ON FUZZIFYING TOPOLOGICAL SPACES

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Abstract

The main aim of this paper is to study the concept of fuzzifying proximity and fuzzifying uniformity in the framework of fuzzifying topology. Some fundamental properties of them are established.

Key words: Fuzzifying topologies, Fuzzifying proximity and Fuzzifying uniformity

1. Introduction

Since the generalization of the notion of an ordinary set into a fuzzy set by a Zadeh [15], many authors work for construction new branches of fuzzy mathematics. Proximities and uniformities have been studied in details [1, 2, 3, 4, 5, 6]. Samanta et al. [9, 10] give a new definition of fuzzy topology by introducing a concept of openness of fuzzy subsets. He introduced a new definition of fuzzy proximity [11]. In (1991) M. Ying [12] defined a fuzzifying topology on a set X as a mapping from $2^X$, the family of all subsets to the closed unit interval I, satisfying the natural axioms. He also developed the fuzzifying topology in [13, 14] by using fuzzy logic and established.

2. Preliminaries

For the sake of fixing notation, we recall some basic definitions. We shall let X be a nonempty and I be the closed unit interval and we let $I_0=I-0=(0,1)$, $I_1=I-1=[0,1]$. We denote the characteristic function of a subset A of $2^X$ by $1_A$.

Definition 2.1([12]). A function $r : 2^X \times I \rightarrow I$ is called a fuzzifying topology on X if it satisfies the following conditions:

(FP1) $r(X, \emptyset) = r(\emptyset) = 1$.

(FP2) $r(A \cap B, r(A, B) \leq r(A) \wedge r(B)$ for each $A, B \in 2^X$.

(FP3) $r(\bigcup_{i=1}^{n} A_i) \geq \bigwedge_{i=1}^{n} r(A_i)$ for any $(A_i)_{i=1}^{n} \in 2^X$.

The pair $(X, r)$ is called a fuzzifying topological space.

3. On fuzzifying proximities

Definition 3.1. A function $\delta : 2^X \times 2^X \rightarrow I$ is called a fuzzifying proximity on X, if it satisfies the following axioms:

(FP1) $\delta(X, \emptyset) = 0$.

(FP2) $\delta(A, B) = \delta(B, A)$

(FP3) if $\delta(A, B) \neq 1$, then $A \subseteq B^c$.

(FP4) $\delta(A \cup B, C) = \delta(A, C) \vee \delta(B, C)$.

(FP5) For any $A, B \subseteq X$, there exists $C \subseteq X$ such that $\delta(A, B) \geq \bigwedge_{C \subseteq X} \left\{ \delta(A, C) \vee \delta(C^c, B) \right\}$.

The pair $(X, \delta)$ is a fuzzifying proximity space.

Theorem 3.2. Let $\delta$ be a fuzzifying proximity on X. The mapping $C_\delta : 2^X \times 1 \rightarrow 2^X$, is defined by $C_\delta(A, n) = \bigcap \left\{ B \in 2^X \mid \delta(A, B) < 1 - n \right\}$. 

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Then it has the following properties:

1. \( C_\delta(\emptyset, \eta) = \emptyset \)
2. \( A \subseteq C_\delta(A, \eta) \).
3. If \( A_1 \subseteq A_2 \), then \( C_\delta(A_1, \eta) \subseteq C_\delta(A_2, \eta) \).
4. \( C_\delta(A_1 \cup A_2, \eta) = C_\delta(A_1, \eta) \cup C_\delta(A_2, \eta) \).
5. If \( r_1 \leq r_2 \), then \( C_\delta(A, r_1) \subseteq C_\delta(A, r_2) \).
6. \( C_\delta(C_\delta(A, \eta), \eta) = C_\delta(A, \eta) \).

**Proof.** (1), (2), (3) and (5) are easily proved.

(4) From (3), we have
\[
C_\delta(A_1 \cup A_2, \eta) \supseteq C_\delta(A_1, \eta) \cup C_\delta(A_2, \eta).
\]
Conversely, suppose there exist \( A_1, A_2 \subseteq 2^X \) and \( r \in I \) such that
\[
C_\delta(A_1) \subseteq C_\delta(A_1, \eta) \subseteq C_\delta(A_2, \eta),
\]
there exist \( x \in X \) and \( i \in I \) such that
\[
C_\delta(1_{A_i} \setminus 1_{A_i}, \eta)(x) > r \times C_\delta(1_{A_i}, \eta)(x) \sqcup C_\delta(1_{A_i}, \eta)(x).
\]
Since \( C_\delta(1_{A_i}, \eta)(x) < t \), for each \( i \in \{1, 2\} \), there exist \( 1_{B_i} \subseteq 2^X \) with \( \delta(1_{B_i}, 1_{A_i}) < 1 - r \) such that
\[
C_\delta(1_{A_i}, \eta)(x) \leq (1_{B_i} \setminus 1_{A_i})(x) < t.
\]
On the other hand, since
\[
\delta(1_{B_i} \setminus 1_{B_i} \setminus 1_{A_i}) \\
\leq \delta(1_{B_i} \setminus 1_{B_i} \setminus 1_{A_i}) \sqcup \delta(1_{B_i} \setminus 1_{B_i} \setminus 1_{A_i}) \\
< 1 - r
\]
It implies
\[
C_\delta(1_{A_i}, \eta)(x) \leq (1_{B_i} \setminus 1_{A_i})(x).
\]
Thus, \( r_1 \leq r_2 \).

(6) Let \( \delta(B, A) < 1 - r \), then \( B^\delta \supseteq C_\delta(A, \eta) \).

By (FF5) of Definition 3.1, there exists \( \delta \) such that
\[
\delta \supseteq \delta \cdot \delta(B, A).
\]
It follows
\[
\delta(B, A) \supseteq \delta \cdot \delta(B, A).
\]
Since \( \delta \cdot \delta(B, A) < 1 - r \), there exists \( C_\subseteq 2^X \) such that
\[
1 - r > \delta \cdot \delta(B, A) \supseteq \delta(B, C) \sqcup \delta(C,C, A).
\]
Hence
\[
B^\delta \supseteq C_\delta(C, \eta), C \subseteq C_\delta(A, \eta).
\]

Thus
\[
B^\delta \supseteq C_\delta(C, \eta), C \subseteq C_\delta(A, \eta).
\]

**Theorem 3.3.** Let \((X, \delta)\) be a fuzzifying proximity space. Define a map \( r_\delta : 2^X \rightarrow I \) by
\[
r_\delta(A) = \sup \{ r \in I \mid C_\delta(A, \eta) \subseteq A^\delta \}.
\]
Then \( r_\delta \) is a fuzzifying topology on \( X \) induced by \( \delta \).

**Proof.** (O1) Since \( C_\delta(\emptyset, \eta) = \emptyset \) and \( C_\delta(X, \eta) = X \), for all \( r \in I \), \( r_\delta(\emptyset) = r_\delta(X) = 1 \).

(02) Suppose there exist \( A_1, A_2 \subseteq 2^X \) and \( r \in (0, 1) \) such that
\[
r_\delta(A_1 \setminus A_2) < t \times r_\delta(A_1) \sqcup r_\delta(A_2).
\]
Since \( r_\delta(A_1) > t \) and \( r_\delta(A_2) > t \), there exist \( r_1, r_2 > t \) such that
\[
A_i = C_\delta(A, \eta), i = 1, 2.
\]
Put \( r = r_1 \setminus r_2 \). We have
\[
C_\delta(A_1 \setminus A_2, \eta) = (A_1 \cap A_2)^\delta.
\]
Consequently, \( r_\delta(A_1 \setminus A_2) > r > t \). Hence
\[
r_\delta(A_1 \setminus A_2) > r_\delta(A_1) \sqcup r_\delta(A_2).
\]
(03) Suppose there exists a family \( \{ A_j \subseteq 2^X \mid j \in I \} \) and \( t \in (0, 1) \) such that
\[
\bigwedge_{j \in I} r_\delta(A_j) < t \times \bigcup_{j \in I} r_\delta(A_j).
\]
Since \( \bigwedge_{j \in I} r_\delta(A_j) < t \), for each \( j \in I \), there exists \( r_j > t \) such that
\[
A_j = C_\delta(A, \eta), j \in I.
\]
Put \( r = \bigwedge_{j \in I} r_j \). We have
\[
C_\delta\left( \bigcup_{j \in I} A_j, \eta \right) = \left( \bigcup_{j \in I} A_j \right)^\delta.
\]
Consequently, \( r_\delta(\bigcup_{j \in I} A_j) > r > t \). Hence
\[
r_\delta(\bigcup_{j \in I} A_j) > r_\delta(A_j).
\]

**Theorem 3.4.** Let \((X, \delta)\) be a fuzzifying proximity space. A mapping \( \tau_\delta : 2^X \rightarrow I \) defined by
\[
\tau_\delta(A) = \inf_{x \in A}(1 - \delta(A, x))
\]
is a fuzzifying cotopology on \( X \).

**Proof.** (01) Clear.
4. Fuzzifying uniform spaces.

Definition 4.1. A nonzero function $F : 2^{X \times X} \to I$ is called a fuzzifying filter on $X \times X$ if it satisfies the following condition:

SF1) If $A \neq \emptyset$, then $F(A) = 0$.
SF2) $F(A \land B) = F(A) \land F(B)$.
SF3) $F(X \times X) = 1$.

Definition 4.2. A function $U : 2^{X \times X} \to I$ is called a fuzzifying normal fuzzy uniformity on $X$ if it satisfies for $\mu, \nu \in 2^{X \times X}$, the following condition:

(FU1) $U$ is a fuzzifying filtering on $X \times X$.
(FU2) $U(u) \leq U(U^{-1}(y, x))$, where $U^{-1}(x, y) = (y, x)$.
(FU3) $U(u) \leq \sup \{U(\omega) \mid \omega \in u\}$.

The pair $(X, U)$ is said to be a fuzzifying uniform space.

Let $U_1$ and $U_2$ be fuzzifying uniformities on $X$.

We say $U_1$ is finer than $U_2$ or $U_2$ is coarser than $U_1$ iff $U_2(u) \leq U_1(u)$ for all $u \in 2^{X \times X}$.

Theorem 4.3. Let $(X, U)$ be a fuzzifying uniform space. For each $\nu \in I$, let $U^\nu = \{u \in 2^{X \times X} \mid U(u) > \nu\}$.

Then $U^\nu$ is a uniformity on $X$.

Lemma 4.4. Let $(X, U)$ be a fuzzifying uniform space. For each $u, u_1, u_2 \in 2^{X \times X}$ and $A, A_1, A_2 \in 2^X$, we have

(1) $A \subseteq u[A]$, for each $U(u) > 0$.
(2) $u \subseteq u_1 \cup u_2$, for each $U(u) > 0$.
(3) $(u \cdot u)[A] = u[u[A]]$.

(4) $(u_1 \cap u_2)[A_1 \cup A_2] \subseteq u_1[A_1] \cap u_2[A_2]$.
(5) If $f : X \to Y$ is a function, for each $v \in 2^{Y \times Y}$, we have

$f^{-1}(u[f(A)]) = (f \times f^{-1}(v))[A]$.

Theorem 4.5. Let $(X, U)$ be a fuzzifying uniform space. The mapping $C_u : 2^X \times I_1 \to 2^X$, is defined by

$C_u(A, \eta) = \cap \{u(A) \mid U(u) > \eta\}$.

For each $A, A_1, A_2 \in 2^X$ and $\eta_1, \eta_2 \in I_1$, we have the following properties:

(1) $C_u(\emptyset, \eta) = \emptyset$.
(2) $A \subseteq C_u(A, \eta)$.
(3) If $A_1 \subseteq A_2$, then $C_u(A_1, \eta) \subseteq C_u(A_2, \eta)$.
(4) $C_u(A_1 \cup A_2, \eta) = C_u(A_1, \eta) \cup C_u(A_2, \eta)$.
(5) If $\eta_1 \leq \eta_2$, then $C_u(A, \eta_1) \subseteq C_u(A, \eta_2)$.
(6) $C_u(C_u(A, \eta), \eta) = C_u(A, \eta)$.

Proof: (1) Since $u(\emptyset) = \emptyset$, $C_u(\emptyset, \eta) = \emptyset$.
(2) For $U(u) > 0$, by Lemma 4.4(1), $A \subseteq u[A]$ implies $A \subseteq C_u(A, \eta)$.
(3) and (5) are easily proved.
(4) From (3), we have

$C_u(A_1 \cup A_2, \eta) \subseteq C_u(A_1, \eta) \cup C_u(A_2, \eta)$.

Conversely, suppose there exist $A_1, A_2 \in 2^X$ and $\eta \in I$ such that

$C_u(A_1 \cup A_2, \eta) \not\subseteq C_u(A_1, \eta) \cup C_u(A_2, \eta)$.

There exist $x \in X$ and $\nu \in I_1$ such that

$C_u(A_1, \eta)(x) \not\subseteq C_u(A_1, \eta) \cup C_u(A_2, \eta)(x)$.

Since $C_u(A_i, \eta)(x) \leq \eta$ for each $i \in \{1, 2\}$, there exist $u_i \in 2^{X \times X}$ with $U(u_i) > \eta$ such that

$C_u(A_i, \eta)(x) \leq u_i[A_i](x) < \eta$.

On the other hand, since $U(u_1 \cap u_2) > \eta$ and from Lemma 4.4(4),

$(u_1 \cap u_2)[A_1 \cup A_2] \subseteq u_1[A_1] \cap u_2[A_2]$,

we have

$C_u(A_1 \cup A_2, \eta)(x) \leq u_1[A_1](x) \cup u_2[A_2](x)$

$< \eta$.

It is a contradiction.

(6) Suppose there exist $A \in 2^X$ and $\eta \in I_1$ such that

$C_u(C_u(A, \eta), \eta) \not\subseteq C_u(A, \eta)$.

There exist $x \in X$ and $\nu \in I_1$ such that
\[
C_d(\{ \bigvee_{j \in \Gamma} A_j \}^c, r) = \{ \bigvee_{j \in \Gamma} A_j \}^c
\]

Consequently, \( r_s(\bigvee_{j \in \Gamma} A_j) \geq \bigvee_{j \in \Gamma} r_s(A_j) \).

**Definition 4.7.** Let \((X, U)\) and \((Y, V)\) be fuzzifying uniform spaces. A function \(f : (X, U) \to (Y, V)\) is said to be fuzzifying uniform continuous if
\[
V(u) \leq U((f \times f)^{-1}(v)), \forall u \in 2_X^Y.
\]

**Theorem 4.8.** Let \((X, U), (Y, V)\) and \((Z, W)\) be fuzzifying spaces. If \(f : (X, U) \to (Y, V)\) and \(g : (Y, V) \to (Z, W)\) are fuzzifying uniform continuous, then \(g \cdot f : (X, U) \to (Z, W)\) is fuzzifying uniform continuous.

**Proof.** It follows that, for each \(w \in 2_Z^Z\),
\[
W((g \cdot f) \times (g \cdot f)^{-1}(w)) = W((g \times g) \cdot (f \times f)^{-1}(w))
\]
\[
\geq V((f \times f)^{-1}(w))
\]
\[
\geq U(w).
\]

**Theorem 4.9.** Let \((X, U)\) and \((Y, V)\) be fuzzifying uniform spaces. Let \(f : (X, U) \to (Y, V)\) be fuzzifying uniform continuous. Then:
1. \(f(C_d(A, \eta) \subseteq C_d(f(A), \eta)\),
2. \(C_d(f^{-1}(B), \eta) \subseteq f^{-1}(C_d(B, \eta))\),
3. \(f : (X, U) \to (Y, V)\) is fuzzifying continuous.

**Proof.** (1) Suppose there exist \(A_1, A_2 \in 2_X^X\) and \(t \in (0, 1)\) such that
\[
r_s(A_1 \wedge A_2) \leq r_s(A_1) \wedge r_s(A_2).
\]
Since \(r_s(A_1) > t\) and \(r_s(A_2) > t\), there exist \(\eta_1, \eta_2 > t\) such that
\[
A_i = C_d(A_i, \eta_i), i = 1, 2.
\]
Put \(r = \eta_1 \wedge \eta_2\). We have
\[
C_d((A_1 \wedge A_2)^c, \eta) = (A_1 \wedge A_2)^c
\]
Consequently, \(r_s(A_1 \wedge A_2) \geq \eta \wedge t\). Hence
\[
r_s(A_1 \wedge A_2) \geq r_s(A_1) \wedge r_s(A_2).
\]
(03) Suppose here exists a family \(\{A_j \in 2_X : j \in \Gamma\}\) and \(t \in (0, 1)\) such that
\[
r_s(\bigvee_{j \in \Gamma} A_j) \leq \bigwedge_{j \in \Gamma} r_s(A_j).
\]
Since \(\bigwedge_{j \in \Gamma} r_s(A_j) > t\) for each \(j \in \Gamma\), there exists \(r_j > t\) such that
\[
A_j = C_d(A_j, r_j).
\]
Put \(r = \bigwedge_{j \in \Gamma} r_j\). We have
\[
C_d(\{ \bigvee_{j \in \Gamma} A_j \}^c, r) = \{ \bigvee_{j \in \Gamma} A_j \}^c
\]
It implies

\[ uA(f(x)) = (f \times f)^{-1}(\nu)(a_A(x)) \text{ (by Lemma 4.4(5))} \]
\[ = \sup_{x \in X} \{ A(\nu)(x) \times (f \times f)^{-1}(\nu)(x, x) \} \]
\[ \geq Cu(1_A, 1)(x) \]

Thus, \( Cu(1_A, 1)(x) < t \). It is a contradiction.

(2) For each \( B \in 2^X \) and \( r \in I_1 \), put \( A = f^{-1}(B) \).
From (1),
\[ f(Cu(f^{-1}(B), r)) \subseteq Cu(f^{-1}(B), r) \subseteq Cu(B, r) \]

It implies
\[ Cu(f^{-1}(B), r) \subseteq f^{-1}(Cu(f^{-1}(B), r)) \subseteq f^{-1}(Cu(B, r)) \]

(3) From (2), \( Cu(B, r) = B \) implies \( Cu(f^{-1}(B), r) = f^{-1}(B) \). It is easily proved.

References