Exact Controllability for Abstract Fuzzy Differential Equations in Credibility Space

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Abstract
With reasonable control selections on the space of functions, various application models can take the shape of a well-defined control system on mathematics. In the credibility space, controllability management of fuzzy differential equation is as much important issue as stability. This paper addresses exact controllability for abstract fuzzy differential equations in the credibility space in the perspective of Liu process. This is an extension of the controllability results of Park et al. (Controllability for the semilinear fuzzy integro-differential equations with nonlocal conditions) to fuzzy differential equations driven by Liu process.

Keywords: Abstract fuzzy differential equations, Credibility space, Liu process, Fuzzy process

1. Introduction

The concept of fuzzy set was initiated by Zadeh via membership function in 1965. Fuzzy differential equations are a field of increasing interest, due to their applicability to the analysis of phenomena where imprecision in inherent. Kwon et al. [1-4] and Lee et al. [5] have studied the existence and uniqueness for solutions of fuzzy equations.

The theory of controlled processes is one of the most recent mathematical concepts to enable very important applications in modern engineering. However, actual systems subject to control do not admit a strictly deterministic analysis in view of various random factors that influence their behavior. The theory of controlled processes takes the random nature of a systems behavior into account. Many researchers have studied controlled processes. With regard to fuzzy systems, Kwon and Park [6] proved controllability for the impulsive semilinear fuzzy differential equation in n-dimension fuzzy vector space. Park et al. [7] studied the controllability of semilinear fuzzy integrodifferential equations with nonlocal conditions. Park et al. [8] demonstrated the controllability of impulse semilinear fuzzy integrodifferential equations, while Phu and Dung [9] studied the stability and controllability of fuzzy control set differential equations. Lee et al. [10] examined the controllability of a nonlinear fuzzy control system with nonlocal initial conditions in n-dimensional fuzzy vector space $E_N^m$.

In terms of the controllability of stochastic systems, P. Balasubramaniam [11] studied quasilinear stochastic evolution equations in Hilbert spaces, and the controllability of stochastic control systems with time-variant coefficients was proved by Yuhu [12]. Arapostathis et al. [13] studied the controllability properties of stochastic differential systems that are characterized by a linear controlled diffusion perturbed by a smooth, bounded, uniformly Lipschitz nonlinearity.
Stochastic differential equations driven by Brownian motion have been studied for a long time, and are a mature branch of modern mathematics. A new kind of fuzzy differential equation driven by a Liu process was defined as follows by Liu [14]

\[ dX_t = f(X_t, t)dt + g(X_t, t) dC_t \]

where \( C_t \) is a standard Liu process, and \( f, g \) are some given functions. The solution of such equation is a fuzzy process. You [15] discussed the solutions of some special fuzzy differential equations, and derived an existence and uniqueness theorem for homogeneous fuzzy differential equations. Chen [16] for fuzzy differential equations, and derived an existence and uniqueness theorem for fuzzy differential equations. Liu [17] studied an analytic method for solving uncertain differential equations. In this paper, we extend the result of Liu [17] to fuzzy differential equations driven by a Liu process within a controlled system.

We study the exact controllability of abstract fuzzy differential equations in a credibility space:

\[
\begin{cases}
\begin{align*}
\frac{dx(t, \theta)}{dt} &= Ax(t, \theta) dt + f(t, x(t, \theta)) dC_t + Bu(t) dt, \quad t \in [0, T], \\
x(0) &= x_0 \in E_N,
\end{align*}
\end{cases}
\]

where the state \( x(t, \theta) \) takes values in \( X(\subset E_N) \) and another bounded space \( Y(\subset E_N) \). We use the following notation: \( E_N \) is the set of all upper semi-continuously convex fuzzy numbers on \( R, (\Theta, P, Cr) \) is the credibility space, \( A \) is a fuzzy coefficient, the state function \( x : [0, T] \times (\Theta, P, Cr) \rightarrow X \) is a fuzzy process, \( f : [0, T] \times X \rightarrow X \) is a fuzzy function, \( u : [0, T] \times (\Theta, P, Cr) \rightarrow Y \) is a control function, \( B \) is a linear bounded operator from \( Y \) to \( X \), \( C_t \) is a standard Liu process and \( x_0 \in E_N \) is an initial value.

In Section 2, we discuss some basic concepts related to fuzzy sets and Liu processes.

In Section 3, we show the existence of solutions to the free fuzzy differential equation (1)(\( u \equiv 0 \)).

Finally, in Section 4, we prove the exact controllability of the fuzzy differential Eq. (1).

2. Preliminaries

In this section, we give some basic definitions, terminology, notation, and Lemmas that are relevant to our investigation and are needed in latter sections. All undefined concepts and notions used here are standard.

We consider \( E_N \) to be the space of one-dimensional fuzzy numbers \( u : R \rightarrow [0, 1] \), satisfying the following properties:

1. \( u \) is normal, i.e., there exists an \( u_0 \in R \) such that \( u(t) = 1 \);
2. \( u \) is fuzzy convex, i.e., \( u(\lambda t + (1-\lambda)s) \geq \min\{u(t), u(s)\} \)

for any \( t, s \in R, 0 \leq \lambda \leq 1 \);
3. \( u(t) \) is upper semi-continuous, i.e., \( u(t_0) \geq \lim_{t \to t_0} u(t_k) \)

for any \( t_k \in R \) (\( k = 0, 1, 2, \cdots \)), \( t_k \to t_0 \);
4. \( |u| \) is compact.

The level sets of \( u, [u]_a = \{ t \in R : u(t) \geq a \}, a \in (0, 1] \), and \( [u]_0 \) are nonempty compact convex sets in \( R \).

Definition 2.1 [19] We define a complete metric \( D_L \) on \( E_N \) by

\[ D_L(u, v) = \sup_{0 \leq \alpha \leq 1} \sup_{0 \leq \alpha \leq 1} \left| [u]_\alpha - [v]_\alpha \right| \]

for any \( u, v \in E_N \). Let \( D_L(u, v) = D_L(u, v) \) for each \( w \in E_N \), and \( [u]_\alpha = [u_\alpha, u_\alpha] \), for every \( \alpha \in [0, 1] \) where \( u_\alpha, u_\alpha \in R \) with \( u_\alpha \leq u_\alpha \).

Definition 2.2 [20] For any \( u, v \in C([0, T], E_N) \), the metric \( H_1(u, v) \) on \( C([0, T], E_N) \) is defined by

\[
H_1(u, v) = \sup_{0 \leq t \leq T} D_L(u(t), v(t)).
\]

Let \( \Theta \) be a nonempty set, and let \( \mathcal{P} \) be the power set of \( \Theta \). Each element in \( \mathcal{P} \) is called an event. To present an axiomatic definition of credibility, it is necessary to assign a number \( Cr\{A\} \) to each event \( A \) indicating the credibility that \( A \) will occur. To ensure that the number \( Cr\{A\} \) has certain mathematical properties that we intuitively expect, we accept the following four axioms:

1. (Normality) \( Cr\{\emptyset\} = 1 \).
2. (Monotonicity) \( Cr\{A\} \leq Cr\{B\} \) whenever \( A \subset B \).
3. (Self-Duality) \( Cr\{A\} + Cr\{A^c\} = 1 \) for any event \( A \).
4. (Maximality) \( Cr\{\bigcup_i A_i\} = \sup_i Cr\{A_i\} \) for any events \( \{A_i\} \) with \( \sup_i Cr\{A_i\} < 0.5 \).

Definition 2.5 [21] Let \( \Theta \) be a nonempty set, \( \mathcal{P} \) be the power set of \( \Theta \), and \( Cr \) be a credibility measure. Then the triplet \( (\Theta, \mathcal{P}, Cr) \) is called a credibility space.

Definition 2.6 [14] A fuzzy variable is a function from a credibility space \( (\Theta, \mathcal{P}, Cr) \) to the set of real numbers.
**Definition 2.7** [14] Let \( T \) be an index set and \( (\Theta, \mathcal{P}, C_r) \) be a credibility space. A fuzzy process is a function from \( T \times (\Theta, \mathcal{P}, C_r) \) to the set of real numbers.

That is, a fuzzy process \( x(t, \theta) \) is a function of two variables such that the function \( x(t^*, \theta) \) is a fuzzy variable for each \( t^* \). For each fixed \( \theta^* \), the function \( x(t, \theta^*) \) is called a sample path of the fuzzy process. A fuzzy process \( x(t, \theta) \) is said to be sample-continuous if the sample ping is continuous for almost all \( \theta \). Instead of writing \( x(t, \theta) \), we sometimes we use the symbol \( x_t \).

**Definition 2.8** Let \( (\Theta, \mathcal{P}, C_r) \) be a credibility space. For fuzzy random variable \( x_{t_1}, \ldots, x_{t_n} \) in credibility space, for each \( \alpha \in [0, 1] \), the \( \alpha \)-level set \( [x_{t_1}]^\alpha = \{(x_{t_1})^\alpha, (x_{t_2})^\alpha, \ldots, (x_{t_n})^\alpha\} \) is defined by

\[
(x_{t_i})^\alpha = \inf \{a \in R : x_{t_i}(a) \geq \alpha\},
\]

\[
(x_{t_i})^\alpha = \sup \{a \in R : x_{t_i}(a) \geq \alpha\},
\]

where \( (x_{t_1}), (x_{t_2}), \ldots, (x_{t_n}) \in R \) with \( (x_{t_1})^\alpha \leq (x_{t_2})^\alpha \) when \( \alpha \in [0, 1] \).

**Definition 2.9** [22] Let \( \xi \) be a fuzzy variable and \( r \) is real number. Then the expected value of \( \xi \) is defined by

\[
E\xi = \int_0^{+\infty} Cr(\xi \geq r) dr - \int_{-\infty}^0 Cr(\xi \leq r) dr
\]

provided that at least one of the integrals is finite.

**Lemma 2.1** [22] Let \( \xi \) be a fuzzy vector. The expected value operator \( E \) has the following properties:

(i) if \( f \leq g \), then \( E[f(\xi)] \leq E[g(\xi)] \),

(ii) \( E[-f(\xi)] = -E[f(\xi)] \),

(iii) if functions \( f \) and \( g \) are comonotonic, then for any nonnegative real numbers \( a \) and \( b \), we have

\[
E[af(\xi) + bg(\xi)] = aE[f(\xi)] + bE[g(\xi)].
\]

Where \( f(\xi) \) and \( g(\xi) \) are fuzzy variables.

**Definition 2.10** [? A fuzzy process \( C_t \) is said to be a Liu process if

(i) \( C_0 = 0 \),

(ii) \( C_t \) has stationary and independent increments,

(iii) every increment \( C_{t+s} - C_t \) is a normally distributed fuzzy variable with expected value \( et \) and variance \( \sigma^2 t^2 \), whose membership function is

\[
\mu(x) = 2\left(1 + \exp\left(\frac{\pi|x - et|}{\sqrt{6\sigma^2}}\right)\right)^{-1}, \quad x \in R.
\]

The parameters \( e \) and \( \sigma \) are called the drift and diffusion coefficients, respectively. Liu process is said to be standard if \( e = 0 \) and \( \sigma = 1 \).

**Definition 2.11** [23] Let \( x_t \) be a fuzzy process and let \( C_t \) be a standard Liu process. For any partition of closed interval \( [c, d] \) with \( c = t_0 < \cdots < t_n = d \), the mesh is written as

\[
\Delta = \max_{1 \leq i \leq n} (t_i - t_i-1).
\]

Then the fuzzy integral of \( x_t \) with respect to \( C_t \) is

\[
\int_c^d x_t dC_t = \lim_{\Delta \to 0} \sum_{i=1}^n x(t_{i-1})(C_{t_i} - C_{t_{i-1}})
\]

provided that the limit exists almost surely and is a fuzzy variable.

**Lemma 2.2** [23] Let \( C_t \) be a standard Liu process. For any given \( \theta \) with \( Cr(\theta) > 0 \), the path \( C_t \) is Lipschitz continuous, that is, the following inequality holds

\[
|C_{t_1} - C_{t_2}| < K(\theta)|t_1 - t_2|,
\]

where \( K(\theta) \) is a fuzzy variable called the Lipschitz constant of a Liu process with

\[
K(\theta) = \left\{ \begin{array}{ll}
\sup_{0 \leq s < t} \frac{|C_{t} - C_{s}|}{t - s}, & Cr(\theta) > 0, \\
\infty, & \text{otherwise,}
\end{array} \right.
\]

and \( E[K^p] < \infty, \forall p > 0 \).

**Lemma 2.3** [23] Let \( C_t \) be a standard Liu process, and let \( h(t; c) \) be a continuously differentiable function. Define \( x_t = h(t; C_t) \). Then we have the following chain rule

\[
dx_t = \frac{\partial h(t; C_t)}{\partial t} dt + \frac{\partial h(t; C_t)}{\partial C} dC_t.
\]

**Lemma 2.4** [23] Let \( f(t) \) be continuous fuzzy process, the following inequality of fuzzy integral holds

\[
\left|\int_c^d f(t) dC_t\right| \leq K \int_c^d |f(t)| dt,
\]

where \( K = K(\theta) \) is defined in Lemma 2.2.

### 3. Existence of Solutions for Abstract Fuzzy Differential Equations

In this section, by Definition 2.7, instead of longer notation \( x(t, \theta) \), sometimes we use the symbol \( x_t \). We consider the
existence and uniqueness of solutions for the fuzzy differential Eq. (1) \( u \equiv 0 \).

\[
\begin{align*}
  dx_t &= Ax_t dt + f(t, x_t) dC_t, \quad t \in [0, T], \\
  x(0) &= x_0 \in E_N,
\end{align*}
\]

(2)

where the state \( x_t \) takes values in \( X(\subset E_N) \). \( E_N \) is the set of all upper semi-continuously convex fuzzy numbers on \( R \), \( (\Theta, \mathcal{P}, Cr) \) is credibility space, \( A \) is fuzzy coefficient, the state function \( x : [0, T] \times (\Theta, \mathcal{P}, C_r) \rightarrow X \) is a fuzzy process, \( f : [0, T] \times X \rightarrow X \) is regular fuzzy function, \( C_t \) is a standard Liu process, \( x_0 \in E_N \) is initial value.

**Lemma 3.1** [19] Let \( g \) be a function of two variables and let \( a_t \) be an integrable uncertain process. Then a given uncertain differential equation by

\[
dX_t = a_t X_t dt + g(t, X_t) dC_t
\]

has a solution

\[
X_t = Y_t^{-1} Z_t
\]

where

\[
Y_t = \exp \left( - \int_0^t a_s ds \right)
\]

and \( Z_t \) is the solution of uncertain differential equation

\[
dZ_t = Y_t g(t, Y_t^{-1} Z_t) dC_t
\]

with initial value \( Z_0 = X_0 \).

Using Lemma 3.1, we show that, for fuzzy coefficient \( A \), the Eq. (2) have a solution.

**Lemma 3.2** For \( x(0) = x_0 \), if \( x_t \) is solution of the Eq. (2), then the solution \( x_t \) is given by

\[
x_t = S(t) x_0 + \int_0^t S(t-s) f(s, x_s) dC_s, \quad t \in [0, T],
\]

where \( S(t) \) is continuous with \( S(0) = I \), \( |S(t)| \leq c, c > 0 \), for all \( t \in [0, T] \).

**Proof** For fuzzy coefficient \( A \), the following define inverse of \( S(t) \)

\[
S^{-1}(t) = e^{-At}.
\]

Then it follows that

\[
dS^{-1}(t) = -A e^{-At} dt = -AS^{-1}(t) dt.
\]

Applying the integration by parts to the above equation provides

\[
d(S^{-1}(t)x_t)
= d(S^{-1}(t)x_t) + S^{-1}(t) dx_t
= -A S^{-1}(t) dt x_t + S^{-1}(t) A x_t dt + S^{-1}(t) f(t, x_t) dC_t.
\]

That is,

\[
d(S^{-1}(t)x_t) = S^{-1}(t) f(t, x_t) dC_t.
\]

Defining \( z_t = S^{-1}(t)x_t \), we obtain \( x_t = S(t)z_t \) and

\[
dz_t = S^{-1}(t) f(t, S(t)z_t) dC_t.
\]

Furthermore, we get in virtue of \( S(0) = I \), and \( z_0 = x_0 \),

\[
z_t = x_0 + \int_0^t S^{-1}(s) f(s, S(s)z_s) dC_s.
\]

Therefore the Eq. (2) has the following solution

\[
x_t = S(t) x_0 + \int_0^t S(t-s) f(s, x_s) dC_s, \quad t \in [0, T],
\]

where we write \( S(t-s) \) instead of \( S(t)S^{-1}(s) \).

Assume the following statements:

(H1) For \( x_t, y_t \in C([0, T] \times (\Theta, \mathcal{P}, C_r), X) \), \( t \in [0, T] \), there exists positive number \( m \) such that

\[
d_L([f(t, x_t)]^\alpha, [f(t, y_t)]^\alpha) \leq m d_L([x_t]^\alpha, [y_t]^\alpha)
\]

and \( f(0, x(0)) \equiv 0 \).

(H2) \( 2cmKT \leq 1 \).

By Lemma 3.2, we know that the Eq. (2) have a solution \( x_t \). Thus in Theorem 3.1, we show that uniqueness of solution for Eq. (2).

**Theorem 3.1** For every \( x_0 \in E_N \), if hypotheses (H1), (H2) are hold, then the eEq. (2) have a unique solution \( x_t \in C([0, T] \times (\Theta, \mathcal{P}, C_r), X) \).

**Proof** For each \( \xi_t \in C([0, T] \times (\Theta, \mathcal{P}, C_r), X), \quad t \in [0, T] \) define

\[
\Psi \xi_t = S(t) x_0 + \int_0^t S(t-s) f(s, \xi_s) dC_s.
\]
Thus, one can show that $Ψ : [0, T] \times (Θ, \mathcal{P}, C_r) \to C([0, T] \times (Θ, \mathcal{P}, C_r), X)$ is continuous, then

$$Ψ : C([0, T] \times (Θ, \mathcal{P}, C_r), X) \to C([0, T] \times (Θ, \mathcal{P}, C_r), X).$$

It is also obvious that a fixed point of $Ψ$ is solution for the Eq. (2). For $ξ_t, \eta_t \in C([0, T] \times (Θ, \mathcal{P}, C_r), X)$, by Lemma 2.4 and hypothesis (H1), we have

$$d_L([Ψξ_t]^α, [Ψη_t]^α) = d_L\left(\left[\int_0^t S(t-s)f(s, ξ_s)dC_s\right]^α, \left[\int_0^t S(t-s)f(s, η_s)dC_s\right]^α\right) \leq cmK\int_0^t d_L(ξ_s, η_s)ds.$$ 

Therefore, we obtain that

$$D_L(Ψξ_t, Ψη_t) = \sup_{α \in (0, 1]} d_L([Ψξ_t]^α, [Ψη_t]^α) \leq cmK\int_0^t \sup_{α \in (0, 1]} d_L(ξ_s, η_s)ds = cmK\int_0^t D_L(ξ_s, η_s)ds.$$ 

Hence, for a.s. $θ \in Θ$, by Lemma 2.1,

$$E\left(H_1(Ψξ, Ψη)\right) = E\left(\sup_{t \in [0, T]} D_L(Ψξ_t, Ψη_t)\right) \leq E\left(cmK\sup_{t \in [0, T]} \int_0^t D_L(ξ_s, η_s)ds\right) \leq cmKTE\left(H_1(ξ, η)\right).$$

By hypotheses (H2), $Ψ$ is a contraction mapping. By the Banach fixed point theorem, Eq. (2) have a unique fixed point $x_t \in C([0, T] \times (Θ, \mathcal{P}, C_r), X)$.

4. **Exact Controllability for Abstract Fuzzy Differential Equations**

In this section, we study exact controllability for abstract fuzzy differential Eq. (1).

We consider solution for the Eq. (1), for each $u$ in $Y(⊂ E_N)$. Then substitute this expression into the Eq. (3) yields α-level
of \( x_T \).

\[
[x_T]^\alpha = \left[ S(T)x_0 + \int_0^T S(T-s)f(s,x_s)\,dC_s + \int_0^T S(T-s)Bu_s\,ds \right]^{\alpha} \\
= \left[ S^\alpha_\ell(T)(x_0) + \int_0^T S^\alpha_\ell(T-s)f_\ell^\alpha(s,(x_s)_{\ell}^\alpha)\,dC_s + \int_0^T S^\alpha_\ell(T-s)\tilde{B}(\tilde{G}^\alpha_\ell)^{-1}(x_1)^\alpha_\ell - S^\alpha_\ell(T)(x_0)_{\ell}^\alpha \right] \\
+ \int_0^T S^\alpha_\ell(T-s)\tilde{B}(\tilde{G}^\alpha_\ell)^{-1}(x_1)^\alpha_\ell - S^\alpha_\ell(T)(x_0)_{\ell}^\alpha \right] \,ds, \\
S^\alpha_\ell(T)(x_0)_{\ell}^\alpha + \int_0^T S^\alpha_\ell(T-s)f_\ell^\alpha(s,(x_s)_{\ell}^\alpha)\,dC_s + \int_0^T S^\alpha_\ell(T-s)\tilde{B}(\tilde{G}^\alpha_\ell)^{-1}(x_1)^\alpha_\ell - S^\alpha_\ell(T)(x_0)_{\ell}^\alpha \right] \,ds \\
\nonumber = \left[ (x_1)^\alpha_\ell, (x_1)^\alpha_\ell \right] = [x^1]^\alpha.
\]

Hence this control \( u_t \) satisfies \( x_T = x^1 \), a.s. \( \theta \).

We now set

\[
\Phi x_t = S(t)x_0 + \int_0^t S(t-s)f(s,x_s)\,dC_s + \int_0^t S(t-s)\tilde{B}\tilde{G}^{-1}\{x^1 - S(T)x_0 \}
\]

where the fuzzy mappings \( \tilde{G}^{-1} \) satisfies above statements.

(H3) Assume that the linear system of Eq. (1) \( f \equiv 0 \) is controllable.

**Theorem 4.1** If Lemma 2.4 and the hypotheses (H1), (H2) and (H3) are satisfied, then the Eq. (1) are controllable on \([0,T]\).

**Proof** We can easily check that \( \Phi \) is continuous from \( C([0,T] \times (\Theta, \mathcal{P}, C\tau), X) \) to itself. By Lemma 2.4 and hypotheses (H1) and (H2), for any given \( \theta \) with \( C\tau \{ \theta \} > 0 \), \( x_t, y_t \in C([0,T] \times (\Theta, \mathcal{P}, C\tau), X) \), we have

\[
d_L \left( \left[ \int_0^t S(t-s)f(s,x_s)\,dC_s \right]^\alpha \right) \\
\leq d_L \left( \left[ \int_0^t S(t-s)f(s,x_s)\,dC_s \right]^\alpha \right) \\
+ d_L \left( \left[ \int_0^t S(t-s)\tilde{B}\tilde{G}^{-1}(x_1-x_0)\,dC_s \right]^\alpha \right) \\
\nonumber \leq cmK \int_0^t d_L \left( [x]^\alpha, [y]^\alpha \right) \,ds \\
\nonumber + d_L \left( \left[ \tilde{G}^{-1} \int_0^T S(T-x_0)\,dC_s \right]^\alpha \right) \\
\nonumber \leq cmK \int_0^t d_L \left( [x]^\alpha, [y]^\alpha \right) \,ds \\
\nonumber + cK \int_0^t d_L \left( [f(s,x_s)]^\alpha, [f(s,y_s)]^\alpha \right) \,ds
\]
Therefore by Lemma 2.1,

\[
E(\Phi x, \Phi y) = E\left(\sup_{\alpha \in [0,T]} D_L(\Phi x_t, \Phi y_t)\right) = E\left(\sup_{\alpha \in [0,T]} \sup_{0 < \alpha \leq 1} d_L(\Phi x_t^\alpha, [\Phi y_t^\alpha])\right)
\]

\[
\leq E\left(\sup_{\alpha \in [0,T]} 2cmK \int_0^t d_L([x_t^\alpha], [y_t^\alpha])ds\right)
\]

\[
\leq E\left(\sup_{\alpha \in [0,T]} 2cmK T E H_1(x, y)\right).
\]

We take sufficiently small \(T\), \((2cmKT) < 1\). Hence \(\Phi\) is a contraction mapping. We now apply the Banach fixed point theorem to show that the Eq. (3) have a unique fixed point.

Consequently, the Eq. (1) are controllable on \([0, T]\).

**Example 4.1** We consider the following abstract fuzzy differential equations in credibility space

\[
\left\{\begin{array}{l}
\frac{dx_t}{dt} = Ax_t + Mf(t, x_t)dt + Bu_tdt, \\
x(0) = x_0 \in E_N,
\end{array}\right.
\]

where the state \(x_t\) takes values in \(X(\subset E_N)\) and another bounded space \(Y(\subset E_N)\). \(E_N\) is the set of all upper semi-continuously convex fuzzy numbers on \(C_r(\Theta, \mathcal{P}, C_r)\) is credibility space, \(A\) is a fuzzy coefficient, the state function \(x : [0, T] \times (\Theta, \mathcal{P}, C_r) \rightarrow X\) is a fuzzy process, \(f : [0, T] \times X \rightarrow X\) is a regular fuzzy function, \(u : [0, T] \times (\Theta, \mathcal{P}, C_r) \rightarrow Y\) is a control function, \(B\) is a linear bounded operator from \(Y\) to \(X\). \(C_t\) is a standard Liu process, \(x_0 \in E_N\) is an initial value.

Let \(f(t, x_t) = 2tx_t, S^{-1}(t) = e^{-2t}, \) defining \(z_t = S^{-1}(t)x_t,\) then the balance equations become

\[
\left\{\begin{array}{l}
x_t = S(t)x_0 + \int_0^t S(t-s)2tx_tds, \\
x(0) = x_0 \in E_N.
\end{array}\right.
\]

Therefore Lemma 3.2 is satisfy.

The \(\alpha\)-level set of fuzzy number \(\tilde{2}\) is \([2]^\alpha = [\alpha + 1, 3 - \alpha]\) for all \(\alpha \in [0, 1]\). Then \(\alpha\)-level sets of \(f(t, x_t)\) is \([f(t, x_t)]^\alpha = t[(\alpha + 1)(x_t)^\alpha, (3 - \alpha)(x_t)^\alpha].\) Further, we have

\[
d_L\left([f(t, x_t)]^\alpha, [f(t, y_t)]^\alpha\right) = d_L\left(t[(\alpha + 1)(x_t)^\alpha, (3 - \alpha)(x_t)^\alpha], \right.
\]

\[
\left. t[(\alpha + 1)(y_t)^\alpha, (3 - \alpha)(y_t)^\alpha]\right)
\]

\[
= t \max\{[(\alpha + 1)(x_t)^\alpha - (y_t)^\alpha] + (3 - \alpha)(y_t)^\alpha\}
\]

\[
\leq 3T \max\{[(x_t)^\alpha - (y_t)^\alpha] + (3 - \alpha)(y_t)^\alpha\}
\]

\[
= md_L([x_t]^\alpha, [y_t]^\alpha),
\]

where \(m = 3T\) satisfies the inequality in hypothesis (H1), (H2). Then all the conditions stated in Theorem 3.1 are satisfied.

Let an initial value \(x_0 = 0\). Target set is \(x^1 = \tilde{2}\). The \(\alpha\)-level set of fuzzy number \(\tilde{0}\) is \([0] = [\alpha - 1, 1 - \alpha], \) \(\alpha \in (0, 1]\). We introduce the \(\alpha\)-level set of \(u_t\) of Eq. (4).

\[
[u_t]^\alpha = [(u_t)^\alpha, (u_t)^\alpha]
\]

\[
= \left\{(\tilde{G}_t^\alpha)^{-1}(\alpha + 1) - S_t^\alpha(T)(\alpha - 1) - \int_0^t S_t^\alpha(T-s)s(\alpha + 1)(x_s)^\alpha dC_s\right\},
\]

\[
(\tilde{G}_t^\alpha)^{-1}(3 - \alpha) - S_t^\alpha(T)(3 - \alpha) - \int_0^t S_t^\alpha(T-s)s(3 - \alpha)(x_s)^\alpha dC_s\right\].
\]

Then substituting this expression into the Eq. (5) yields \(\alpha\)-level of \(x_T\).

\[
[x_T]^\alpha
\]

\[
= \left\{\begin{array}{l}
S_t^\alpha(T)(\alpha - 1) + \int_0^T S_t^\alpha(T-s)s(\alpha + 1)(x_s)^\alpha dC_s \\
+ \int_0^T S_t^\alpha(T-s)B(\tilde{G}_t^\alpha)^{-1}(\alpha + 1) dC_s \\
- \int_0^T S_t^\alpha(T-s)s(3 - \alpha)(x_s)^\alpha dC_s
\end{array}\right.
\]

\[
S_t^\alpha(T)(1-\alpha) + \int_0^T S_t^\alpha(T-s)s(3 - \alpha)(x_s)^\alpha dC_s \\
+ \int_0^T S_t^\alpha(T-s)B(\tilde{G}_t^\alpha)^{-1}(3 - \alpha) dC_s.
\]
\[ -S_\alpha^r(T)(1 - \alpha) - \int_0^T S_\alpha^r(T - s)(3 - \alpha)(x_s)^\alpha dC_s \] ds \]

\[ = [(\alpha + 1), (3 - \alpha)] = [2]^\alpha. \]

Then all the conditions stated in Theorem 4.1 are satisfied. So the Eq. (4) are controllable on \([0, T]\).

5. Conclusions

If there is an exact controllability encouraged for the abstract fuzzy differential equations, it can provide a benchmark for an approach to handle controllability about the equations such as fuzzy semilinear integrodifferential equations, fuzzy delay integrodifferential equations on the credibility space. Therefore, the theoretical result of this study can be used to make stochastic extension on the credibility space.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

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References


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