INTUITIONISTIC FUZZY $(t,s)$-CONGRUENCES

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Abstract

We introduce the notion of intuitionistic fuzzy $(t,s)$-congruences on a lattice and study some of its properties. Moreover, we obtain some properties of intuitionistic fuzzy congruences on the direct product of two lattices. Finally, we prove that the set of all intuitionistic fuzzy congruences on a lattice forms a distributive lattice.

Key words: intuitionistic fuzzy set, intuitionistic fuzzy $(t,s)$-equivalence relation, intuitionistic fuzzy $(t,s)$-congruence.

0. Introduction

The subject of fuzzy sets as an approach to a mathematical representation of vagueness in everyday language was introduced by L.A. Zadeh [22] in 1965. He generalized the idea of the characteristic function of a subset of a set $X$ by defining a fuzzy subset of $X$ as a map from $X$ into $[0,1]$. After that time, Sidky and Atallah [21] introduced the concept of $T$-congruences on a lattice and investigated some of its properties.

As a generalization of fuzzy sets, the notion of intuitionistic fuzzy sets was introduced by Atanassov [1]. Since then, Çoker and his colleagues [5,6,9], and Lee and Lee [19], and Hur and his colleagues [14] introduced the concept of intuitionistic fuzzy topological spaces and studied various properties. Moreover, Hur and his colleagues [13] applied the notion of intuitionistic fuzzy set to topological group. Moreover, Banerjee and Basnet [2], Biswas [3], Hur and his colleagues [10,11,12,15] applied to group theory using intuitionistic fuzzy sets. In 1996, Bustince and Burillo [4] introduced the concept of intuitionistic fuzzy relations and investigated some of its properties. In 2003, Deschrijver and Kerre [7] investigated some properties of the composition of intuitionistic fuzzy relations. In particular, Hur and his colleagues [16,18] introduced the concept of intuitionistic fuzzy congruences on a lattice (a semigroup) and studied some of its properties. Also, Hur and his colleagues [17] investigated various properties of intuitionistic fuzzy equivalence relations.

In this paper, we introduce the notion of intuitionistic fuzzy $(t,s)$-congruences on a lattice and study some of its properties. Moreover, we obtain some properties of intuitionistic fuzzy congruences on the direct product of two lattices. Finally, we prove that the set of all intuitionistic fuzzy congruences on a lattice forms a distributive lattice.

1. Preliminaries

In this section, we list some basic concepts and one result which are needed in the later sections.

For sets $X, Y$ and $Z$, $f = (f_1, f_2) : X \rightarrow Y \times Z$ is called a complex mapping if $f_1 : X \rightarrow Y$ and $f_2 : X \rightarrow Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0,1]$ as $I$. And for a general background of lattice theory, refer to [8]. Moreover, we will use $t$ and $s$ to denote a $t$-norm and a $t$-conorm, respectively. For a $t$-norm and a $t$-conorm, we refer to [20].

Definition 1.1[1,5]. Let $X$ be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \rightarrow I \times I$ is called an intuitionistic fuzzy set (in short, IFS) in $X$ if $\mu_A(x) + \nu_A(x) \leq 1$
for each \( x \in X \), where the mappings \( \mu_A : X \rightarrow I \) and \( \nu_A : X \rightarrow I \) denote the degree of membership (namely \( \mu_A(x) \)) and the degree of nonmembership (namely \( \nu_A(x) \)) of each \( x \in X \) to \( A \), respectively. In particular, 0\( _x \) and 1\( _x \) denote the intuitionistic fuzzy empty set and the intuitionistic fuzzy whole set in \( X \) defined by 0\( _x(x) = (0, 1) \) and 1\( _x(x) = (1, 0) \) for each \( x \in X \), respectively.

We will denote the set of all IFSs in \( X \) as IFS(\( X \)).

**Definitions 1.2 [1].** Let \( X \) be a nonempty set and let \( A = (\mu_A, \nu_A) \) and \( B = (\mu_B, \nu_B) \) be IFSs on \( X \). Then

1. \( A \subseteq B \) iff \( \mu_A \leq \mu_B \) and \( \nu_A \geq \nu_B \).
2. \( A = B \) iff \( A \subseteq B \) and \( B \subseteq A \).
3. \( A^* = (\nu_A, \mu_A) \).
4. \( A \cup B = (\mu_A \lor \mu_B, \nu_A \land \nu_B) \).
5. \( A \cap B = (\mu_A \land \mu_B, \nu_A \lor \nu_B) \).

**Definition 1.3 [5].** Let \( \{ A_i \}_{i \in J} \) be an arbitrary family of IFSs in \( X \), where \( A_i = (\mu_{A_i}, \nu_{A_i}) \) for each \( i \in J \). Then

1. \( \bigcap A_i = (\land \mu_{A_i}, \lor \nu_{A_i}) \).
2. \( \bigcup A_i = (\lor \mu_{A_i}, \land \nu_{A_i}) \).

**Definition 1.4 [5].** Let \( X \) be a set. Then a complex mapping \( R = (\mu_R, \nu_R) : X \times X \rightarrow I \times I \) is called an intuitionistic fuzzy relation in short, IFR(\( X \)) on \( X \) if \( \mu_R(x, y) + \nu_R(x, y) \leq 1 \) for each \( (x, y) \in X \times X \), i.e., \( R \in \text{IFR}(X) \).

We will denote the set of all IFRs on a set \( X \) as IFR(\( X \)).

**Definition 1.5 [5,7].** Let \( X \) be a set and let \( P, Q \in \text{IFR}(X) \). Then the composition \( Q \circ P \) of \( P \) and \( Q \), is defined as follows: for any \( x, y \in X \),

\[
\mu_{Q \circ P}(x, y) = \bigwedge_{z \in X} [\mu_P(x, z) \land \mu_Q(z, y)]
\]

and

\[
\nu_{Q \circ P}(x, y) = \bigvee_{z \in X} [\nu_P(x, z) \lor \nu_Q(z, y)].
\]

**Definition 1.6 [5,7].** An intuitionistic fuzzy Relation \( R \) on a set \( X \) is called an intuitionistic fuzzy equivalence relation (in short, IFER) on \( X \) if it satisfies the following conditions:

(i) it is intuitionistic fuzzy reflexive, i.e., \( R(x, x) = (1, 0) \) for each \( x \in X \).
(ii) it is intuitionistic fuzzy symmetric, i.e., \( R(x, y) = R(y, x) \) for any \( x, y \in X \).
(iii) it is intuitionistic fuzzy transitive, i.e., \( R \circ R \subseteq R \).

We will denote the set of all IFERs on \( X \) as IFE(\( X \)).

**Result 1.1 [17, Proposition 2.10].** Let \( \{ R_\alpha \}_{\alpha \in I} \) be a nonempty family of IFERs on a set \( X \). Then \( \bigcap_{\alpha \in I} R_\alpha \in \text{IFE}(X) \). However, in general, \( \bigcup_{\alpha \in I} R_\alpha \) need not be an IFER on \( X \).

**2. Intuitionistic fuzzy \((t, s)\)-equivalence relations**

Throughout this section, let \( X, Y, \) and \( Z \) be nonempty sets.

**Definition 2.1.** Let \( R \in \text{IFS}(X \times Y) \) and let \( S \in \text{IFS}(Y \times Z) \). Then the \((t, s)\)-composition of \( R \) and \( S \), \( S \circ_t R \), is defined as follows: for each \( (x, z) \in X \times Z \),

\[
\mu_{S \circ_t R}(x, z) = \bigvee_{y \in Y} [\mu_R(x, y) \land \mu_S(y, z)]
\]

and

\[
\nu_{S \circ_t R}(x, z) = \bigwedge_{y \in Y} [\nu_R(x, y) \lor \nu_S(y, z)].
\]

**Definition 2.2.** Let \( A \in \text{IFS}(X) \) and let \( (\lambda, \mu) \in I \times I \) with \( \lambda + \mu \leq 1 \).

1. \( A^{(\lambda, \mu)} = \{ x \in X : \mu_A(x) \geq \lambda \} \) and \( \nu_A(x) \leq \mu \) is called the \((\lambda, \mu)\)-level subset of \( A \).
2. \( A^{(\lambda, \mu)} = \{ x \in X : \mu_A(x) > \lambda \} \) and \( \nu_A(x) < \mu \) is called the strong \((\lambda, \mu)\)-level subset of \( A \).

It is clear that \( R \in \text{IFR}(X) \) if and only if \( R^{(\lambda, \mu)} \) and \( R^{(\lambda, \mu)} \) are relations on \( X \).

**Result 2.A [17, Theorem 2.17].** Let \( R \in \text{IFR}(X) \). Then \( R \in \text{IFE}(X) \) if and only if \( R^{(\lambda, \mu)} \) is an equivalence relation for each \( (\lambda, \mu) \in I \times I \) with \( \lambda + \mu \leq 1 \).

**Proposition 2.3.** Let \( A, B \in \text{IFS}(X) \) and let \( (\lambda, \mu) \in I \times I \) with \( \lambda + \mu \leq 1 \). Then

1. \( (A \cap B)^{(\lambda, \mu)} = A^{(\lambda, \mu)} \cap B^{(\lambda, \mu)} \).
2. \( (A \cup B)^{(\lambda, \mu)} = A^{(\lambda, \mu)} \cup B^{(\lambda, \mu)} \).

**Proof.** (1) Let \( x \in X \). Then

\[
x \in (A \cap B)^{(\lambda, \mu)} \iff \mu_{A \cap B}(x) = \mu_A(x) \land \mu_B(x) \geq \lambda
\]

and \( \nu_{A \cap B}(x) = \nu_A(x) \lor \nu_B(x) \leq \mu \)

\[
\iff \mu_A(x) \geq \lambda, \mu_B(x) \geq \lambda
\]

and \( \nu_A(x) \leq \mu, \nu_B(x) \leq \mu \)

\[
\iff \mu_A(x) \geq \lambda, \nu_A(x) \leq \mu
\]

and \( \mu_B(x) \geq \lambda, \nu_B(x) \leq \mu \)

\[
\iff x \in A^{(\lambda, \mu)} \text{ and } x \in B^{(\lambda, \mu)}
\]

\[
\iff x \in A^{(\lambda, \mu)} \cap B^{(\lambda, \mu)}.
\]
(2) Let \( x \in X \). Then
\[
\begin{align*}
\forall x \in (A \cup B)^{(\lambda, \mu)} \\
\iff \\
\mu_{A\cup B}(x) = \mu_A(x) \lor \mu_B(x) \geq \lambda \\
\text{and } \\
\nu_{A\cup B}(x) = \nu_A(x) \land \nu_B(x) \leq \mu \\
\iff \\
\mu_A(x) \geq \lambda \text{ or } \mu_B(x) \geq \lambda \\
\text{and } \\
\nu_A(x) \leq \mu \text{ or } \nu_B(x) \leq \mu \\
\iff \\
\mu_A(x) \geq \lambda, \nu_A(x) \leq \mu \\
\text{or } \\
\mu_B(x) \geq \lambda, \nu_B(x) \leq \mu \\
\iff \\
x \in A^{(\lambda, \mu)} \text{ or } x \in B^{(\lambda, \mu)} \\
\iff x \in A^{(\lambda, \mu)} \cup B^{(\lambda, \mu)}.
\end{align*}
\]

**Definition 2.4.** Let \( R \in \text{IFR}(X) \). Then \( R \) is said to be:

(1) an **intuitionistic fuzzy \( (\lambda, \mu) \)-reflexive** if \( \mu_R(x, x) \geq \lambda \) and \( \nu_R(x, x) \leq \mu \) for each \( x \in X \), where \( (\lambda, \mu) \in I \times I \) with \( \lambda + \mu \leq 1 \).

(2) an **intuitionistic fuzzy \( (t, s) \)-transitive** if \( R \circ^t R \subseteq R \).

(3) an **intuitionistic fuzzy \( (t, s) \)-equivalence relation** on \( X \) if it is intuitionistic fuzzy reflexive, symmetric, and \( (t, s) \)-transitive.

(4) an **intuitionistic fuzzy \( (\lambda, \mu) \)-(t, s)-equivalence relation** on \( X \) if it is intuitionistic fuzzy \( (\lambda, \mu) \)-reflexive, symmetric, and \( (t, s) \)-transitive.

We will denote the set of intuitionistic fuzzy \( (t, s) \)-\( \text{IFR}(X) \) \( \text{IF}(\lambda, \mu)-(t, s)(X) \) as \( \text{IF}(\lambda, \mu)-(t, s)(X) \).

**Proposition 2.5.** The intersection of arbitrary subfamily of \( \text{IF}(\lambda, \mu)-(t, s)(X) \) is an intuitionistic fuzzy \( (t, s) \)-\( \text{IFR}(X) \) \( \text{IF}(\lambda, \mu)-(t, s)-(t, s) \)-equivalence relation on \( X \).

**Proof.** Let \( \{R_\alpha\}_{\alpha \in \Gamma} \) be a family of intuitionistic fuzzy \( (t, s) \)-\( \text{IFR}(X) \) \( \text{IF}(\lambda, \mu)-(t, s)(X) \) and let \( R = \bigcap_{\alpha \in \Gamma} R_\alpha \). By Result 1.4, it is clear that \( R \) is intuitionistic fuzzy reflexive and symmetric. Thus it is sufficient to show that \( R \) is intuitionistic fuzzy \( (t, s) \)-transitive.

Let \( x, y, z \in X \). Then
\[
\mu_R(x, y) = \bigwedge_{\alpha \in \Gamma} \mu_{R_\alpha}(x, y)
\geq \bigwedge_{\alpha \in \Gamma} \mu_{R_\alpha}(x, y)
\text{(Since } R_\alpha \text{ is } (t, s) \text{-transitive} \}
= \bigwedge_{\alpha \in \Gamma} \left\{ \bigvee_{a \in X} [\mu_{R_\alpha}(x, a) \land \mu_{R_\alpha}(a, y)] \right\}
\geq \bigwedge_{\alpha \in \Gamma} \left\{ \mu_{R_\alpha}(x, z) \land \mu_{R_\alpha}(z, y) \right\}
= \left\{ \bigwedge_{\alpha \in \Gamma} \mu_{R_\alpha}(x, z) \right\} \land \left\{ \bigwedge_{\alpha \in \Gamma} \mu_{R_\alpha}(z, y) \right\}
= \mu_R(x, z) \land \mu_R(z, y)
\]

And
\[
\nu_R(x, y) = \bigvee_{\alpha \in \Gamma} \nu_{R_\alpha}(x, y) \leq \bigvee_{\alpha \in \Gamma} \nu_{R_\alpha}(x, y)
\leq \bigvee_{\alpha \in \Gamma} \left\{ \nu_{R_\alpha}(x, z) \land \nu_{R_\alpha}(z, y) \right\}
= \bigvee_{\alpha \in \Gamma} \left\{ \nu_{R_\alpha}(x, z) \land \nu_{R_\alpha}(z, y) \right\}
= \nu_R(x, z) \land \nu_R(z, y).
\]

So \( R \) is intuitionistic fuzzy \( (t, s) \)-transitive. On the other hand, we can easily show that \( R \) is intuitionistic fuzzy \( (\lambda, \mu) \)-reflexive. Hence \( R = \bigcap_{\alpha \in \Gamma} R_\alpha \in \text{IF}(\lambda, \mu)-(t, s)(X) \). This completes the proof.

### 3. Intuitionistic fuzzy congruences on a lattice

Let \( L \) be a lattice with least element 0 and greatest element 1.

**Definition 3.1.** Let \( R \in \text{IFR}(L) \). Then \( R \) is said to be:

(i) \( 16 \text{ an } \text{intuitionistic fuzzy compatible if it satisfies the following conditions: for any } x_1, x_2, y_1, y_2 \in L, \)
\begin{align*}
\text{(i) } & \mu_R(x_1 \land x_2, y_1 \land y_2) \geq \mu_R(x_1, y_1) \land \mu_R(x_2, y_2), \\
\text{(ii) } & \mu_R(x_1 \lor x_2, y_1 \lor y_2) \leq \mu_R(x_1, y_1) \lor \mu_R(x_2, y_2), \\
\text{(iii) } & \nu_R(x_1 \land x_2, y_1 \land y_2) \leq \nu_R(x_1, y_1) \land \nu_R(x_2, y_2), \\
\text{(iv) } & \nu_R(x_1 \lor x_2, y_1 \lor y_2) \geq \nu_R(x_1, y_1) \lor \nu_R(x_2, y_2).
\end{align*}

(ii) \( 16 \text{ an } \text{intuitionistic fuzzy congruence on } L \) if it is an IF on \( L \) and is intuitionistic fuzzy compatible.

(iii) \( 16 \text{ an } \text{intuitionistic fuzzy } (t, s) \)-compatible if it satisfies the following conditions: for any \( x_1, x_2, y_1, y_2 \in L, \)
\begin{align*}
\text{(i) } & \mu_R(x_1 \land x_2, y_1 \land y_2) \geq \mu_R(x_1, y_1) \lor \mu_R(x_2, y_2), \\
\text{(ii) } & \mu_R(x_1 \lor x_2, y_1 \lor y_2) \leq \mu_R(x_1, y_1) \land \mu_R(x_2, y_2), \\
\text{(iii) } & \nu_R(x_1 \land x_2, y_1 \land y_2) \leq \nu_R(x_1, y_1) \lor \nu_R(x_2, y_2), \\
\text{(iv) } & \nu_R(x_1 \lor x_2, y_1 \lor y_2) \geq \nu_R(x_1, y_1) \land \nu_R(x_2, y_2).
\end{align*}

(4) an intuitionistic fuzzy \( (t, s) \)-congruence on \( L \) if \( R \in \text{IF}(t, s)(L) \) and \( R \) is intuitionistic fuzzy \( (t, s) \)-compatible.

(5) an intuitionistic fuzzy \( (\lambda, \mu) \)-(t, s)-congruence on \( L \) if \( R \in \text{IF}(\lambda, \mu)-(t, s)(L) \) and \( R \) is intuitionistic fuzzy \( (t, s) \)-compatible.

We will denote the set of all intuitionistic fuzzy congruences \( \text{IFC}(L) \text{, } \text{IFC}(t, s)(L) \text{, and } \text{IFC}(\lambda, \mu)-(t, s)(L) \).

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Proposition 3.2. Let \( R \in \text{IFR}(L) \), let \( \alpha = \bigvee_{x,y \in L} \mu_R(x, y) \), \( \beta = \bigwedge_{x,y \in L} \nu_R(x, y) \) and let \((\lambda, \mu) \in \text{Im} R^\lambda \). Then

1. \( R \) is intuitionistic fuzzy \((\alpha, \beta)\)-reflexive if and only if \( R^\lambda \) is reflexive.
2. \( R \) is intuitionistic fuzzy symmetric if and only if \( R^\lambda \) is symmetric.
3. If \( R \) is intuitionistic fuzzy transitive, then \( R^\lambda \) is transitive.
4. If \( R^\lambda \) is transitive, then \( R \) is intuitionistic fuzzy \((t, s)\)-transitive.
5. If \( R \) is intuitionistic fuzzy compatible, then \( R^\lambda \) is compatible.
6. If \( R^\lambda \) is compatible, then \( R \) is intuitionistic fuzzy \((t, s)\)-compatible.

Proof. (1) \((\Rightarrow)\) : Suppose \( R \) is intuitionistic fuzzy \((\alpha, \beta)\)-reflexive and let \( x \in L \). Then clearly \( \mu_R(x, x) = \alpha \) and \( \nu_R(x, x) = \beta \). Since \((\lambda, \mu) \in \text{Im} R^\lambda \), \( \mu_R(x, x) > \lambda \) and \( \nu_R(x, x) < \mu \). Thus \( (x, x) \in R^\lambda \). So \( R^\lambda \) is reflexive.

(\(<\)=) : Suppose \( R^\lambda \) is reflexive and let \( x \in L \). Then \( \mu_R(x, x) > \lambda \) and \( \nu_R(x, x) < \mu \) for each \((\lambda, \mu) \in \text{Im} R^\lambda \). Thus \( \mu_R(x, x) > \alpha \) and \( \nu_R(x, x) \leq \beta \). By the definition of \((\alpha, \beta)\), \( \mu_R(x, x) = \alpha \) and \( \nu_R(x, x) = \beta \). So \( R \) is intuitionistic fuzzy \((\alpha, \beta)\)-reflexive.

(2) \((\Rightarrow)\) : Suppose \( R \) is intuitionistic fuzzy symmetric. For any \( x, y \in L \), let \((x, y) \in R^\lambda \). Then \( \mu_R(x, y) = \mu_R(y, x) > \lambda \) and \( \nu_R(x, y) = \nu_R(y, x) < \mu \). Thus \((x, y) \in R^\lambda \). So \( R^\lambda \) is symmetric.

(\(<\)=) : \( R^\lambda \) is symmetric. Assume that \( R(x, y) = (\lambda_1, \mu_1) \) and \( R(y, x) = (\lambda_2, \mu_2) \) such that \( \lambda_1 \geq \lambda_2 \) and \( \mu_1 < \mu_2 \). Then \((x, y) \in R^\lambda \). By the hypothesis, \((y, x) \in R^\lambda \). Thus \( \lambda_2 = \mu_R(y, x) > \lambda_2 \) and \( \mu_2 = \nu_R(y, x) < \mu_2 \). This is a contradiction. So \((x, y) \in R(x, y) \). Hence \( R \) is intuitionistic fuzzy symmetric.

(3) Suppose \( R \) is intuitionistic fuzzy transitive. For any \( x, y, z \in L \), let \((x, y) \in R^\lambda \) and \((y, z) \in R^\lambda \). Then \( \mu_R(x, y) > \lambda \), \( \nu_R(x, y) < \mu \) and \( \mu_R(y, z) > \lambda \), \( \nu_R(y, z) < \mu \). By the hypothesis,

\[
\mu_R(x, z) \geq \bigvee_{a \in L} \mu_R(x, a) \land \mu_R(a, z)
\]

\[
\geq \mu_R(x, y) \land \mu_R(y, z)
\]

\[
> \lambda
\]

and

\[
\nu_R(x, z) \leq \bigwedge_{a \in L} \nu_R(x, a) \lor \nu_R(a, z)
\]

\[
\leq \nu_R(x, y) \lor \nu_R(y, z)
\]

\[
< \mu.
\]

Thus \((x, z) \in R^\lambda \). Hence \( R^\lambda \) is transitive.

(4) Suppose \( R^\lambda \) is transitive. For any \( x, y, z \in L \), let \( R(x, y) = (\lambda_1, \mu_1) \) and \( R(y, z) = (\lambda_2, \mu_2) \) such that \( \lambda_1 \leq \lambda_2 \) and \( \mu_1 \geq \mu_2 \). Then

\[
\mu_R(x, y) \land \mu_R(y, z) \leq \lambda_1 \land \nu_R(x, y) \land \nu_R(y, z) \geq \mu_1.
\]

Assume that \( R(x, z) = (\lambda_3, \mu_3) \) such that \( \lambda_3 < \lambda_1 \) and \( \mu_3 > \mu_1 \). Then \( (x, y) \in R^{(\lambda_3, \mu_3)} \) and \( (y, z) \in R^{(\lambda_3, \mu_3)} \). By the hypothesis, \((x, z) \in R^{(\lambda_3, \mu_3)} \). Then \( \lambda_3 = \mu_R(x, z) > \lambda_3 \) and \( \mu_3 = \nu_R(x, z) < \mu_3 \). This contradicts the fact that \( R(x, z) = (\lambda_3, \mu_3) \). Hence \( R \) is intuitionistic fuzzy \((t, s)\)-transitive.

(5) Suppose \( R \) is intuitionistic fuzzy compatible and let \( x_1, x_2, y_1, y_2 \in L \). Then

\[
\mu_R(x_1 \land x_2, y_1 \land y_2) \geq \mu_R(x_1, y_1) \land \mu_R(x_2, y_2)
\]

and

\[
\nu_R(x_1 \land x_2, y_1 \land y_2) \leq \nu_R(x_1, y_1) \lor \nu_R(x_2, y_2).
\]

Also, \( \mu_R(x_1 \lor x_2, y_1 \lor y_2) \geq \mu_R(x_1, y_1) \land \mu_R(x_2, y_2) \) and

\[
\nu_R(x_1 \lor x_2, y_1 \lor y_2) \leq \nu_R(x_1, y_1) \lor \nu_R(x_2, y_2).
\]

So \((x_1 \land x_2, y_1 \land y_2) \in R^{(\lambda_3, \mu_3)} \) and \((x_1 \lor x_2, y_1 \lor y_2) \in R^{(\lambda_3, \mu_3)} \). Hence \( R^\lambda \) is compatible.

(6) Suppose \( R^\lambda \) is compatible. For any \( x_1, x_2, y_1, y_2 \in L \), let \( R(x_1, y_1) = (\lambda_1, \mu_1) \) and \( R(x_2, y_2) = (\lambda_2, \mu_2) \) such that \( \lambda_1 \leq \lambda_2 \) and \( \mu_1 \geq \mu_2 \). Assume that \( \mu_R(x_1 \land x_2, y_1 \land y_2) = \lambda_3 < \lambda_1 \) and \( \nu_R(x_1 \land x_2, y_1 \land y_2) = \mu_3 > \mu_1 \). Then \((x_1, y_1) \in R^{(\lambda_3, \mu_3)} \) and \((x_2, y_2) \in R^{(\lambda_3, \mu_3)} \). By the hypothesis, \((x_1 \land x_2, y_1 \land y_2) \in R^{(\lambda_3, \mu_3)} \). Thus \( \lambda_3 = \mu_R(x_1 \land x_2, y_1 \land y_2) > \lambda_2 \) and \( \mu_3 = \nu_R(x_1 \land x_2, y_1 \land y_2) < \mu_3 \). This is a contradiction. Then \( \lambda_3 \geq \lambda_1 \) and \( \mu_3 \leq \mu_1 \). So

\[
\mu_R(x_1 \land x_2, y_1 \land y_2) = \lambda_3 \geq \lambda_1
\]

\[
= \mu_R(x_1, y_1) \land \mu_R(x_2, y_2) \geq \mu_R(x_1, y_1) \land \mu_R(x_2, y_2)
\]

and

\[
\nu_R(x_1 \land x_2, y_1 \land y_2) = \mu_3 \leq \mu_1
\]

\[
= \nu_R(x_1, y_1) \lor \nu_R(x_2, y_2) \leq \nu_R(x_1, y_1) \lor \nu_R(x_2, y_2).
\]

By the similar arguments, we have

\[
\mu_R(x_1 \lor x_2, y_1 \lor y_2) \geq \mu_R(x_1, y_1) \land \mu_R(x_2, y_2)
\]

and

\[
\nu_R(x_1 \lor x_2, y_1 \lor y_2) \leq \nu_R(x_1, y_1) \lor \nu_R(x_2, y_2).
\]
Hence $R$ is intuitionistic fuzzy $(t,s)$-compatible. This
completes the proof.

The following is the immediate result of Definition 3.1
and Proposition 3.2.

**Corollary 3.2.** Let $R \in IFR(L)$, let $\alpha = \bigvee_{x,y \in L} \mu_R(x,y)$, 
$\beta = \bigwedge_{x,y \in L} \nu_R(x,y)$ and let $(\lambda, \mu) \in \text{Im}R \setminus \{(\alpha, \beta)\}$.

1. $R \in IF\text{E}_{(\alpha, \beta)}((t,s))(L)$ if and only if $R^{(\lambda, \mu)}$ is an 
equivalence relation on $L$.

2. $R \in IF\text{C}_{(\alpha, \beta)}((t,s))(L)$ if and only if $R^{(\lambda, \mu)}$ is a
congruence.

**Remark 3.3.** If $R \in IF\text{C}_{(t,s)}(L)$, then $\mu_R(x \land z, y \land z) \geq 
\mu_R(x, y), \nu_R(x \land z, y \land z) \leq \nu_R(x, y)$ and $\mu_R(x \lor z, y \lor z) \geq 
\mu_R(x, y), \nu_R(x \lor z, y \lor z) \leq \nu_R(x, y)$ for any $x, y, z \in L$.

**Proposition 3.4.** The intersection of family of intuitionistic fuzzy 
$(t,s)$-congruences [resp. $(\lambda, \mu)$-$(t,s)$-congruences] on $L$ is also so.

**Proof.** Let $\{R_\alpha\}_{\alpha \in \Gamma}$ be a family of intuitionistic fuzzy 
$(t,s)$-congruences [resp. $(\lambda, \mu)$-$(t,s)$-congruences] on $L$ and let $R = \bigcap_{\alpha \in \Gamma} R_\alpha$. Then, by Proposition 2.5, $R \in 
IF\text{E}_{(t,s)}(L)$ [resp. $IF\text{C}_{(t,s)}(L)$]. Let $x_1, x_2, y_1, y_2 \in L$. Then

\[
\begin{align*}
\mu_R(x_1 \land x_2, y_1 \land y_2) \\
\geq \bigwedge_{\alpha \in \Gamma} \mu_{R_\alpha}(x_1, y_1) \land \mu_{R_\alpha}(x_2, y_2)
\end{align*}
\]

(Since $R_\alpha$ is $(t,s)$-compatible)

\[
\begin{align*}
&= \bigwedge_{\alpha \in \Gamma} \mu_{R_\alpha}(x_1, y_1) \\
&= \bigwedge_{\alpha \in \Gamma} \mu_{R_\alpha}(x_1, y_1) \land \mu_{R_\alpha}(x_2, y_2)
\end{align*}
\]

and

\[
\begin{align*}
\nu_R(x_1 \land x_2, y_1 \land y_2) \\
\leq \bigwedge_{\alpha \in \Gamma} \nu_{R_\alpha}(x_1, y_1)
\end{align*}
\]

By the similar arguments, we have

\[
\begin{align*}
\mu_R(x_1 \lor x_2, y_1 \lor y_2) \geq \mu_R(x_1, y_1) \lor \mu_R(x_2, y_2) \\
\nu_R(x_1 \lor x_2, y_1 \lor y_2) \leq \nu_R(x_1, y_1) \lor \nu_R(x_2, y_2).
\end{align*}
\]

So $R$ is intuitionistic fuzzy $(t,s)$-compatible. Hence $R \in 
IF\text{E}_{(t,s)}(L)$ [resp. $IF\text{C}_{(\lambda, \mu)}((t,s))(L)$].

**Proposition 3.5.** Let $R \in IFR(L)$. Then $R \in IF\text{C}(L)$ if
and only if $R^{(\lambda, \mu)}$ is an equivalence relation on $L$ for each $(\lambda, \mu) \in 
\text{Im}R$.

**Proof.** By Result 1.4, $R \in IF\text{E}(L)$ if and only if $R^{(\lambda, \mu)}$ is an 
equivalence relation on $L$ for each $(\lambda, \mu) \in \text{Im}R$. Then it is sufficient to show that $R$ is intuitionistic fuzzy compatible if and only if $R^{(\lambda, \mu)}$ is compatible.

$(\Rightarrow)$ Suppose $R$ is intuitionistic fuzzy compatible. For any $x_1, x_2, y_1, y_2 \in L$, let $(x_1, y_1) \in R^{(\lambda, \mu)}$ and $(x_2, y_2) \in R^{(\lambda, \mu)}$. Then

\[
\mu_R(x_1, y_1) \geq \lambda, \nu_R(x_1, y_1) \leq \mu
\]

and

\[
\mu_R(x_2, y_2) \geq \lambda, \nu_R(x_2, y_2) \leq \mu.
\]

Since $R$ is intuitionistic fuzzy compatible,

\[
\begin{align*}
\mu_R(x_1 \land x_2, y_1 \land y_2) &\geq \mu_R(x_1, y_1) \land \mu_R(x_2, y_2) \\
\nu_R(x_1 \land x_2, y_1 \land y_2) &\leq \nu_R(x_1, y_1) \lor \nu_R(x_2, y_2)
\end{align*}
\]

Thus $(x_1 \land x_2, y_1 \land y_2) \in R^{(\lambda, \mu)}$. By the similar arguments, we have $(x_1 \lor x_2, y_1 \lor y_2) \in R^{(\lambda, \mu)}$. So $R^{(\lambda, \mu)}$ is compatible.

$(\Leftarrow)$ Suppose $R^{(\lambda, \mu)}$ is compatible for each $(\lambda, \mu) \in \text{Im}R$. For any $x_1, x_2, y_1, y_2 \in L$, let $R(x_1, y_1) = (\lambda_1, \mu_1)$ and $R(x_2, y_2) = (\lambda_2, \mu_2)$ such that $\lambda_1 \geq \lambda_2$ and $\mu_1 \leq \mu_2$. Then $(x_1, y_1) \in R^{(\lambda_1, \mu_1)}$ and $(x_2, y_2) \in R^{(\lambda_2, \mu_2)}$. By the hypothesis, $(x_1 \land x_2, y_1 \land y_2) \in R^{(\lambda_1, \mu_1)}$ and $(x_1 \lor x_2, y_1 \lor y_2) \in R^{(\lambda_2, \mu_2)}$. Thus

\[
\begin{align*}
\mu_R(x_1 \land x_2, y_1 \land y_2) &\geq \lambda_1 \land \lambda_2 \\
\mu_R(x_1 \lor x_2, y_1 \lor y_2) &\geq \mu_1 \lor \mu_2
\end{align*}
\]

and

\[
\begin{align*}
\nu_R(x_1 \land x_2, y_1 \land y_2) &\leq \mu_1 \land \lambda_2 \\
\nu_R(x_1 \lor x_2, y_1 \lor y_2) &\leq \mu_1 \lor \mu_2.
\end{align*}
\]

So $R$ is intuitionistic fuzzy compatible. This completes the proof.

**Definition 3.6.** Let $L$ and $M$ be lattices. We define two operations $\land$ and $\lor$ on $L \times M$, respectively as follows:

\[
(a, b) \land (a_1, b_1) = (a \land a_1, b \land b_1)
\]

and

\[
(a, b) \lor (a_1, b_1) = (a \lor a_1, b \lor b_1).
\]

Then $L \times M$ is a lattice. In this case, $L \times M$ is called the direct product of $L$ and $M$ denoted by $L \times M$.
Result 3.A[8, Theorem 13 in P.28]. Let $L$ and $M$ be lattices, let $P$ be a congruence on $L$ and let $Q$ a congruence on $M$. We define the relation $P \times Q$ on $L \times M$ by

$$(a, b) \equiv (c, d)(P \times Q)$$

if and only if

$$a \equiv c(P) \quad \text{and} \quad b \equiv d(Q).$$

Then $P \times Q$ is congruence on $L \times M$. Conversely, every congruence on $L \times M$ is of this form.

Result 3.B[8, Lemma 8 in P.24]. Let $R$ be a reflexive and symmetric relation on a lattice $L$. Then $R$ is a congruence on $L$ if and only if the following conditions hold: for any $x, y, z, t \in L$,

(i) $x \equiv y(R)$ iff $x \land y \equiv x \lor y(R)$.

(ii) $x \leq y \leq z \equiv y(R)$ and $y \leq z(R)$ imply that $x \equiv z(R)$.

(iii) $x \leq y$ and $y \equiv z(R)$ imply that $x \land t \equiv y \land t(R)$ and $x \lor t \equiv y \lor t(R)$.

Result 3.C[8, Theorem 9 in P.25]. Let $C(L)$ denote the set of all congruences on a poset $L$. For any $P, Q \in C(L)$, we define $P \land Q \equiv P \cap Q$ and the join, $P \lor Q$, as follows: for any $x, y \in L$, $x \equiv y(P \lor Q)$ if and only if there is a sequence $z_0 = x \land y, z_1, \ldots, z_{n-1} = x \lor y$ in $L$ such that $z_0 \leq z_1 \leq \cdots \leq z_{n-1}$ and for each $i$, $0 \leq i < n-1, z_i \equiv z_{i+1}(P)$ or $z_i \equiv z_{i+1}(Q)$. Then $C(L)$ is called the congruence lattice of $L$, where $C(L)$ denotes the set of all congruences on $L$ partially ordered by set inclusion.

Proposition 3.7. Let $L$ and $M$ be lattices, and let $P \in \text{IFR}(L), Q \in \text{IFR}(M)$. We define a complex mapping $P \times Q = (\mu_{P \times Q}, \nu_{P \times Q}) : (L \times M) \times (L \times M) \rightarrow I \times I$ as follows: for any $(x_1, y_1), (x_2, y_2) \in L \times M$,

$$\mu_{P \times Q}((x_1, y_1), (x_2, y_2)) = \mu_P(x_1, x_2) \land \mu_Q(y_1, y_2)$$

and

$$\nu_{P \times Q}((x_1, y_1), (x_2, y_2)) = \nu_P(x_1, x_2) \lor \nu_Q(y_1, y_2).$$

Then $(P \times Q)^{(\lambda, \mu)} = P^{(\lambda, \mu)} \times Q^{(\lambda, \mu)}$ for each $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$.

Proof. It is clear that $P \times Q$ is an IFR on $L \times M$. Let $(x_1, y_1), (x_2, y_2) \in L \times M$ and let $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$. Then

$$(x_1, y_1), (x_2, y_2) \in (P \times Q)^{(\lambda, \mu)} \iff \mu_{P \times Q}((x_1, y_1), (x_2, y_2)) = \mu_{P \times Q}((x_1, y_1), (x_2, y_2)) \geq \lambda$$

and

$$\nu_{P \times Q}((x_1, y_1), (x_2, y_2)) = \nu_{P \times Q}((x_1, y_1), (x_2, y_2)) \geq \mu_{P \times Q}((x_1, y_1), (x_2, y_2)) \leq \mu$$

$$\Rightarrow \mu_P(x_1, x_2) \geq \lambda, \mu_Q(y_1, y_2) \geq \lambda$$

and

$$\nu_P(x_1, x_2) \leq \mu, \nu_Q(y_1, y_2) \leq \mu$$

$$\Rightarrow \mu_P(x_1, x_2) \geq \lambda, \nu_P(x_1, x_2) \leq \mu$$

and

$$\mu_Q(y_1, y_2) \geq \lambda, \nu_Q(y_1, y_2) \leq \mu$$

$$\Rightarrow (x_1, x_2) \in P^{(\lambda, \mu)} \quad \text{and} \quad (y_1, y_2) \in Q^{(\lambda, \mu)}$$

$$\Rightarrow ((x_1, x_2), (y_1, y_2)) \in (P \times Q)^{(\lambda, \mu)}.$$
and

\[ \nu_R((x_1, y), (x_2, y)) \]
\[ = \nu_R((x_1, y), (x_2, y)) \lor \nu_R((x_1 \land x_2, y'), (x_1 \land x_2, y')) \]
\[ \geq \nu_R((x_1, y) \lor (x_1 \land x_2, y'), (x_2, y) \lor (x_1 \land x_2, y')) \]
\[ = \nu_R((x_1 \lor (x_1 \land x_2), y \lor y'), (x_2 \lor (x_1 \land x_2), y \lor y')) \]
\[ = \nu_R((x_1 \lor (x_1 \land x_2), y \lor y'), (x_2, y \lor y')) \]
\[ = \nu_R((x_1 \lor (x_1 \land x_2), y \lor y'), (x_2, y \lor y') \lor \nu_R((x_1 \lor (x_1 \land x_2), y \lor y'), (x_2, y \lor y')) \]
\[ \geq \nu_R((x_1 \lor (x_1 \land x_2), y \lor y'), (x_2, y \lor y') \lor \nu_R((x_1 \lor (x_1 \land x_2), y \lor y'), (x_2, y \lor y')) \]
\[ = \nu_R((x_1 \land (x_1 \lor (x_1 \land x_2), y \lor y'), (x_2, y \lor y') \land (x_1 \lor (x_1 \land x_2), y \lor y')) \]
\[ = \nu_R((x_1 \land (x_1 \lor (x_1 \land x_2), y \lor y'), (x_2, y \lor y')) \land (x_1 \lor (x_1 \land x_2), y \lor y')) \]

By the similar arguments, we have \( \mu_R((x_1, y), (x_2, y)) \geq \mu_R((x_1, y'), (x_2, y')) \) and \( \nu_R((x_1, y), (x_2, y)) \leq \nu_R((x_1, y'), (x_2, y')) \) for each \((y, y') \in M \times M\). Hence \( R((x_1, y), (x_2, y)) = R((x_1, y'), (x_2, y')) \) for any \(x_1, x_2 \in L\) and any \(y, y' \in M\).

(2) By the similar arguments of the proof of (1), we can see that (2) holds.

**Proposition 3.10.** Let \( R \in IFC(L \times M) \). Then there exist \( P \in IFC(L) \) and \( Q \in IFC(M) \) such that \( R = P \times Q \).

**Proof.** We define two complex mappings \( P = (\mu_P, \nu_P) : L \times L \rightarrow I \times I \) and \( Q = (\mu_Q, \nu_Q) : M \times M \rightarrow I \times I \) as follows:

(i) \( P((x_1, x_2), (x_1, x_2)) = R((x_1, y_1), (x_2, y_2)) \) for any \(x_1, x_2 \in L\) and each \(y, y' \in M\),

(ii) \( Q((y_1, y_2), (y_1, y_2)) = R((x_1, y_1), (x_2, y_2)) \) for each \(x, y \in L\) and any \(y, y' \in M\).

Then clearly \( P \in IFR(L) \) and \( Q \in IFR(M) \). Moreover, by Lemma 3.9, \( P \) and \( Q \) are well-defined. Let \((\lambda, \mu) \in I \times I \) with \( \lambda + \mu \leq 1 \).

Then \( (x_1, x_2) \in R^{(\lambda, \mu)} \)
\[ \Leftrightarrow ((x_1, y_1), (x_2, y_2)) \in R^{(\lambda, \mu)} \] for each \(y \in M\)
\[ \Leftrightarrow ((x_1, y_1), (x_2, y_2)) \in R^{(\lambda, \mu)} \] for each \(x \in L\).

By Proposition 3.5, \( R^{(\lambda, \mu)} \) and \( Q^{(\lambda, \mu)} \) are congruences on \( L \) and \( M \), respectively. Moreover, by Result 3.B, \( R^{(\lambda, \mu)} = P^{(\lambda, \mu)} \times Q^{(\lambda, \mu)} \). By proposition 3.8, \( P^{(\lambda, \mu)} \times Q^{(\lambda, \mu)} = (P \times Q)^{(\lambda, \mu)} \).

So \( R = P \times Q \). Moreover, by Proposition 3.5, \( P \in IFC(L) \) and \( Q \in IFC(M) \). This completes the proof.

**Proposition 3.11.** Let \( R \in IFR(L) \) be intuitionistic fuzzy \((t, s)\)-transitive such that \( \mu_R((x \land z, y \land z)) \geq \mu_R((x, y)) \) and \( \nu_R((x \lor z, y \lor z)) \leq \nu_R((x, y)) \) for any \(x, y, z \in L\). Then \( R \) is intuitionistic fuzzy \((t, s)\)-compatible.

**Proof.** Let \( x_1, x_2, y_1, y_2 \in L \).

\[ \mu_R((x_1 \land x_2, y_1 \land y_2)) \]
\[ \geq \bigvee_{z \in L} [\mu_R((x_1 \land x_2, z) \land \mu_R((z, y_1 \land y_2))] \]
(Since \( R \) is \((t, s)\)-transitive)
\[ \geq \mu_R((x_1 \land x_2, y_1 \land y_2) \land \mu_R((y_1 \land y_2, x_1 \land x_2)) \]
\[ \geq \mu_R((x_1, y_1) \land \mu_R((x_2, y_2))) \]
(By the hypotheses)
\[ \land \mu_R((x_1 \land x_2, y_1 \land y_2)) \]
\[ \leq \mu_R((x_1 \land x_2, y_1 \land y_2)) \]
\[ \leq \nu_R((x_1 \land x_2, y_1 \land y_2)) \]
\[ \leq \nu_R((x_1 \land x_2, y_1 \land y_2)) \]
\[ \leq \nu_R((x_1 \land x_2, y_1 \land y_2)) \]
\[ \leq \nu_R((x_1 \land x_2, y_1 \land y_2)) \]
\[ \leq \nu_R((x_1 \land x_2, y_1 \land y_2)) \]
\[ \leq \nu_R((x_1 \land x_2, y_1 \land y_2)) \]
\[ \leq \nu_R((x_1 \land x_2, y_1 \land y_2)) \]
\[ \leq \nu_R((x_1 \land x_2, y_1 \land y_2)) \]
\[ \leq \nu_R((x_1 \land x_2, y_1 \land y_2)) \]
\[ \leq \nu_R((x_1 \land x_2, y_1 \land y_2)) \]
\[ \leq \nu_R((x_1 \land x_2, y_1 \land y_2)) \]

Similarly, we have

\[ \mu_R((x_1 \lor x_2, y_1 \lor y_2)) \geq \mu_R((x_1, y_1) \land \mu_R((x_2, y_2))) \land \mu_R((x_1 \lor x_2, z) \land \mu_R((z, y_1 \lor y_2))) \]
\[ \geq \mu_R((x_1, y_1) \land \mu_R((x_2, y_2))) \land \mu_R((x_1 \lor x_2, y_1 \lor y_2)) \]
\[ \leq \nu_R((x_1 \lor x_2, y_1 \lor y_2)) \]
\[ \leq \nu_R((x_1 \lor x_2, y_1 \lor y_2)) \]
\[ \leq \nu_R((x_1 \lor x_2, y_1 \lor y_2)) \]

Hence \( R \) is intuitionistic fuzzy \((t, s)\)-compatible.

The following is the immediate result of Proposition 3.11.

**Corollary 3.11.** Let \( R \in IFR(L) \) be intuitionistic fuzzy transitive such that \( \mu_R((x \land z, y \land z)) \geq \mu_R((x, y)) \) and \( \mu_R((x \lor z, y \lor z)) \leq \mu_R((x, y)) \) for any \(x, y, z \in L\). Then \( R \) is intuitionistic fuzzy transitive.

**Proposition 3.12.** If \( R \in IFC(L) \), then \( R((x, y)) = R((x \land y, x \lor y)) \) for any \(x, y \in L\).

**Proof.** Let \( x, y \in L \).

\[ \mu_R((x \land y, x \lor y)) \]
\[ = \mu_R((x \land y, x \lor y) \land \mu_R((x, x)) \]
(Since \( R \) is intuitionistic fuzzy reflexive)
\[ \leq \mu_R((x \land y) \land (x \lor y) \land (x \lor y)) \]
(Since \( R \) is intuitionistic fuzzy transitive)
\[ \leq \mu_R((x \land y) \land (x \lor y) \land (x \lor y)) \]
(Since \( R \) is intuitionistic fuzzy symmetric)
and
\[
\nu_R(x \land y, x \lor y) \\
\geq \nu_R((x \land y) \land x, (x \lor y) \land x) \\
\geq \nu_R(x \land y, x \lor y) = \nu_R(x, x \lor y).
\]

Also
\[
\mu_R(x \land y, x \lor y) = \mu_R(x \land y, x \lor y) \land \mu_R(y, y) \\
(\text{Since } R \text{ is intuitionistic fuzzy reflexive}) \\
\leq \mu_R((x \land y) \lor y, (x \lor y) \lor y) \\
(\text{Since } R \text{ is intuitionistic fuzzy compatible}) \\
\leq \mu_R(y, x \lor y) \\
= \mu_R(x \lor y, y) \\
(\text{Since } R \text{ is intuitionistic fuzzy symmetric})
\]

and
\[
\nu_R(x \land y, x \lor y) \\
= \nu_R(x \land y, x \lor y) \lor \nu_R(y, y) \\
\geq \nu_R((x \land y) \lor y, (x \lor y) \lor y) \\
\geq \nu_R(y, x \lor y) = \nu_R(x \lor y, y).
\]

Thus
\[
\mu_R(x, y) \\
\geq \bigvee_{z \in L} \left[\mu_R(x, z) \land \mu_R(z, y)\right] \\
(\text{Since } R \text{ is intuitionistic fuzzy transitive}) \\
\geq \mu_R(x, x \land y) \land \mu_R(x \land y, x \lor y) \land \mu_R(x \lor y, y) \\
= \mu_R(x \land y, x \lor y)
\]

and
\[
\nu_R(x, y) \\
\leq \bigwedge_{z \in L} \left[\nu_R(x, z) \lor \nu_R(z, y)\right] \\
\leq \nu_R(x, x \land y) \lor \nu_R(x \land y, x \lor y) \lor \nu_R(x \lor y, y) \\
= \nu_R(x \land y, x \lor y).
\]

On the other hand,
\[
\mu_R(x, y) = \mu_R(x, y) \land \mu_R(y, y) \\
(\text{Since } R \text{ is intuitionistic fuzzy reflexive}) \\
\leq \mu_R(x \land y, y \land y) \\
(\text{Since } R \text{ is intuitionistic fuzzy compatible}) \\
= \mu_R(x \land y, y)
\]

and
\[
\nu_R(x, y) = \nu_R(x, y) \lor \nu_R(y, y) \geq \nu_R(x \land y, y \land y) \\
= \nu_R(x \land y, y).
\]

Similarly, we have
\[
\mu_R(x, y) \leq \mu_R(x \lor y, y)
\]
and
\[
\nu_R(x, y) \geq \nu_R(x \lor y, y).
\]

Thus
\[
\mu_R(x, y) \leq \mu_R(x \land y, y) \land \mu_R(x \lor y, y) \\
= \mu_R(x \land y, y) \land \mu_R(y, x \lor y) \\
\leq \mu_R(x \land y, x \lor y)
\]

and
\[
\nu_R(x, y) \geq \nu_R(x \land y, y) \lor \nu_R(x \lor y, y) \\
= \nu_R(x \land y, y) \lor \nu_R(y, x \lor y) \\
\geq \nu_R(x \land y, x \lor y).
\]

Hence \(R(x, y) = R(x \land y, x \lor y)\) for any \(x, y \in L\).

**Proposition 3.13.** Let \(R \in \text{IFR}(L)\) be intuitionistic fuzzy reflexive and compatible such that \(R(b, c) = R(b \land c, b \lor c)\) for any \(b, c \in [a, d] \subset L\). Then \(\mu_R(b, c) \geq \mu_R(a, d)\) and \(\nu_R(b, c) \leq \nu_R(a, d)\).

**Proof.** Let \(b, c \in [a, d] \subset L\). Then
\[
\mu_R(b \land c, d) = \mu_R(a \lor (b \land c), d \lor (b \land c)) \\
(\text{Since } b \land c \geq a \text{ and } b \land c \leq d) \\
\geq \mu_R(a, d) \land \mu_R(b \land c, b \land c) \\
(\text{Since } R \text{ is intuitionistic fuzzy compatible}) \\
= \mu_R(a, d) \\
(\text{Since } R \text{ is intuitionistic fuzzy reflexive})
\]

and
\[
\nu_R(b \land c, d) = \nu_R(a \lor (b \land c), d \lor (b \land c)) \\
\leq \nu_R(a, d) \lor \nu_R(b \land c, b \land c) \\
= \nu_R(a, d).
\]

Thus
\[
\mu_R(b, c) = \mu_R(b \land c, b \lor c) \\
(\text{By the hypothesis}) \\
= \mu_R((b \land c) \land (b \lor c), d \land (b \lor c)) \\
(\text{Since } b \land c \leq b \lor c, \text{ and } b \lor c \leq d) \\
\geq \mu_R(b \land c, d) \land \mu_R(b \lor c, b \lor c) \\
(\text{Since } R \text{ is intuitionistic fuzzy compatible}) \\
= \mu_R(b \land c, d) \\
(\text{Since } R \text{ is intuitionistic fuzzy reflexive}) \\
\geq \mu_R(a, d)
\]
and

\[ \nu_R(b, c) = \nu_R(b \wedge c, b \vee c) = \nu_R((b \wedge c) \wedge (b \vee c), d \wedge (b \vee c)) \leq \nu_R(b \wedge c, d) \vee \nu_R(b \vee c, b \vee c) \leq \nu_R(b \wedge c, d) \leq \nu_R(a, d). \]

This completes the proof.

**Question.** Can we find an analogue of Result 3.B by intuitionistic fuzzy setting? That is, let \( R \) be an intuitionistic fuzzy reflexive and symmetric relation on a lattice \( L \). Then, under certain conditions, \( R \in IFC(L) \) [resp. \( R \in IFC_{(t,s)}(L) \)]?

It is clear that \( ITC(L) \) is closed with respect to \( \cap \) from Proposition 3.4. So \( P \wedge Q = P \cap Q \) for any \( P, Q \in ITC(L) \).

**Definition 3.14.** Let \( P, Q \in ITC(L) \). The supremum of \( P \) and \( Q \) is defined to be the IFS \( P \vee Q = \cap \{ R \in ITC(L) : P \cup Q \subseteq R \} \).

It is clear that \( P \vee Q \in ITC(L) \). So \( (ITC(L), \leq, \wedge, \vee) \) is a lattice.

**Proposition 3.15.** Let \( P, Q \in IFC(L) \). Then \( (P \vee Q)^{(\lambda, \mu)} = P^{(\lambda, \mu)} \vee Q^{(\lambda, \mu)} \) for each \( (\lambda, \mu) \in \text{Im}(P \vee Q) \).

**Proof.** Let \( R \in IFR(L) \) such that \( \text{Im}R = \text{Im}(P \vee Q) \) and let \( R^{(\lambda, \mu)} = P^{(\lambda, \mu)} \vee Q^{(\lambda, \mu)} \) for each \( (\lambda, \mu) \in \text{Im}R \). Then \( R^{(\lambda, \mu)} \) is a congruence on \( L \) for each \( (\lambda, \mu) \in \text{Im}R \). By Proposition 3.5, \( R \in IFC(L) \). Moreover, by Proposition 2.3, \( P \cup Q \subseteq R \). Let \( R \in IFC(L) \) with \( P \cup Q \subseteq D \). Then \( (P \cup Q)^{(s, t)} = P^{(s, t)} \cup Q^{(s, t)} \subseteq D^{(s, t)} \) for each \( (s, t) \in I \times I \) with \( s + t \leq 1 \). Thus \( R^{(s, t)} \subseteq D^{(s, t)} \) for each \( (s, t) \in I \times I \) with \( s + t \leq 1 \). So \( R \in C, D \), i.e., \( R = P \vee Q \). Hence \( (P \vee Q)^{(\lambda, \mu)} = P^{(\lambda, \mu)} \vee Q^{(\lambda, \mu)} \) for each \( (\lambda, \mu) \in \text{Im}(P \vee Q) \).

From Proposition 2.3, Proposition 3.15 and Result 3.C, we obtain the following result

**Proposition 3.16.** Let \( P, Q \in IFC(L) \) and \( (s, t) \in I \times I \) with \( s + t \leq 1 \). Then \( \mu_{P \vee Q}(x, y) \geq s \) and \( \nu_{P \vee Q}(x, y) \leq t \) for any \( x, y \in L \) if and only if there is a sequence \( x \wedge y = z_1 \leq \cdots \leq z_n = x \vee y \) in \( L \) such that \( \mu_{P \vee Q}(z_i, z_{i+1}) \geq s \) and \( \nu_{P \vee Q}(z_i, z_{i+1}) \leq t \) for each \( i \in \{1, 2, \ldots, n - 1\} \).

**Proposition 3.17.** The lattice \( IFC(L) \) is distributive.

**Proof.** Let \( P, Q, R \in IFC(L) \). For any \( x, y \in L \), let \( P(x, y) = (s_1, t_1) \) and \( Q \breve{\cap} R(x, y) = (s_2, t_2) \). Then, by Proposition 3.16, there is a sequence \( x \wedge y = z_1 \leq \cdots \leq z_n = x \vee y \) in \( L \) such that \( \mu_{P \breve{\cap} Q}(z_i, z_{i+1}) \geq s_2 \) and \( \nu_{P \breve{\cap} Q}(z_i, z_{i+1}) \leq t_2 \) for each \( i \in \{1, 2, \ldots, n - 1\} \).

Since \( (x, y) \in P^{(s_1, t_1)} \) and \( Q \breve{\cap} R \in P^{(s_1, t_1)} \) is a congruence on \( L \), by Result 3.B, \( (x \wedge y, x \vee y) \in P^{(s_1, t_1)} \). By Result 3.C, \( (z_i, z_{i+1}) \in P^{(s_1, t_1)} \) for each \( i \in \{1, 2, \ldots, n - 1\} \). Let \( i \in \{1, 2, \ldots, n - 1\} \). Then

\[
\begin{align*}
\mu_{P \breve{\cap} Q}(x, y) &= \mu_{P}(x, y) \wedge \mu_{Q \breve{\cap} R}(x, y) \\
&= s_1 \wedge s_2 \\
&\leq \mu_{P}(z_i, z_{i+1}) \wedge (\mu_{Q}(z_i, z_{i+1}) \vee \mu_{R}(z_i, z_{i+1})) \\
&= \mu_{P \breve{\cap} Q}(z_i, z_{i+1}) \vee \mu_{P \vee R}(z_i, z_{i+1}) \\
&= \mu_{P \breve{\cap} Q}(x, y) \vee \mu_{P \vee R}(x, y),
\end{align*}
\]

and

\[
\begin{align*}
\nu_{P \breve{\cap} Q}(x, y) &= \nu_{P}(x, y) \vee \nu_{Q \breve{\cap} R}(x, y) = t_1 \vee t_2 \\
&\leq \nu_{P}(z_i, z_{i+1}) \vee (\nu_{Q}(z_i, z_{i+1}) \vee \nu_{R}(z_i, z_{i+1})) \\
&= \nu_{P \breve{\cap} Q}(z_i, z_{i+1}) \vee \nu_{P \vee R}(z_i, z_{i+1}) \\
&= \nu_{P \breve{\cap} Q}(x, y) \vee \nu_{P \vee R}(x, y).
\end{align*}
\]

Thus

\[
\mu_{P \breve{\cap} Q}(x, y) \leq \mu_{P \vee Q \breve{\cap} R}(x, y)
\]

and

\[
\nu_{P \breve{\cap} Q}(x, y) \geq \nu_{P \vee Q \breve{\cap} R}(x, y).
\]

So \( P \wedge (Q \vee R) \subseteq (P \wedge Q) \vee (P \wedge R) \). Similarly, we have \( (P \wedge Q) \vee (P \wedge R) \subseteq P \wedge (Q \vee R) \). Hence \( P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R) \). This completes the proof.

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