Fuzzy Separability and Axioms of Countability in Fuzzy Hyperspaces

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ABSTRACT

We study some relations between separability in fuzzy topological spaces and one in fuzzy hyperspaces. And we investigate some properties of axiom of countability in fuzzy hyperspaces.

Key words : fuzzy separability, axioms of countability, compactly Cb, fuzzy hyperspace.

1. Introduction and preliminaries

In 2000, K.Hur, J.R.Moon and J.H.Ryou[5] introduced the concept of a fuzzy hyperspace and studied some of its properties. In this paper, we study some relations between separability in fuzzy topological space and we in fuzzy hyperspaces. And we investigate some properties of axioms of countability in fuzzy hyperspaces.

We will list some concepts and properties needed in the later section.

Let $I_1 = [0, 1]$ and let $I_2 = (0, 1]$. For a set $X$, let $I^X$ be the collection of all the mappings from $X$ into $I$. Then each member of $I^X$, $A: X \rightarrow I$, is called a fuzzy set in $X$(cf. [2,10,13]). In particular, $\emptyset$ and $X$ can be considered as fuzzy sets in $X$ defined by $\phi(x) = 0$ and $X(x) = 1$ for each $x \in X$, respectively.

The concept of a fuzzy point and its properties refer to [8,10,12]. And we will denote the set of all fuzzy points in a set $X$ as $F(X)$.

Definition 1.1[1]. Let $X$ be a nonempty set. Then a fuzzy set $A$ in $X$ is called :

(1) an upper fuzzy set if $A(x) > 1/2$ whenever $A(x) \neq 0$

for each $x \in X$.

(2) a lower fuzzy set if $A(x) < 1/2$ whenever $A(x) \neq 1$

for each $x \in X$.

It is clear that the only fuzzy sets in $X$ which are both upper and lower fuzzy sets are $\emptyset$ and $X$.

Throughout this paper, we use the fuzzy topological space defined by Chang[2]. For a fts $X$, we will denote the family of all F-open sets and F-closed sets in $X$ as $FO(X)$ and $FC(X)$, respectively.

Definition 1.2[3]. A fts $X$ is said to be fuzzy $T_1$(in short, $FT_1$) if for any two fuzzy points $x_i$ and $y_i$ in $X$ : $x \neq y$

(case 1) When $x = y$ and $\lambda \notin \mu$(say), there exists a $U \subseteq FO(X)$ such that $y_i \notin U$ and $x_i \notin \overline{U}$.

(case 2) When $x = y$, there exists a $U \subseteq FO(X)$ such that $y_i \notin U$ and $x_i \notin \overline{U}$.

Definition 1.3[4]. A fuzzy set $A$ in a fts $X$ is said to be fuzzy compact(in short, $F$-compact) in $X$ if for each F-filter base $\beta$ such that for any finite subcollection $(B_i : i = 1, \cdots, n)$ of $\beta$, $\bigcap_{i=1}^{n} B_i \cap A \neq \emptyset$.

Definition 1.4. Let $X$ be a fts.

(1) $\mathcal{Q} \subseteq F(X)$ is said to be dense(resp. Q-dense) in $X$ [11] if for each $\emptyset \neq U \subseteq FO(X)$, there exists $x_i \in \mathcal{Q}$ such that $x_i \in U$(resp. $x_i \notin \overline{U}$).

(2) $A \subseteq X$ is said to be fuzzy dense(in short, F-dense) in $X$[9] if $\overline{A} = X$.

It is clear that the concept of being dense and that of being Q-dense do not imply each other.

Definition 1.5.

(1) Separable(i)(resp. Q-separable) [11] if there exists a sequence $(x_{n, i})_{n \in N}$ of fuzzy points in $X$ such that $(x_{n, i})_{n \in N}$ is dense(resp. Q-dense) in $X$.

(2) Separable(ii)[12] if there exists a sequence $(x_{n, i})_{n \in N}$ of fuzzy points in $X$ such that for each $\emptyset \neq U \subseteq FO(X)$, there exists an $x_{n, i}$ such that $x_{n, i} \in U$. 
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It is clear that $X$ is separable(i) if and only if it is separable(ii).

Although the concept of being dense and that of being $Q$-dense do not imply each other, but we have the following.

**Result 1.A[11, Proposition 5.11].** A fts $X$ is separable (i) if and only if it is $Q$-separable.

In the light of Result 1.A, from now on, we shall make no difference between separable(i)(F-separable) spaces and $Q$-separable spaces. For convenience, they are both called fuzzy separable(F-separable) spaces.

**Definition 1.6.** A fts $X$ is said to:

1. satisfy the first axiom of countability or be $C_1[12]$ if every fuzzy point in $X$ has a countable local base.
2. satisfy the $Q$-first axiom of countability or be $Q-C_1[10]$ if every fuzzy point in $X$ has a countable $Q$-local base.

**Result 1.B[10, Proposition 3.1].** If $X$ is a $C_1$-space, then it is a $Q-C_1$-space.

**Definition 1.7[12].** A fts $(X, T)$ is said to satisfy the second axiom of countability or to be $C_0$ if there exists a countable base $B$ for $T$.

2. Separability and axioms of countability in fuzzy hyperspaces

**Notations 2.1.** For a fts $X$, let $I^X = \{ E \in I^X : E \text{ is a nonempty F-closed set in } X \}$, $L^X = \{ E \in I^X : E \subset A \}$, where $A \in I^X$, $K(X) = \{ E \in I^X : E \text{ is F-compact in } X \}$, $F_\ast(X) = \{ E \in I^X : E \text{ has at most } n \text{ elements } \}$, $F(X) = \{ E \in I^X : E \text{ is finite } \}$.

**Definition 2.2[5].** Let $X$ be a fts, Then the fuzzy Vietoris topology $T_\ast$ on $I^X$ is the generated by the collection of the forms $\langle U_1, \ldots, U_n \rangle$ with $U_i \in FO(X)$ for each $i = 1, \ldots, n$. The pair $(I^X, T_\ast)$ is called a fuzzy hyperspace with fuzzy Vietoris topology (in short, fuzzy hyperspace).

It is clear that $K(X)$, $F_\ast(X)$ and $F(X)$ are subspaces of $I^X$.

**Result 2.A[6, Theorem 3.7].** Let $X$ be a $FT_1$-space and let $U_i, V_i \in I^X$ upper fuzzy sets in $X$ for each $i = 1, \ldots, n$ and each $j = 1, \ldots, m$. Then $\langle U_1, \ldots, U_n \rangle \subset \langle V_1, \ldots, V_m \rangle$ if and only if $\bigcup_{i=1}^n U_i \subset \bigcup_{j=1}^m V_j$ and for each $V_j$ there exists $U_i$ such that $U_i \subset V_j$.

**Result 2.B[5, Theorem 3.7].** Let $X$ be a $FT_1$-space. $F(X)$ is dense in $I^X$.

**Definition 2.3[7].** A fts $X$ is said to be finitely $F$-compact if each finite fuzzy set in $X$ is $F$-compact in $X$.

**Theorem 2.4.** If $X$ is a finitely $F$-compact $FT_1$-space, then $K(X)$ is dense in $I^X$.

**Proof.** Let $E \in I^X$ and let $U = \langle U_1, \ldots, U_n \rangle \cap K(X) \cap K(X)$ such that $E \in U$, where $\langle U_1, \ldots, U_n \rangle$ is a base member for $T$. Then $E \subset \bigcup_{i=1}^n U_i$ and $E \cup U_i$ for each $i = 1, \ldots, n$. Let $x_{i, j} \in U_i$ for each $j = 1, \ldots, n$. Let $F = \{ x_{i, j} \}$. Since $X$ is $FT_1$, $F \in I^X$.

Moreover, $E \subset \bigcup_{i=1}^n U_i$ and $F \cup U_i$ for each $i = 1, \ldots, n$. Then $E \subset \langle U_1, \ldots, U_n \rangle$. Since $F$ is finite, by the hypothesis, $F \in K(X)$. Then $E \subset \langle U_1, \ldots, U_n \rangle \cap (\bigcup_{i=1}^n U_i)$. Thus $E \in cl K(X)$, i.e., $I^X \subset cl K(X)$. So $cl K(X) = I^X$. Hence $K(X)$ is dense in $I^X$.

**Definition 2.5[6].** A fts $X$ is called a $(q, \varepsilon)$-fuzzy topological space (in short, $(q, \varepsilon)$-fts) if for each $U \in FO(X)$, $U$ is a $(q, \varepsilon)$-fuzzy set in $X$, i.e., there exists $\varepsilon F_\ast(X)$ such that $\varepsilon F_\ast U$ if and only if $\varepsilon U = U$.

**Theorem 2.6.** Let $X$ be a $(q, \varepsilon)$-space. Then $X$ is $F$-separable if and only if $I^X$ is separable.

**Proof.** ($\Rightarrow$): Suppose $X$ is $F$-separable. Then, by Result 1.A, $X$ is $Q$-separable. By Definition 1.5, there exists a sequence $D = \{ x_{i, j} \} \subset X$ of fuzzy points in $X$ such that $D$ is $Q$-dense in $X$. Let $B$ be the collection of finite subsets of $D$. Then $B$ is countable. Let $\langle U_1, \ldots, U_n \rangle$ be a base element for $T_\ast$. Since $D$ is $Q$-dense in $X$ and $\varnothing \neq U_i \subset FO(X)$ for each $i = 1, \ldots, m$ there exists $n \in N$ such that $x_{i, n} \in D$ and $x_{i, n} \in U_i$ for each $i = 1, \ldots, m$. Since $X$ is a $(q, \varepsilon)$-fts, $x_{i, n} \in U_i$ for each $i = 1, \ldots, m$. Let $E = \{ x_{i, n} \} \subset X$. Since $X$ is $FT_1$, $E \in I^X$. Moreover, $E \cup U_i$ for each $i = 1, \ldots, n$ and $E \subset \bigcup_{i=1}^n U_i$. Then $E \subset B \cap \langle U_1, \ldots, U_n \rangle$. Thus $B$ is countable dense in $I^X$. Hence
$I_\beta^N$ is separable.

(⇐): Suppose $I_\beta^N$ is separable. Let $\beta = \{A_n\}_{n \in \mathbb{N}}$ be a countable dense subset of $I_\beta^N$. For each $n \in \mathbb{N}$, choose a fuzzy point $a_{n,i} \in A_n$. Let $D = \{a_{n,i}\}_{n \in \mathbb{N}}$. Now let $U \in FO(X)$. Then clearly $\langle U \rangle$ is open in $I_\beta^N$. Since $\beta$ is dense in $I_\beta^N$, $\beta \cap \langle U \rangle \neq \emptyset$. Let $A_n \in \beta \cap \langle U \rangle$.

Then $A_n \in \langle U \rangle$. Thus $A_n \cup U$. Let $a_{n,i} \in A_n$ such that $a_{n,i} \cup U$. Then $a_{n,i} \in D$ such that $a_{n,i} \cup U$. Thus $D$ is Q-dense in $X$. So $D$ is countable Q-dense in $X$. Hence, by Result 1.D, $X$ is F-separable.

**Theorem 2.7.** Let $X$ be a $(q, \infty)$-F$T_1$-space. Then $X$ is F-separable if and only if $F(X)$ is separable.

**Proof.** (⇐): Suppose $X$ is F-separable. Then, by Result 1.A, $X$ is Q-separable. Let $D = \{x_{n,i}\}_{n \in \mathbb{N}}$ be a sequence of fuzzy points in $X$ such that $D$ is Q-dense in $X$. Let $\beta$ be the collection of finite subset of $D$. Then clearly $\beta$ is countable. Let $U = \bigcup_{i=1}^m U_i \in F(X)$, where $\langle U_1, \ldots, U_m \rangle$ is a base member for $T_\beta$. Since $D$ is Q-dense in $X$ and $\emptyset \neq U \in FO(X)$ for each $i = 1, \ldots, m$, there exists $n_i \in \mathbb{N}$ such that $x_{n_i,i} \in D$ and $x_{n_i,i} \cup U_i$ for each $i = 1, \ldots, m$. Let $E = \{x_{n_i,i}, \ldots, x_{n_i,i}\}$. Since $X$ is F$T_1$, $E \in I_\beta^N$. Since $X$ is a $(q, \infty)$-fuzzy, $x_{n_i,i} \in U_i$ for each $i = 1, \ldots, m$. Then $E \subseteq \bigcup_{i=1}^m U_i$ and $E \cup U_i$ for each $i = 1, \ldots, m$. Thus $E \in \langle U_1, \ldots, U_m \rangle$. So $E \in \beta \cap \langle U \rangle$. Hence, $F(X)$ is separable.

(⇒): Suppose $F(X)$ is separable. Let $\beta = \{A_n\}_{n \in \mathbb{N}}$ be a countable dense subset of $F(X)$. For each $n \in \mathbb{N}$, choose $a_{n,i} \in A_n$. Let $D = \{a_{n,i}\}_{n \in \mathbb{N}}$. Now let $U \in FO(X)$. Then $\langle U \rangle \cap F(X)$ is open in $F(X)$. Since $\beta$ is dense in $F(X)$, $\beta \cap (\langle U \rangle \cap F(X)) \neq \emptyset$. Let $A_n \in \beta \cap (\langle U \rangle \cap F(X))$. Then $A_n \in \langle U \rangle$. Thus $A_n \cup U$. Let $a_{n,i} \in A_n$ such that $a_{n,i} \cup U$. Then $a_{n,i} \in D$ such that $a_{n,i} \cup U$. Thus $D$ is Q-dense in $X$. So $D$ is countable Q-dense in $X$. Hence, by Result 1.A, $X$ is F-separable.

**Theorem 2.8.** Let $X$ be a finitely F-compact $(q, \infty)$-F$T_1$-space. Then $X$ is F-separable if and only if $K(X)$ is separable.

**Proof.** (⇐): Suppose $X$ is F-separable. Then, by Result 1.A, $X$ is Q-separable. Let $D = \{x_{n,i}\}_{n \in \mathbb{N}}$ be a sequence of fuzzy points in $X$ such that $D$ is Q-dense in $X$. Let $\beta$ be the collection of finite subsets of $D$. Let $U = \langle U_1, \ldots, U_m \rangle \cap K(X)$, where $\langle U_1, \ldots, U_m \rangle$ is a base member for $T_\beta$. Since $D$ is Q-dense in $X$ and $\emptyset \neq U \in FO(X)$ for each $i = 1, \ldots, m$, there exists $n_i \in \mathbb{N}$ such that $x_{n_i,i} \in D$ and $x_{n_i,i} \cup U_i$ for each $i = 1, \ldots, m$. Let $E = \{x_{n_i,i}, \ldots, x_{n_i,i}\}$. Since $X$ is $T_{\infty}$, $E \in I_\beta^N$. Since $X$ is a $(q, \infty)$-fuzzy, $x_{n_i,i} \in U_i$ for each $i = 1, \ldots, m$. Then $E \subseteq \bigcup_{i=1}^m U_i$ and $E \cup U_i$ for each $i = 1, \ldots, m$. Thus $E \in \langle U_1, \ldots, U_m \rangle$. Since each finite subset in $X$ is F-compact in $X$, $E \in K(X)$. Then $E \in \beta \cap (\langle U_1, \ldots, U_m \rangle \cap K(X)) \neq \emptyset$. Hence $K(X)$ is separable.

(⇒): Suppose $K(X)$ is separable. Let $\beta = \{A_n\}_{n \in \mathbb{N}}$ be a countable dense subset of $K(X)$. For each $n \in \mathbb{N}$, choose $a_{n,i} \in A_n$. Let $D = \{a_{n,i}\}_{n \in \mathbb{N}}$. Let $\emptyset \neq U \in FO(X)$. Then $\langle U \rangle \cap K(X)$ is open in $K(X)$. Since $\beta$ is dense in $K(X)$, $\langle U \rangle \cap K(X) \neq \emptyset$. Let $A_n \in \beta \cap (\langle U \rangle \cap K(X))$. Then $A_n \cup U$. Let $a_{n,i} \cup U$. Thus $D$ is Q-dense in $X$. So $D$ is countable Q-dense in $X$. Hence, by Result 1.A, $X$ is F-separable.

**Theorem 2.9.** Let $X$ be a F$T_1$-space. If $D = \{x_{n,i}\}_{n \in \mathbb{N}}$ is Q-dense (resp. dense) in $X$, then $D^* = D \cup \cdots \cup D (n \text{ factors})$ is Q-dense (resp. dense) in $X^* = X \times \cdots \times X$.

**Proof.** Let $U$ be a nonempty F-open set in $X^*$, where $U = U_1 \times U_2 \times \cdots \times U_n$ and $U_i \in FO(X)$ for each $i = 1, \ldots, n$. Since $U \neq \emptyset$, $U_i \neq \emptyset$ for each $i = 1, \ldots, n$. Since $D$ is Q-dense (resp. dense) in $X$, there exists $m_i \in \mathbb{N}$ such that $x_{m_i,i} \cup D$. Let $A_n \in \beta \cap (U \cap K(X))$. Then $A_n \in \langle U \rangle$. Thus $A_n \cup U$. Let $a_{n,i} \in A_n$ such that $a_{n,i} \cup U$. Then $a_{n,i} \in D$ such that $a_{n,i} \cup U$. Thus $D$ is Q-dense in $X$. So $D$ is countable Q-dense in $X$. Hence, by Result 1.A, $X$ is F-separable.

**Theorem 2.10.** Let $X$ and $Y$ be fts's and let $f : X \to Y$ a F-continuous surjection. If $D = \{x_{n,i}\}_{n \in \mathbb{N}}$ is Q-dense (resp. dense) in $X$, then $f(D)$ is Q-dense (resp. dense) in $Y$.

**Proof.** It is obvious.

**Theorem 2.11.** Let $X$ be a F$T_1$-space. If $F_n(X)$ is separable, then $X$ is F-separable.

**Proof.** Suppose $F_n(X)$ is separable. Let $\beta = \{A_n\}_{n \in \mathbb{N}}$ be a countable dense subset of $F_n(X)$. For each $n \in \mathbb{N}$, choose $a_{n,i} \in A_n$. Let $D = \{a_{n,i}\}_{n \in \mathbb{N}}$. Then $D$ is countable. Let $\emptyset \neq U \in FO(X)$. Then $U = \langle U \rangle \cap F_n(X)$ is open in $F_n(X)$. B is dense in $F_n(X)$, $\beta \cap U \neq \emptyset$. Let $A_n \in \beta \cap U$. Then $A_n \in U$. Thus
$A_\varepsilon q U$. Let $a_{\varepsilon}, q U$ be such that $a_{\varepsilon}, q U$. Then $a_{\varepsilon}, q U \in D$ such that $\lambda_{\varepsilon}, q U$. Thus $D$ is $q$-dense in $X$. So $X$ is $q$-separable. Hence, by Result 1.1, $X$ is $q$-separable.

**Theorem 2.12.** Let $X$ be a finitely $F$-compact space. If $K(X)$ is first countable, then $X$ is $C_1$.

**Proof.** Let $x_\varepsilon \in F_\varepsilon(X)$. Since each finite fuzzy set is $F$-compact in $X$, $(x_\varepsilon) \subseteq K(X)$. Since $K(X)$ is first countable, there exists a countable local base $U$ at $(x_\varepsilon)$. Without loss of generality, let $U = \{U_a\}_{a \in N}$ be a countable local base at $(x_\varepsilon)$. Then clearly $\{U_a\}_{a \in N}$ is a countable local base at $x_\varepsilon$. Hence $X$ is $C_1$.

From Theorem 2.12 and Result 1.1, we can easily obtain the following.

**Corollary 2.12.** Let $X$ be a finitely $F$-compact space. If $K(X)$ is finite, then $X$ is $q$-separable.

**Theorem 2.13.** If $I_0^X$ is first countable, then each one of the subspaces of $I_0^X$ is first countable.

**Theorem 2.14.** Let $X$ be a $(q, \varepsilon)$-FT$_t$ space. If $X$ is $q$-separable, then $F(X)$ is first countable.

**Proof.** Suppose $X$ is $q$-separable and let $x_\varepsilon \in x_\varepsilon \subseteq F(X)$, where $x_{i, \varepsilon} \subseteq F(X)$ for each $i = 1, \ldots, n$. Since $X$ is $q$-separable, for each $i = 1, \ldots, n$, there exists a countable $q$-local base $B_i(x_{i, \varepsilon})$ at $x_{i, \varepsilon}$. Let $B$ be the collection of all open sets of the form $\langle V_1, \ldots, V_n \rangle \subseteq F(X)$, where $V_i \subseteq B(x_{i, \varepsilon})$ for each $i = 1, \ldots, n$. Then clearly $B$ is countable. We show that $B$ is a local base at $x_\varepsilon$. Let $x_\varepsilon \subseteq B$. Then $B = \langle V_1, \ldots, V_n \rangle \subseteq F(X)$, where $V_i \subseteq B_i(x_{i, \varepsilon})$. Since $B_i(x_{i, \varepsilon})$ is a $q$-local base at $x_{i, \varepsilon}$, for each $i = 1, \ldots, n$, $E \subseteq \bigcup_{j=1}^n V_j$. Hence, $E \subseteq B$. Now let $U = \cup_{i=1}^n U_i \subseteq F(X)$ such that $E \subseteq U$. Then $E \subseteq \bigcup_{i=1}^n U_i$ and $E \subseteq F(X)$, for each $i = 1, \ldots, n$. Thus, for each $i = 1, \ldots, n$, there exists $j \subseteq \{1, \ldots, m\}$ such that $x_{i, \varepsilon} \subseteq q U$. Let $U = \bigcap_{i=1}^n (U_i \setminus x_{i, \varepsilon} \setminus q U)$. Then clearly, $x_{i, \varepsilon} \subseteq Q U$ and $x_{i, \varepsilon} \subseteq F(X)$. Since $B_i(x_{i, \varepsilon})$ is a $q$-local base at $x_{i, \varepsilon}$, for each $i = 1, \ldots, n$, there exists $B_i(x_{i, \varepsilon})$ such that $x_{i, \varepsilon} \subseteq B_i(x_{i, \varepsilon}) \subseteq U$. Since $X$ is a $(q, \varepsilon)$-fins, $x_{i, \varepsilon} \subseteq B_i$, $x_{i, \varepsilon} \subseteq U$ and $x_{i, \varepsilon} \subseteq U_i$, for each $i = 1, \ldots, n$. Then $\bigcup_{i=1}^n B_i \subseteq \bigcup_{i=1}^n U_i$, and for each $U_i$, there exists a $B_i$, such that $B_i \subseteq U_i$. Thus, by Result 2.1, $\langle B_1, \ldots, B_n \rangle \subseteq F(X) \subseteq \langle U_1, \ldots, U_n \rangle \subseteq F(X)$. Moreover, $E \subseteq \bigcup_{i=1}^n B_i$. Thus, $E \subseteq \langle B_1, \ldots, B_n \rangle$. So $E$ is a countable local base at $E$. Hence $F(X)$ is first countable.

From Result 1.1 and Theorem 2.14, we can easily obtain the following.

**Corollary 2.14-1.** Let $X$ be a $(q, \varepsilon)$-FT$_t$ space. If $X$ is $C_1$, then $F(X)$ is first countable.

From Theorem 2.14 and Theorem 2.13, we can easily obtain the following.

**Corollary 2.14-2.** Let $X$ be a $(q, \varepsilon)$-FT$_t$ space. If $X$ is $q$-separable, then $F(X)$ is first countable.

From Theorem 2.14 and Result 1.1, we can easily obtain the following.

**Corollary 2.14-3.** Let $X$ be a $(q, \varepsilon)$-FT$_t$ space. If $X$ is $C_1$, then $F(X)$ is first countable.

**Theorem 2.15.** Let $X$ be a FT$_t$ space. If $X$ is $C_2$, then $K(X)$ is second countable.

**Proof.** Suppose $X$ is $C_2$. Let $\beta = \{U_a\}_{a \subseteq X}$ be a countable base for $X$. Let $\beta = \{U_a, \ldots, U_{a}\}$ be a countable base for $K(X)$. Hence $K(X)$ is second countable.

**Definition 2.16.** Let $X$ be a fts and let $A \subseteq I^X$.

1. A subcollection $U$ of $I^X$ is called a proper cover of $A$ if
   (i) $U$ is a cover of $A$.
   (ii) For each $U \subseteq U$, $\cup U = A$.
2. $X$ is said to be compactly $C_6$ if for each $F$-compact set $K$ in $X$ there exists a countable collection $\beta$ of $F$-open sets in $X$ such that if $U \subseteq U$, $\cup U$ is a proper cover of $K$ then there exists a proper cover $\{V_1, \ldots, V_m\} \subseteq \beta$ of $K$ such that $\langle V_1, \ldots, V_m \rangle \subseteq \langle U_1, \ldots, U_m \rangle$.

**Theorem 2.17.** Let $X$ be a fts. Then $X$ is compactly $C_6$ if and only if $K(X)$ is first countable.

**Proof.** $(\Rightarrow)$ Suppose $X$ is compactly $C_6$ and let $K \subseteq K(X)$. Let $\beta$ be the countable collection of $F$-open sets in $X$ satisfying Definition 2.16. Let $B(K)$ be the collection of open sets in $K(X)$ which are constructed
from the finite proper covers of $K$ contained in $\beta$. Then clearly $B(K)$ is countable. We shall show that $B(K)$ is a local base at $K$. Let $K \in \{U_1, \ldots, U_n\} \cap K(X)$, where $\{U_1, \ldots, U_n\}$ is a base member for $T_x$. Then clearly $\{U_1, \ldots, U_n\}$ is a proper cover of $K$. By Definition 2.16, there exists a proper cover $\{V_1, \ldots, V_m\}$ of $K$ such that $\{V_1, \ldots, V_m\} \cap K(X) \subset \{U_1, \ldots, U_n\} \cap K(X)$. Hence $K(X)$ is first countable.

$\implies$ Suppose $K(X)$ is first countable and let $K \in K(X)$. Then there exists a countable local base $B(K)$ at $K$. We may assume that each member $B_i$ of $B(K)$ is of the form $B_i = \{U_1^i, \ldots, U_n^i\} \cap K(X)$. Let $\beta = \{U_i^i: k \geq 1 \text{ and } 1 \leq i \leq n\}$. If $\{U_1, \ldots, U_n\}$ is a proper cover of $K$, then $K \subseteq \bigcup U_i$ and $K \subseteq U_i$ for each $i = 1, \ldots, n$. Thus $K \in \{U_1, \ldots, U_n\} \cap K(X)$. So there exists $B_i \in B(K)$ such that $B_i \subseteq \{U_1, \ldots, U_n\} \cap K(X)$ and $(U_1^i, \ldots, U_n^i)$ is a desired proper cover in $\beta$. Hence $X$ is compactly $C_0$.

### References


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