilde{g} \) are epic, then the sequence of fuzzy \( R \)-maps,

\[
\begin{array}{ccc}
\chi_{Ker\tilde{a}} & \xrightarrow{\tilde{f}} & \eta_{Ker\tilde{b}} \\
\uparrow\mu_V & \xrightarrow{\nu} & \nu_p \\
\tilde{h} \quad & \xrightarrow{\tilde{g}} & \rho_{Ker\gamma} \\
\end{array}
\]

is exact.

**Proof.** By Theorem 3.4 and Theorem 3.8, we must show that

1. \( \rho_{\tilde{Im}\tilde{g}} = \rho_{Ker\tilde{h}} \) and
2. \( \tilde{\mu}_{\tilde{Im}\tilde{h}} = \tilde{\mu}_{Ker\tilde{f}} \).

(1). Let \( x \) be any element of \( \rho_{\tilde{Im}\tilde{g}} \). Then there exists \( y \in \eta_{Ker\tilde{\beta}} \) such that

\[
g_1(y) = \tilde{g}(y) = x.
\]

Since \( \tilde{h} \) is a zero map,

\[
\tilde{g}(y) = \tilde{g}(y) = x.
\]

Since \( \tilde{h} \) is a zero map, \( \tilde{g}(y) = \tilde{g}(y) = x \).

Thus \( x \in \rho_{Ker\tilde{h}} \). Hence

\[
\rho_{\tilde{Im}\tilde{g}} \subseteq \rho_{Ker\tilde{h}}.
\]

Conversely let \( x \) be any element of \( \rho_{Ker\tilde{h}} \). Then

\[
\tilde{h}(x) = 1.
\]

Since \( \tilde{g} \) is epic, there exists \( y \in \eta_N \) such that \( g(y) = x \).

Since \( \tilde{g} \) is epic, then there exists \( y \in \eta_N \) such that \( g(y) = x \).

Thus

\[
\beta(y) = \nu g_1(y) = 1,
\]

we have

Since \( g_1 \) is quasi-monic, \( \nu_{Ker g_1} = \nu_{P'} \), where \( P' = \{ x \mid \nu(x) = 1 \} \).

Thus

\[
\nu(\beta) = 1.
\]

So

\[
y \in \eta_{Ker\tilde{\beta}}
\]

and \( g(y) = x \). Thus

\[
x \in \rho_{\tilde{Im}\tilde{g}}.
\]

Hence

\[
\rho_{Ker\tilde{h}} \subseteq \rho_{\tilde{Im}\tilde{g}}.
\]

This completes the proof of (1).

(2). To prove (2), we must show that

\[
\tilde{\mu}_{\tilde{Im}\tilde{f}} = \{ 0 \},
\]

since \( \tilde{h} \) is a zero map. Let \( v + Im\tilde{a} \) be any element of \( \rho_{\tilde{Im}\tilde{f}} \). Then

\[
\nu f_1^*(v + Im\tilde{a}) = \tilde{v}(f_1(v) + Im\tilde{b}) = 1.
\]

If \( f_1(v) \in Im\tilde{b} \), then there exists \( n \in N \) such that \( \beta(n) = f_1(v) \). Since \( \tilde{f} \) is epic, there exists \( m \in M \) such that \( f(m) = n \). Thus

\[
f_1(v) = \beta(n) = \beta f(m) = f_1\alpha(m).
\]

Since \( f_1 \) is monic, \( v = \alpha(m) \). Thus \( v + Im\tilde{a} = \alpha(m) + Im\tilde{a} = Im\tilde{a} = 0 \).

If \( f_1(v) \notin Im\tilde{b} \), then

\[
1 = \tilde{v} f_1^*(v + Im\tilde{a}) = \inf\{ \nu f_1(v) + z \mid z \in Im\tilde{b} \}
\]

Thus

\[
\nu f_1(v) = 1.
\]

Thus

\[
v \in Ker\tilde{f}_1.
\]

Since \( \tilde{f}_1 \) is monic, we have \( v = 0 \). Hence

\[
v + Im\tilde{a} = Im\tilde{a} = 0.
\]

In any case, we have

\[
v + Im\tilde{a} = Im\tilde{a} = 0.
\]

Hence

\[
\tilde{\mu}_{\tilde{Im}\tilde{f}} = \{ 0 \}.
\]

This completes the proof of (2).  \( \square \)
References


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