Topologies induced by two types uniformities

Yong Chan Kim\(^1\) and Young Sun Kim\(^2\)

\(^1\) Department of Mathematics, Kangnung National University, Gangneung, 201-702, Korea
\(^2\) Department of Applied Mathematics, Pai Chai University, Daejeon, 302-735, Korea

Abstract

In strictly two-sided, commutative biquantale, we show that Hutton \((L, \otimes)\)-uniform spaces and \((L, \odot)\)-uniform spaces induce enriched \((L, \odot)\)-topological spaces and enriched \((L, \odot)\)-interior spaces.

**Key words:** Hutton \((L, \otimes)\)-uniform spaces, \((L, \odot)\)-uniform spaces, enriched \((L, \odot)\)-topological spaces, enriched \((L, \odot)\)-interior spaces

1. Introduction

Recently, Gutiérrez García et al.[2] introduced \(L\)-valued Hutton uniformity where a quadruple \((L, \leq, \otimes, *)\) is defined by a GL-monoid \((L, *)\) dominated by \(\otimes\), a cl-quasi-monoid \((L, \leq, \otimes)\). Kubik et al.[12] studied the relationships between the categories of \(I(L)\)-uniform spaces and \(L\)-uniform spaces. Kim et al. [9,10], as a somewhat different aspect in [2], introduced the notion of Hutton \((L, \odot)\)-uniformities as a view point of the approach using uniform operators defined by Rodabaugh [15] and \((L, \odot)\)-uniformities in a sense Lowen [12] and Höhle [3] based on powersets of the form \(L^{X \times X}\). Furthermore, the category \(\text{HUnif}\) of all Hutton \((L, \odot)\)-uniform spaces and \(H\)-uniformly continuous maps and the category \(\text{HUnif}\) of all \((L, \odot)\)-uniform spaces and uniformly continuous maps are isomorphic.

In this paper, we introduce the notion of enriched \((L, \odot)\)-topologies and enriched \((L, \odot)\)-interior spaces. We investigate the relations between them. Moreover, we show that Hutton \((L, \otimes)\)-uniform spaces and \((L, \odot)\)-uniform spaces induce enriched \((L, \odot)\)-topologies and enriched \((L, \odot)\)-interior spaces.

2. Preliminaries

**Definition 2.1**[4-7, 14] A triple \((L, \leq, \odot)\) is called a strictly two-sided, commutative biquantale (stsc-biquantale, for short) iff it satisfies the following properties:

\((L1)\) \(L = (L, \leq, \vee, \wedge, T, \bot)\) is a completely distributive lattice where \(T\) is the universal upper bound and \(\bot\) denotes the universal lower bound;

\((L2)\) \((L, \odot)\) is a commutative semigroup;

\((L3)\) \(a = a \odot T\), for each \(a \in L\);

\((L4)\) \(\odot\) is distributive over arbitrary joins, i.e.

\[\bigvee_{i \in \Gamma} (a_i) \odot b = \bigvee_{i \in \Gamma} (a_i \odot b).\]

\((L5)\) \(\odot\) is distributive over arbitrary meets, i.e.

\[\bigwedge_{i \in \Gamma} (a_i) \odot b = \bigwedge_{i \in \Gamma} (a_i \odot b).\]

In this paper, we always assume that \((L, \leq, \odot^*)\) is a stsc-biquantale with strong negation * where \(a^* = a \rightarrow 0\) unless otherwise specified.

All algebraic operations on \(L\) can be extended point-wise to the set \(L^X\) as follows: for all \(x \in X, f, g \in L^X\) and \(\alpha \in L\),

\((1)\) \(f \leq g\) iff \(f(x) \leq g(x)\);

\((2)\) \((f \odot g)(x) = f(x) \odot g(x)\);

\((3)\) \(1_X(x) = T\), \(\alpha \odot 1_X(x) = \alpha\) and \(1_\phi(x) = \bot\);

\((4)\) \((\alpha \rightarrow \lambda)(x) = \alpha \rightarrow \lambda(x)\) and \((\lambda \rightarrow \alpha)(x) = \lambda(x) \rightarrow \alpha\);

\((5)\) \((\alpha \odot \lambda)(x) = \alpha \odot \lambda(x)\).

**Definition 2.2**[9,10] Let \(\Omega(X)\) be a subset of \((L^X)^{L^X}\) such that

\((O1)\) \(\lambda \leq \phi(\lambda)\), for each \(\lambda \in L^X\),

\((O2)\) \(\phi(\bigvee_{i \in \Gamma} \lambda_i) = \bigvee_{i \in \Gamma} \phi(\lambda_i)\), for \(\{\lambda_i\}_{i \in \Gamma} \subseteq L^X\),

\((O3)\) \(\alpha \odot \phi(\lambda) = \phi(\alpha \odot \lambda)\), for each \(\lambda \in L^X\).
Lemma 2.3 [9,10] For \( \Phi, \Phi_1, \Phi_2, \Phi_3 \in \Omega(X) \), we define, for all \( \lambda \in L^X \),
\[
\phi^{-1}(\lambda) = \{ x \in X \mid \phi(x) \in \lambda \},
\]
\[
\phi \circ \phi_2(\lambda) = \phi_1(\phi_2(\lambda)),
\]
\[
\phi_1 \cap \phi_2(\lambda) = \bigwedge \{ \phi_1(\lambda_1) \cap \phi_2(\lambda_2) \mid \lambda_1 = \lambda_1 \cap \lambda_2 \}.
\]
Then the following properties hold:
1. If \( \phi_1(1_{x_1}) = \rho_2 \) for all \( x \in X \), then \( \phi(\lambda) = \bigvee_{x \in X} \lambda(x) \cap \rho_2 \).
2. If \( \phi_1(1_{x_1}) = \phi_2(1_{x_1}) \) for all \( x \in X \), then \( \phi_1 = \phi_2 \).
3. \( \phi^{-1} \cap \phi_2 \cap \phi_1 \cap \phi_2 \in \Omega(X) \).
4. \( \phi_1 \cap \phi_2 \leq \phi_1 \) and \( \phi_1 \cap \phi_2 \leq \phi_2 \).
5. \( \phi_1 \cap \phi_2 \cap \phi_3 \cap \phi_2 \in \Omega(X) \).
6. \( \phi_1 \cap \phi_2 \cap \phi_3 \cap \phi_2 \leq \phi_1 \cap \phi_1 \cap \phi_2 \cap \phi_2 \).
7. Define \( \phi_1 \in \Omega(X) \) as \( \phi_1(1_{x_1}) = 1 \), for all \( x \in X \).
8. Then \( \phi \in \Omega(X) \) for all \( \Phi \in \Omega(X) \).

Definition 2.4 [9,10] A nonempty subset \( U \) of \( \Omega(X) \) is called a Hutton \( (L, \circ) \)-uniformity on \( X \) if it satisfies the following conditions:
1. If \( \phi \leq \psi \) with \( \phi \in U \) and \( \psi \in \Omega(X) \), then \( \psi \in U \).
2. For each \( \phi \in U \), \( \phi \circ \psi \in \Omega(X) \).
3. For each \( \phi \in U \), there exists \( \psi \in \psi \) such that \( \psi \circ \phi \leq \phi \).
4. For each \( \phi \in U \), there exists \( \phi^{-1} \in U \).

The pair \( (X, U) \) is said to be a Hutton \( (L, \circ) \)-uniform space.

Definition 2.5 [9,10] Let \( E(X \times X) = \{ u \in L^X \times X \mid u(x, z) = 1 \} \) be a subset of \( L^X \times X \). A nonempty subset \( D \) of \( E(X \times X) \) is called an \( (L, \circ) \)-uniformity on \( X \) if it satisfies the following conditions:
1. If \( u \leq v \) with \( u \in D \) and \( v \in E(X \times X) \), then \( v \in D \).
2. For each \( u, v \in D \), \( u \circ v \in D \).
3. For each \( u \in D \), there exists \( v \in D \) such that \( u \circ v \leq u \).
4. For each \( u \in D \), there exists \( u^* \in U \) where \( u^*(x, y) = u(y, x) \).

The pair \( (X, D) \) is said to be an \( (L, \circ) \)-uniform space.

Theorem 2.6 [9,10] We define a mapping \( \Gamma : E(X \times X) \rightarrow \Omega(X) \) as follows:
\[
\Gamma(u)(\lambda)(y) = \bigwedge_{x \in X} \lambda(x) \circ u(x, y).
\]
Then we have the following properties:
1. If \( u \in E(X \times X), \Gamma(u) \in \Omega(X) \).
2. \( \Gamma \) has a right adjoint mapping \( \Lambda : \Omega(X) \rightarrow E(X \times X) \) as follows:
\[
\Lambda(\phi)(x, y) = (\phi(1_{x_1}))(y).
\]
3. \( \Gamma \circ \Lambda = 1_{\Omega(X)} \) and \( \Lambda \circ \Gamma = E(X \times X) \).

Theorem 2.7 [9,10] Let \( u, u_1, u_2 \in E(X \times X) \). Then we have the following properties:
1. If \( u_1 \leq u_2 \) then \( \Gamma(u_1) \leq \Gamma(u_2) \).
2. \( \Gamma(u_1 \circ u_2) \leq \Gamma(u_1) \circ \Gamma(u_2) \).
3. \( \Gamma(1_{\Omega}) = 1_{\Omega} \).
4. \( \Gamma(\mu)^{-1} = \Gamma(u^*) \).
5. \( \Gamma(u)^{-1}(\lambda \rightarrow 1) = \Gamma(u^*)^{-1}(\lambda) \rightarrow 1 \) for all \( \lambda \in L^X \).
6. \( \Gamma(u_1 \circ u_2) = \Gamma(u_1) \circ \Gamma(u_2) \).
7. \( \Gamma(\alpha \circ u) = \alpha \circ \Gamma(u) \).
8. If \( u \) is an \( \circ \)-equivalence relation on \( X \), then \( \Gamma(u)^{-1} = \Gamma(u^*) = \Gamma(u) \).

Theorem 2.8 [9,10] Let \( \phi, \phi_1, \phi_2 \in \Omega(X) \). Then we have the following properties:
1. If \( \phi_1 \leq \phi_2 \), then \( \Lambda(\phi_1) \leq \Lambda(\phi_2) \).
2. \( \Lambda(\phi_1) \circ \Lambda(\phi_2) = \Lambda(\phi_1 \circ \phi_2) \).
3. \( \Lambda(1_{\Omega}) = 1_{\Omega} \).
4. \( \Lambda(\phi)^{\circ} = \Lambda(\phi)^{-1} \).
5. \( \Lambda(\phi_1) \circ \Lambda(\phi_2) = \Lambda(\phi_2 \circ \phi_1) \).
6. \( \Lambda(\alpha \circ \phi) = \alpha \circ \Lambda(\phi) \).
7. \( \Lambda(\phi \circ \phi) = \phi \circ \phi \), \( \Lambda(\phi) = \phi^{-1} \), then \( \Lambda(\phi) \) is an \( \circ \)-equivalence relation.

Theorem 2.9 [9,10] Let \( D \) be an \( (L, \circ) \)-uniform space. We define a subset \( U_D \) of \( \Omega(X) \) as follows:
\[
U_D = \{ \phi \in \Omega(X) \mid \exists u \in D, (u)(\alpha) \leq \phi \}.
\]
Then \( U_D \) is a Hutton \( (L, \circ) \)-uniformity on \( X \).

Theorem 2.10 [9,10] Let \( U \) be a Hutton \( (L, \circ) \)-uniformity on \( X \). We define a subset \( D_U \) of \( E(X \times X) \) as follows:
\[
D_U = \{ u \in E(X \times X) \mid \exists \phi \in U, \Lambda(\phi) \leq u \}.
\]
Then:
1. \( D_U \) is an \( (L, \circ) \)-uniformity on \( X \).
2. \( U \subseteq D \) and \( U \subseteq \Omega \).

Lemma 2.11 [10] Let \( f : X \rightarrow Y \) be a function. We define the inverse and preimage operators
\[
f^\circ : (L^X)^{L^X} \rightarrow (L^Y)^{L^Y}, \quad f^\circ \phi = (L^f)^{-1}(L^y)^{L^X}
\]
such that for each \( \phi \in (L^X)^{L^X} \) and \( \psi \in (L^Y)^{L^Y} \) for all \( \mu, \mu_1, \mu_2 \in L^X, \rho_1, \rho_2 \in L^Y \),
\[
f^\circ(\phi)(\mu) = (f^{-1} \circ \phi \circ f^{-1})(\rho) = f^{-1}(\phi(f^{-1}(\rho))).
\]

253
\[ f^{\circ}(\psi)(\mu) = (f^{-} \circ \psi \circ f^{-})(\mu) = f^{-}(\psi(f^{-}(\mu))). \]

For each \( \psi, \psi_1, \psi_2 \in \Omega(Y) \) and \( \phi_1, \phi_2 \in \Omega(X) \), we have the following properties.

1. The pair \( (f^{\circ}, f^{\circ}) \) is a Galois connection; i.e.,
   \( f^{\circ} \circ f^{\circ} = f^{\circ} \).
2. \( f^{-}(\mu_1 \circ \mu_2) \leq f^{-}(\mu_1) \circ f^{-}(\mu_2) \) with equality if \( f \) is injective and \( f^{-}(\rho_1 \circ \rho_2) = f^{-}(\rho_1) \circ f^{-}(\rho_2) \).
3. \( f^{\circ}(\psi) \in \Omega_X \).
4. \( \psi, \psi_1 \leq \psi_2 \), then \( f^{\circ}(\psi_1) \leq f^{\circ}(\psi_2) \).
5. \( f^{\circ}(\psi_1) \circ f^{\circ}(\psi_2) \leq f^{\circ}(\psi_1 \circ \psi_2) \) with equality if \( f^{-} \) is onto.
6. \( (f^{\circ}(\psi))^{-1} = f^{\circ}(\psi^{-1}) \in \Omega_X \).
7. \( f^{\circ}(\psi_1) \circ f^{\circ}(\psi_2) = f^{\circ}(\psi_1 \circ \psi_2) \) and \( f^{-}(\phi_1) \circ f^{-}(\phi_2) \geq f^{-}(\phi_1 \circ \phi_2) \).
8. \( f^{-}(\psi^{-1})^{-1} = \psi^{-1}(f^{-}(\mu)) \), for all \( \mu \in L^X \).

Def. 2.12 Let \( f : X \to Y \) be a function. For each \( v, v_1, v_3 \in E(Y \times Y) \), \( \phi \in \Omega(Y) \) and \( \lambda \in L^X \), we have:

1. \( f^{\circ}(\Gamma(v)) = f^{-} \circ \Gamma(v) \circ f^{-} = \Gamma((f \times f)^{-}(v)). \)
2. \( (f \times f)^{-}(\Lambda(\phi)) = \Lambda(f^{\circ}(\phi)). \)
3. \( \Gamma((f \times f)^{-}(v^2)) = \Gamma(((f \times f)^{-}(v))^{2}) = \Gamma((f \times f)^{-}(v)) \).
4. \( (f \times f)^{-}(v_1 \circ v_2) = (f \times f)^{-}(v_1) \circ (f \times f)^{-}(v_2) \).
5. \( (f \times f)^{-}(v) \circ (f \times f)^{-}(v) \leq (f \times f)^{-}(v \circ v) \).

Def. 2.13 (9,10) Let \( (X, U_1) \) and \( (Y, U_2) \) be Hutton \((L, \circ)\)-uniform spaces. A function \( f : (X, U_1) \to (Y, U_2) \) is \( H \)-uniformly continuous if \( f^{\circ}(\psi) \in U_1 \), for every \( \psi \in U_2 \).

(2) Let \( (X, D_1) \) and \( (Y, D_2) \) be \((L, \circ)\)-uniform spaces.
A function \( f : (X, D_1) \to (Y, D_2) \) is uniformly continuous if \( f^{-}(v) \in D_1 \), for every \( v \in D_2 \).

3. Topologies induced by two types uniform spaces

Def. 3.1 A subset \( T \) of \( L_X^X \) is called an \((L, \circ)\)-topology on \( X \) if it satisfies the following conditions:

1. \( 1_X, 1_T \in T \).
2. If \( \lambda_1, \lambda_2 \in T \), then \( \lambda_1 \circ \lambda_2 \in T \).
3. If \( \lambda_1 \circ \lambda_2 \in T \), then \( \lambda_1 \circ \lambda_2 \in T \).
4. If \( \lambda_1 \in T \) for all \( \lambda \in T \), then \( (\forall \lambda \in T) \in T \).
The pair \((X, T)\) is called an \((L, \circ)\)-topology.
An \((L, \circ)\)-topological space is called enriched if it satisfies:

1. \( \lambda \in T \), then \( \alpha \circ \lambda \in T \).
2. \( (X, T_1) \) and \((Y, T_2)\) be \((L, \circ)\)-topological spaces.
A function \( f : (X, T_1) \to (Y, T_2) \) is \( L \)-continuous if \( f^{-}(\lambda) \in T_1 \), for every \( \lambda \in T_2 \).

Def. 3.2 A function \( I : L^X \to L^X \) is called an \((L, \circ)\)-interior operator on \( X \) iff \( I \) satisfies the following conditions:

1. \( I(1_X) = 1_X \).
2. \( I(\lambda) \leq \lambda \).
3. \( I(\lambda \circ \mu) \geq I(\lambda) \circ I(\mu) \).
4. \( I(\lambda \circ \mu) \leq I(\lambda) \circ I(\mu) \).
The pair \((X, I)\) is called an \((L, \circ)\)-interior space.
An \((L, \circ)\)-interior space \((X, I)\) is called topological if \( I(I(\lambda)) \geq I(\lambda), \forall \lambda \in L^X \).
An \((L, \circ)\)-interior space \((X, I)\) is called enriched if \( E \) \( I(\alpha \circ \lambda) \geq \alpha \circ I(\lambda), \forall \alpha \in L, \lambda \in L^X \).

Thm. 3.3 (1) Let \((X, T)\) be an enriched \((L, \circ)\)-topological space. Define a map \( I_T : L^X \to L^X \) as follows:

\[ I_T(\lambda) = \bigvee\{\rho \in L^X \mid \rho \leq \lambda, \ \rho \in T\}. \]

Then \( I_T \) is an enriched \((L, \circ)\)-interior operator on \( X \) induced by \( T \).

(2) Let \((X, I)\) be an enriched \((L, \circ)\)-interior space. Define a subset \( T_I \) of \( L_X^X \) by

\[ T_I = \{\lambda \in L^X \mid \lambda \leq I(\lambda)\}. \]

Then \( T_I \) is an enriched \((L, \circ)\)-topology on \( X \) induced by \( I \).

Proof. (1) For all \( \alpha \in L, \lambda_1, \lambda_2, \lambda \in L^X \), we have

\[ I_T(I_{\lambda_1}) \circ I_T(I_{\lambda_2}) \]
\[ = \bigvee \{\rho_1 \in L^X \mid \rho_1 \leq \lambda_1, \ \rho_1 \in T\} \]
\[ \circ \bigvee \{\rho_2 \in L^X \mid \rho_2 \leq \lambda_2, \ \rho_2 \in T\} \]
\[ \leq \bigvee \{\rho_1 \circ \rho_2 \mid \rho_1 \circ \rho_2 \leq \lambda_1 \circ \lambda_2, \ \rho_1 \circ \rho_2 \in T\} \]
\[ \leq I_T(\lambda_1 \circ \lambda_2). \]

\[ \alpha \circ I_T(\lambda) \]
\[ = \alpha \circ \bigvee \{\rho \in L^X \mid \rho \leq \lambda, \ \rho \in T\} \]
\[ = \bigvee \{\alpha \circ \rho \mid \alpha \circ \rho \leq \alpha \circ \lambda, \ \alpha \circ \rho \in T\} \]
\[ \leq I_T(\alpha \circ \lambda). \]

Other cases and (2) are similarly proved.

(3) Since \( I(\lambda) \in T_I \) and \( I(\lambda) \leq \lambda \), by the definition of \( I_{T_1}, I_{T_1}(\lambda) \geq I(\lambda) \).

Suppose there exists \( \lambda \in L^X \) such that \( I_{T_1}(\lambda) \not\leq I(\lambda) \).
Then there exists \( \rho \in L^X \) with \( \rho \in T_I \) and \( \rho \leq \lambda \) such that \( \rho \not\leq I(\lambda) \).
On the other hand, since \( \rho = I(\rho) \leq I(\lambda) \), then \( \rho = I(\rho) = I(I(\rho)) \leq I(\lambda) \).
It is a contradiction. Hence \( I_{T_1} \leq I \).

Let \( \mu \in T \). Then \( I_T(\mu) = \mu \). Thus \( \mu \in T_{I_T} \). Let \( \mu \in T_{I_T} \). Then \( I_T(\mu) = \mu \in T \). Hence \( T_{I_T} = T \).
Theorem 3.4 Let $U$ be a Hutton $(L, \odot)$-uniformity on $X$. We define a mapping $I_U : L^X \to L^X$ as follows:

$$I_U(\lambda) = \bigvee \{ \rho \in L^X \mid \exists \phi \in U, \phi(\rho) \leq \lambda \}.$$

Then:

1. $I_U$ is an enriched topological $(L, \odot)$-interior operator on $X$.
2. $I_1(\lambda) = \bigvee \{ \rho \in L^X \mid \exists \phi \in U, \phi(\rho) \leq \lambda \}$
3. $I_2(\lambda) = \bigvee \{ \phi(\rho) \in L^X \mid \exists \phi \in U, \phi(\rho) \leq \lambda \}$
4. $I_3(\lambda) = \bigvee \{ \alpha \odot 1_{\{x\}} \mid \exists \phi \in U, \phi(\alpha \odot 1_{\{x\}}) \leq \lambda \}$
5. $I_4(\lambda) = \bigvee \{ \alpha \odot 1_{\{x\}} \mid \exists \phi \in U, \phi(1_{\{x\}}) \leq \alpha \to \lambda \}$

Then $I_U(\lambda) = I_i(\lambda)$ for $i = 1, 2, 3, 4$.

(3) $T_{I_U}$ is an enriched $(L, \odot)$-topology induced by $U$.

Proof. (1) (11) Since $\phi(1_X) = 1_X$, we have $I_U(1_X) = 1_X$.

(12) Since $\rho \leq \phi(\rho) \leq \lambda$, $I_U(\lambda) \leq \lambda$ for all $\lambda \in L^X$.

(13) Suppose $I_U(\lambda \odot \mu) \geq I_U(\lambda) \odot U(\mu)$. By the definition of $I_U(\lambda)$ and (L4), there exist $\rho, \gamma \in L^X$ and $\phi, \psi \in U$ with $\phi(\rho) \leq \lambda, \psi(\gamma) \leq \mu$ such that

$$I_U(\lambda \odot \mu) \not\leq \rho \odot \gamma.$$

Since $\phi \odot \psi \in U$,

$$(\phi \odot \psi)(\rho \odot \gamma) \leq \phi(\rho) \odot \psi(\gamma) \leq \lambda \odot \mu,$$

Thus, $I_U(\lambda \odot \mu) \not\geq \rho \odot \gamma$. It is a contradiction. Thus (13) holds.

(14) Suppose $I_U(\lambda \wedge \mu) \geq I_U(\lambda) \wedge U(\mu)$. By the definition of $I_U(\lambda)$ and a completely distributive lattice $L$, there exist $\rho, \gamma \in L^X$ and $\phi, \psi \in U$ with $\phi(\rho) \leq \lambda, \psi(\gamma) \leq \mu$ such that

$$I_U(\lambda \wedge \mu) \not\geq \rho \wedge \gamma.$$

Since $\phi \odot \psi \in U$, we have $(\phi \odot \psi)(\rho \wedge \gamma) \leq \phi(\rho \wedge \gamma) \odot \psi(\delta) = \phi(\rho \wedge \gamma)$, similarly $(\phi \odot \psi)(\rho \wedge \gamma) \leq \psi(\delta) \wedge \gamma$. It implies

$$(\phi \odot \psi)(\rho \wedge \gamma) \leq \phi(\rho \wedge \gamma) \wedge \psi(\delta) \leq \phi(\rho) \wedge \psi(\delta) \leq \lambda \wedge \mu.$$

Thus, $I_U(\lambda \wedge \mu) \not\geq \rho \wedge \gamma$. It is a contradiction. Thus (14) holds.

(5) Suppose there exists $\lambda \in L^X$ such that $I_U(I_U(\lambda)) \not\geq I_U(\lambda)$. By the definition of $I_U(\lambda)$, there exist $\rho \in L^X, \phi \in U$ with $\phi(\rho) \leq \lambda$ such that $I_U(I_U(\lambda)) \not\geq \rho$.

On the other hand, since $\phi \in U$, there exists $\psi \in U$ with $\psi \circ \phi \leq \phi$. It implies $\psi(\phi(\rho)) \leq \phi(\rho) \leq \lambda$. By the definition of $I_U(\lambda)$, we have $\psi(\phi(\rho)) \leq I_U(\lambda)$. By the definition of $I_U(I_U(\lambda))$, it follows that $I_U(I_U(\lambda)) \geq \rho$. It is a contradiction. Hence, $I_U(I_U(\lambda)) \geq I_U(\lambda)$.

(E) $\alpha \odot I_U(\lambda) = \bigvee \{ \rho \in L^X \mid \exists \phi \in U, \phi(\rho) \leq \lambda \}$

$$\leq \bigvee \{ (\phi \odot \rho) \in L^X \mid \exists \phi \in U, \phi(\alpha \odot \rho) \leq \lambda \} \leq I_U(\alpha \odot \lambda).$$

(2) Let $\rho \in L^X$ and $\phi \in U$ such that $\phi(\rho) \leq \lambda$. Since $\phi \in U$, there exists $\psi \in U$ with $\psi \circ \phi \leq \phi$ such that $(\psi \odot \psi)(\rho) \leq \phi(\rho) \leq \lambda$. Thus, $I_U(\lambda) \leq I_1(\lambda)$. Since $\rho \leq \psi(\rho), I_1 \leq I_2$. Hence $I_U(\lambda) \leq I_1(\lambda) \leq I_2(\lambda)$. Trivially, $I_2(\lambda) \leq I_U(\lambda)$. Hence $I_U(\lambda) = I_1(\lambda) = I_2(\lambda)$.

Trivially, $I_3(\lambda) \leq I_4(\lambda)$. Suppose $I_3(\lambda) \not\geq I_4(\lambda)$. By the definition of $I_U(\lambda)$, there exist $\rho \in L^X, \phi \in U$ with $\phi(\rho) \leq \lambda$ such that $I_3(\lambda) \not\geq \rho$. Since $\rho = \bigvee_{x \in X} \rho(x) \odot 1_{\{x\}},$

$$\phi(\rho) = \phi\left( \bigvee_{x \in X} \rho(x) \odot 1_{\{x\}} \right) = \bigvee_{x \in X} \phi(\rho(x) \odot 1_{\{x\}}) \leq \lambda$$

Put $\alpha_z = \rho(z)$. Since $\phi(\alpha_z \odot 1_{\{z\}}) \leq \lambda$ for all $z \in X, \rho = \bigvee_{z \in X} \rho(z) \odot I_3(\lambda).$ It is a contradiction.

Hence $I_3(\lambda) = I_4(\lambda)$.

Since $\phi(\alpha \odot 1_{\{z\}}) = \alpha \circ \phi(1_{\{z\}}) \leq \lambda$ if $\phi(1_{\{z\}}) \leq \alpha \to \lambda,$ we have $I_3 = I_4$.

Theorem 3.5 Let $D$ be an $(L, \odot)$-uniformity on $X$. We define a mapping $I_D : L^X \to L^X$ as follows:

$$I_D(\lambda) = \bigvee \{ \rho \in L^X \mid \exists u \in D, \Gamma(u)(\rho) \leq \lambda \}.$$

Then:

1. $I_D$ is an enriched topological $(L, \odot)$-interior operator on $X$.
2. $I_D(\lambda) = \bigvee \{ \alpha \odot 1_{\{y\}} \mid \exists u \in D, \alpha \circ u(y, -) \leq \lambda \}.$

(3) $T_{I_D}$ is an enriched $(L, \odot)$-topology induced by $D$. Moreover, $I_U = I_D$.

(4) If $U$ is a Hutton $(L, \odot)$-uniformity on $X$, then $I_U = I_D$.

Proof. (1) (II) Since $\Gamma(u)(1_X) \leq 1_X$, we have $I_D(1_X) = 1_X$.

(12) Since $\rho \leq \Gamma(u)(\rho) \leq \lambda, I_D(\lambda) \leq \lambda$ for all $\lambda \in L^X$.

(13) Suppose $I_D(\lambda \odot \mu) \not\geq I_D(\lambda) \odot I_D(\mu)$. By the definition of $I_D$ and (L4), there exist $\rho, \gamma \in L^X$ and $u, v \in D$ with $\Gamma(u)(\rho) \leq \lambda, \Gamma(v)(\gamma) \leq \mu$ such that

$$I_D(\lambda \odot \mu) \not\geq \rho \odot \gamma.$$

Since $\Gamma(u), \Gamma(v) \in \Omega(x)$ from Theorem 2.6(1) and $u \circ v \in D$, by Theorem 2.7(2), we have:

$$\Gamma(u \odot v)(\rho \odot \gamma) \leq \Gamma(u) \odot \Gamma(v)(\rho \odot \gamma)$$

$$\leq \Gamma(u)(\rho) \odot \Gamma(v)(\gamma) \leq \lambda \odot \mu.$$
Thus, \( I_D(\lambda \odot \mu) \geq \rho \odot \gamma \). It is a contradiction. Thus (I3) holds.

(I4) Suppose \( I_D(\lambda \wedge \mu) \nleq I_D(\lambda) \wedge I_D(\mu) \). By the definition of \( I_D(\lambda) \) and a completely distributive lattice \( L \), there exist \( \rho, \gamma \in L^X \) and \( u, v \in D \) with \( \Gamma(u)(\rho) \leq \lambda \), \( \Gamma(v)(\gamma) \leq \mu \) such that

\[
I_D(\lambda \wedge \mu) \ngeq \rho \wedge \gamma.
\]

Since \( u \circ v \in D \), by Theorem 2.7(2),

\[
\Gamma(u \circ v)(\rho \wedge \gamma) \leq \Gamma(u) \circ \Gamma(v)(\rho \wedge \gamma) \\
\leq \Gamma(u)(\rho \wedge \gamma) \wedge \Gamma(v)(\rho \wedge \gamma) \leq \Gamma(u)(\rho) \wedge \Gamma(v)(\gamma) \\
\leq \lambda \wedge \mu.
\]

Thus, \( I_U(\lambda \wedge \mu) \geq \rho \wedge \gamma \). It is a contradiction. Thus (I4) holds.

(T) Suppose there exists \( \lambda \in L^X \) such that \( I_D(I_D(\lambda)) \nleq I_D(\lambda) \). By the definition of \( I_D(\lambda) \), there exists \( \rho \in L^X \), \( u \in D \) with \( \Gamma(u)(\rho) \leq \lambda \) such that \( I_D(I_D(\lambda)) \nleq \rho \).

On the other hand, since \( u \in D \), there exists \( v \in D \) with \( v \circ u \leq u \). It implies, by Theorem 2.7(6),

\[
\Gamma(v) \circ \Gamma(u)(\rho) = \Gamma(v \circ u)(\rho) \leq \Gamma(u)(\rho) \leq \lambda.
\]

By the definition of \( I_D(\lambda) \), we have \( \Gamma(v)(\rho) \leq I_D(\lambda) \). By the definition of \( I_D(I_D(\lambda)) \), it follows that \( I_D(I_D(\lambda)) \nleq \rho \). It is a contradiction. Hence, \( I_D(I_D(\lambda)) \geq I_D(\lambda) \).

(E) For \( \alpha \in L \) and \( \lambda \in L^X \),

\[
\begin{align*}
\alpha \odot I_D(\lambda) &= \alpha \odot \bigvee \{ \rho \in L^X \mid \exists u \in D, \Gamma(u)(\rho) \leq \lambda \} \\
&\leq \bigvee \{ (\alpha \odot \rho) \in L^X \mid \exists u \in D, \alpha \odot \Gamma(u)(\rho) \leq \alpha \odot \lambda \} \\
&\leq \bigvee \{ (\alpha \odot \rho) \in L^X \mid \exists u \in D, \Gamma(u)(\alpha \odot \rho) \leq \alpha \odot \lambda \} \\
&\leq I_D(\alpha \odot \lambda).
\end{align*}
\]

(2) Since \( \Gamma(u) \in \Omega(X) \), by I3 of Theorem 3.4(2), we have

\[
\begin{align*}
\Gamma(u)(\alpha \odot I_1(y)) &= \bigvee_{x \in X} (\alpha \odot I_1(y))(x) \odot u(x, -) \\
&= \alpha \odot u(y, -).
\end{align*}
\]

It implies

\[
\begin{align*}
I_D(\lambda) &= \bigvee \{ \alpha \odot I_1(y) \mid \exists u \in D, \Gamma(u)(\alpha \odot I_1(y)) \leq \lambda \} \\
&= \bigvee \{ \alpha \odot I_1(y) \mid \exists u \in D, \alpha \odot u(y, -) \leq \lambda \}.
\end{align*}
\]

(3) By Theorem 3.3 and (1), \( T_{I_U} \) is an enriched \( (L, \odot) \)-topology induced by \( D \).

For \( u \in D \) with \( \Gamma(u) \leq \phi \), \( \phi(\rho) \leq \lambda \) implies \( \Gamma(u)(\rho) \leq \phi(\rho) \leq \lambda \). Thus, \( I_U(\phi) \leq I_D \).

For \( u \in D \) with \( \Gamma(u)(\rho) \leq \lambda \) since \( \Gamma(u) \in U_D \), \( I_U(\phi) \geq I_{U_D} \).

(4) For \( \phi \in U \) with \( \phi(\rho) \leq \lambda \), \( \Lambda(\phi) \in D_U \). So, \( \Gamma(\Lambda(\phi))(\rho) = \phi(\rho) \leq \lambda \). Thus, \( I_U(\phi) \geq I_{U_D} \).

For \( u \in D \) with \( \Gamma(u)(\rho) \leq \lambda \) by the definition of \( D_U \), there exists \( \phi \in U \) such that \( \Lambda(\phi) \leq u \). So, \( \Gamma(\Lambda(\phi))(\rho) = \phi(\rho) \leq \Gamma(u)(\rho) \leq \lambda \). Hence \( I_D(\lambda) \leq I_{U_D} \).

**Example 3.6** Let \( X = \{ x, y, z \} \) be a set and \( \{ 0, 1 \} \) an biquantale defined by \( x \circ y = \max \{ 0, x + y - 1 \} \) (ref.[4-6, 14]).

Define \( \phi \in \Omega(X) \) as follows:

\[
\phi(1_{(x)}) = \phi(1_{(y)}) = 1_{(x,y)}, \quad \phi(1_{(z)}) = \phi(1_{(z)}).
\]

Since

\[
\phi \circ \phi(1_{(z)}) = \phi \circ \phi(1_{(y)}) = 1_{(x,y)}, \quad \phi \circ \phi(1_{(z)}) = 1_{(z)},
\]

by Lemma 2.3(2), \( \phi \circ \phi = \phi \). We have \( \phi \circ \phi = \phi \) because

\[
\phi \circ \phi(1_{(z)}) = \phi \circ \phi(1_{(y)}) = 1_{(x,y)}, \quad \phi \circ \phi(1_{(z)}) = 1_{(z)}.
\]

Since

\[
\phi^{-1}(1_{(x)}) = \phi^{-1}(1_{(y)}) = 1_{(x,y)}, \quad \phi^{-1}(1_{(z)}) = 1_{(z)},
\]

Hence \( \phi^{-1} = \phi \).

(1) Define \( U = \{ \psi \in \Omega(X) \mid \phi \leq \psi \} \). Then \( U \) is a Hutton \( (L, \odot) \)-uniformity on \( X \).

For each \( \lambda \in L^X \), by I3 of Theorem 3.4,

\[
I_U(\lambda)(x) = I_U(\lambda)(y) = \lambda(x) \wedge \lambda(y), \quad I_U(\lambda)(z) = \lambda(z).
\]

We obtain

\[
T_{I_U} = \{ \alpha \circ I_X, \lambda \in L^X \mid \lambda(x) = \lambda(y) = a, \lambda(y) = b, \forall a, b, \alpha \in L \}.
\]

(2) We obtain \( D_U = \{ u \in E(X \times X) \mid \Lambda(\phi) \leq u \} \). Since \( \phi \circ \phi = \phi = \phi \circ \phi^{-1} \), by Theorem 2.8(7), \( \Lambda(\phi) \) is an \( \circ \)-equivalence relation such that

\[
\begin{align*}
\Lambda(\phi)(x, x) &= 1, \quad \Lambda(\phi)(x, y) = 1, \quad \Lambda(\phi)(x, z) = 0 \\
\Lambda(\phi)(y, x) &= 1, \quad \Lambda(\phi)(y, y) = 1, \quad \Lambda(\phi)(y, z) = 0 \\
\Lambda(\phi)(z, x) &= 0, \quad \Lambda(\phi)(z, y) = 0, \quad \Lambda(\phi)(z, z) = 1.
\end{align*}
\]

Furthermore, \( \Lambda(\phi) \circ \Lambda(\phi) = \Lambda(\phi) \), \( \Lambda(\phi^{-1}) = \Lambda(\phi) \) and \( \Lambda(\phi) \circ \Lambda(\phi) = \Lambda(\phi \circ \phi) = \Lambda(\phi) \). Hence \( D_U \) is an \( (L, \odot) \)-uniformity on \( X \).

(3) For each \( \lambda \in L^X \), by Theorem 3.5(2),

\[
I_{D_U}(\lambda) = (\alpha \circ I_1(x)) \cup (\alpha \circ I_1(y)) \cup (\beta \circ I_1(z)),
\]

where \( \alpha = \lambda(x) \wedge \lambda(y), \beta = \lambda(z) \). Hence \( I_{D_U} = I_U \).

**Theorem 3.7** Let \( (X, U) \) and \( (Y, V) \) be \( (L, \odot) \)-Hutton-uniform spaces. Let \( f : (X, U) \to (Y, V) \) be \( H \)-uniformly continuous. Then we have the following properties.

(1) \( f^{-1}(I_V(\lambda)) \leq I_U(f^{-1}(\lambda)) \), for each \( \lambda \in L^Y \).

(2) \( f : (X, T_{I_U}) \to (Y, T_{I_V}) \) is \( L \)-continuous.
Proof. (1) Since \( f^-((\psi))(f^-((\rho))) \leq f^-((\psi(\rho))) \), we have
\[
\begin{align*}
f^-((\mathbf{I}_V(\lambda)) &= f^- \left( \bigvee \{ \rho \mid \exists \psi \in \mathbf{V}, \psi(\rho) \leq \lambda \} \right) \\
&\leq \bigvee \{ f^-((\rho)) \mid f^-((\psi(\rho))) \leq f^-((\lambda)), f^\leq((\psi)) \in \mathbf{U} \} \\
&\leq \bigvee \{ f^-((\rho)) \mid f^\leq((\psi))(f^-((\rho))) \leq f^-((\lambda)), f^\leq((\psi)) \in \mathbf{U} \} \\
&\leq \mathbf{I}_U(f^-((\lambda))) \}
\end{align*}
\]
(2) Since \( \lambda \leq \mathbf{I}_V(\lambda) \) implies \( f^-((\lambda)) \leq \mathbf{I}_U(f^-((\lambda))) \) from (1), for each \( \lambda \in \mathbf{T}_{\mathbf{I}_V} \), we have \( f^-((\lambda)) \in \mathbf{T}_{\mathbf{I}_U} \).

Theorem 3.8 Let \((X, D_1)\) and \((Y, D_2)\) be \((L, \odot)\)-uniform spaces. Let \( f : (X, D_1) \to (Y, D_2) \) be \( L \)-uniformly continuous. Then:

(1) \( \mathbf{I}_{D_2}(f^-((\rho))) \geq f^-((\mathbf{I}_{D_2}(\rho))) \), for each \( \rho \in \mathbf{L}^Y \).
(2) \( f : (X, \mathbf{T}_{\mathbf{I}_{D_1}}) \to (Y, \mathbf{T}_{\mathbf{I}_{D_2}}) \) is \( L \)-continuous.

Proof. (1) Put \( \lambda = f^-((\gamma)) \) from Lemma 2.12(1), \( \Gamma((v))(\gamma) \leq \rho \) implies
\[
\Gamma((f \times f)^-((v))(f^-((\gamma)))) = f^-((\Gamma((v)))(f^-((\Gamma((\gamma))))) \\
\leq f^-((\Gamma((v)(\gamma)))) \\
\leq f^-((\rho)).
\]
Since \((f \times f)^-((v)) \in \mathbf{U} \) for \( v \in \mathbf{V} \), we have
\[
f^-((\mathbf{I}_{D_2}(\rho))) \\
= f^- \left( \bigvee \{ (\gamma) \in \mathbf{L}^X \mid \Gamma((v))(\gamma) \leq \rho, v \in \mathbf{V} \} \right) \\
= \bigvee \{ f^-((\gamma)) \in \mathbf{L}^X \mid \Gamma((v))(\gamma) \leq \rho, v \in \mathbf{V} \} \\
\leq \bigvee \{ f^-((\gamma)) \in \mathbf{L}^X \mid ((f \times f)^-((v))(f^-((\gamma))) \leq f^-((\rho)), (f \times f)^-((v)) \in \mathbf{U} \} \\
\leq \mathbf{I}_{D_2}(f^-((\rho))).
\]
(2) From (1) and Theorem 3.3, \( \mathbf{I}_{D_2}(\rho) \geq \rho \) implies \( \mathbf{I}_{D_2}(f^-((\rho))) \geq \phi^-((\rho)) \). It is easily proved.

References


Yong Chan Kim
He received the M.S and Ph.D. degrees in Department of Mathematics from Yonsei University, in 1984 and 1991, respectively. From 1991 to present, he is a professor in the Department of Mathematics, Kangnung University. His research interests are fuzzy topology and fuzzy logic.

Young Sun Kim
He received the M.S and Ph.D. degrees in Department of Mathematics from Yonsei University, in 1985 and 1991, respectively. From 1988 to present, he is a professor in the Department of Applied Mathematics, Pai Chai University. His research interests are fuzzy topology and fuzzy logic.