INTUITIONISTIC FUZZY G-CONGRUENCES

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Abstract

We introduce the concept of intuitionistic fuzzy G-equivalence relations (congruence), and we obtain some results. Furthermore, we prove that IFCG(K) is isomorphic to IFN*(K) for any group K. Also, we prove that (IFCGL(λ,μ)/, ∼, *) and (IFNGL(λ,μ)(K), o) are isomorphic.

Key words: intuitionistic fuzzy set, intuitionistic fuzzy G-equivalence relation, intuitionistic fuzzy G-congruence, intuitionistic fuzzy right (left) conformable.

1. 0. Introduction

The concept of a fuzzy sets was introduced by Zadeh[21] in 1965, and since then these has been a tremendous interest in the subject due to its diverse applications ranging from engineering and computer science to social behavior studies. In particular, many researchers [7,17,19,20,22] applied the notion of a fuzzy set to relations and congruences.

As a generalization of fuzzy sets, the concept of intuitionistic fuzzy sets was introduced by Atanassov[1] in 1986. After that time, various researchers [2-6,9-12,14] applied the notion of intuitionistic fuzzy sets to relation group theory and topology. In particular, Hur and his colleagues [13,15] introduce the notion of intuitionistic fuzzy congruences on a lattice and a semigroup, and investigate some of its properties, respectively. Moreover, Hur and his colleagues [16] studied intuitionistic fuzzy congruences in the sense of lattice.

In this paper, we introduce the concept of intuitionistic fuzzy G-equivalence relations (congruence), and we obtain some results. Furthermore, we prove that IFCG(K) is isomorphic to IFN*(K) for any group K, where IFCG(K) [resp. IFN*(K)] denotes the set of all intuitionistic fuzzy G-congruences on K [resp. intuitionistic fuzzy nonempty normal subgroups of G]. Also, we prove that (IFCGL(λ,μ)/, ∼, *) and (IFNGL(λ,μ)(K), o) are isomorphic.

2. Preliminaries

In this section, we list some basic concepts and one result which are needed in the later sections.

For sets X, Y and Z, f = (f₁, f₂) : X → Y × Z is called a complex mapping if f₁ : X → Y and f₂ : X → Z are mappings.

Throughout this paper, we will denote the unit interval [0, 1] as I.

Definition 2.1[1,5]. Let X be a nonempty set. A complex mapping A = (μ_A, ν_A) : X → I × I is called an intuitionistic fuzzy set (in short, IFS) in X if μ_A(x) + ν_A(x) ≤ 1 for each x ∈ X, where the mappings μ_A : X → I and ν_A : X → I denote the degree of membership (namely μ_A(x)) and the degree of nonmembership (namely ν_A(x)) of each x ∈ X to A, respectively. In particular, 0₁ and 1₁ denote the intuitionistic fuzzy empty set and the intuitionistic fuzzy whole set in X defined by 0₁(x) = (0, 1) and 1₁(x) = (1, 0) for each x ∈ X, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definitions 2.2[1]. Let X be a nonempty set and let A = (μ_A, ν_A) and B = (μ_B, ν_B) be IFSs in X. Then:

(1) A ⊆ B iff μ_A ≤ μ_B and ν_A ≥ ν_B.
(2) A = B iff A ⊆ B and B ⊆ A.
(3) A^c = (ν_A, μ_A).

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100
(4) \( A \cap B = (\mu_A \land \mu_B, \nu_A \lor \nu_B) \).
(5) \( A \cup B = (\mu_A \lor \mu_B, \nu_A \land \nu_B) \).

**Definition 2.3.** Let \( \{A_i\}_{i \in J} \) be an arbitrary family of IFSs in \( X \), where \( A_i = (\mu_{A_i}, \nu_{A_i}) \) for each \( i \in J \) then:
(1) \( \bigcap_{i \in J} A_i = (\bigwedge_{i \in J} \mu_{A_i}, \bigvee_{i \in J} \nu_{A_i}) \).
(2) \( \bigcup_{i \in J} A_i = (\bigvee_{i \in J} \mu_{A_i}, \bigwedge_{i \in J} \nu_{A_i}) \).

**Definition 2.4.** Let \( X \) be a set. Then a complex mapping \( R = (\mu_R, \nu_R) : X \times X \to I \times I \) is called an intuitionistic fuzzy relation (in short, IFR) on \( X \) if \( \mu_R(x, y) + \nu_R(x, y) \leq 1 \) for each \( (x, y) \in X \times X \), i.e., \( R \in \text{IFS}(X \times X) \).

We will denote the set of all IFRs on a set \( X \) as \( \text{IFR}(X) \).

**Definition 2.5.** Let \( R \in \text{IFR}(X) \). Then the inverse of \( R \), \( R^{-1} \), is defined as by \( R^{-1}(y, x) = R(y, x) \) for any \( x, y \in X \).

**Definition 2.6.** Let \( X \) be a set and let \( R, Q \in \text{IFR}(X) \). Then the composition of \( R \) and \( Q \), \( Q \circ R \), is defined as follows:
For any \( x, y \in X \),
\[
\mu_{Q \circ R}(x, y) = \bigvee_{z \in X} [\mu_R(x, z) \land \mu_Q(z, y)]
\]
and
\[
\nu_{Q \circ R}(x, y) = \bigwedge_{z \in X} [\nu_R(x, z) \lor \nu_Q(z, y)].
\]

**Definition 2.7.** An intuitionistic fuzzy relation \( R \) on a set \( X \) is called an intuitionistic fuzzy equivalence relation (in short, IFER) on \( X \) if it satisfies the following conditions:
(i) it is intuitionistic fuzzy reflexive, i.e., \( R(x, x) = (1, 0) \) for each \( x \in X \).
(ii) it is intuitionistic fuzzy symmetric, i.e., \( R^{-1} = R \).
(iii) it is intuitionistic fuzzy transitive, i.e., \( R \circ R \subseteq R \).

We will denote the set of all IFERs on \( X \) as \( \text{IFER}(X) \).

Let \( R \) be an intuitionistic fuzzy equivalence relation on a set \( X \) and let \( a \in X \). We define a complex mapping \( Ra : X \to I \times I \) as follows: For each \( x \in X \),
\[
Ra(x) = (R(a, x), 1).
\]
Then clearly \( Ra \in \text{IFS}(X) \). The intuitionistic fuzzy set \( Ra \) in \( X \) is called an intuitionistic fuzzy equivalence class of \( R \) containing \( a \) in \( X \). The set \( \{Ra : a \in X\} \) is called the intuitionistic fuzzy quotient set of \( X \) by \( R \) and denoted by \( X/R \).

**Result 2.1[A, Theorem 2.15].** Let \( R \) be an intuitionistic fuzzy equivalence relation on \( X \). Then the following hold:
(1) \( Ra = Rb \) if and only if \( R(a, b) = (1, 0) \) for any \( a, b \in X \).
(2) \( R(a, b) = (0, 1) \) if and only if \( Ra \cap Rb = \emptyset \), for any \( a, b \in X \).
(3) \( \bigcup_{a \in X} Ra = 1 \).
(4) There exists the surjection \( p : X \to X/R \) defined by \( p(x) = Rx \) for each \( x \in X \).

**3. Intuitionistic fuzzy \( G \)-equivalence relations**

**Definition 3.1.** Let \( R \) be an intuitionistic fuzzy relation on a set \( X \). Then \( R \) is said to be \( G \)-reflexive if for any \( x, y \in X \) with \( x \neq y \),
(i) \( \mu_R(x, x) > 0 \) and \( \nu_R(x, x) < 1 \),
(ii) \( \mu_R(x, y) \leq \delta_1(R) \) and \( \nu_R(x, y) \geq \delta_2(R) \), where \( \delta_1(R) = \bigwedge_{t \in X} \mu_R(t, t) \) and \( \delta_2(R) = \bigvee_{t \in X} \nu_R(t, t) \).

An intuitionistic fuzzy \( G \)-reflexive and transitive relation on \( X \) is called an intuitionistic fuzzy \( G \)-preorder on \( S \). An intuitionistic fuzzy symmetric \( G \)-preorder on \( X \) is called an intuitionistic fuzzy \( G \)-equivalence relation on \( X \). We will denote the set of all intuitionistic fuzzy \( G \)-equivalence relations on \( X \) as \( \text{IFER}_G(X) \).

**Proposition 3.2.** (1) If \( H \) and \( K \) are intuitionistic fuzzy \( G \)-reflexive relations on a set \( X \), then \((K \circ H)(x, x) = (H \cap K)(x, x) \) for each \( x \in X \).
(2) If \( R \) is an intuitionistic fuzzy \( G \)-preorder on a set \( X \), then \( R \circ R = R \).

**Proof.** (1) Let \( x \in X \). Then
\[
\mu_{H \circ K}(x, x) = \bigwedge_{t \in X} [\mu_H(x, t) \land \mu_K(t, x)]
\]
\[
= \mu_K(x, x) \land \mu_H(x, x) (\text{Since } H \text{ and } K \text{ are intuitionistic fuzzy } G \text{-reflexive})
\]
and
\[
\nu_{H \circ K}(x, x) = \bigwedge_{t \in X} [\nu_H(x, t) \lor \nu_K(t, x)]
\]
\[
= \nu_K(x, x) \lor \nu_H(x, x)
\]
Hence \((K \circ H)(x, x) = H \cap K(x, x) \) for each \( x \in X \).
(2) Since \( R \) is intuitionistic fuzzy transitive, \( R \circ R \subseteq R \).

Let \( x, y \in X \). Then
\[
\mu_{R \circ R}(x, y) = \bigwedge_{t \in X} [\mu_R(x, t) \land \mu_R(t, y)]
\]
\[
\geq \mu_R(x, x) \land \mu_R(x, y) (\text{Since } R \text{ is intuitionistic fuzzy } G \text{-reflexive})
\]
and
\[
\nu_{R \circ R}(x, y) = \bigwedge_{t \in X} [\nu_R(x, t) \lor \nu_R(t, y)]
\]
\[
\leq \nu_R(x, x) \lor \nu_R(x, y)
\]
Thus \( R \subseteq R \circ R \). Hence \( R \circ R = R \).

**Proposition 3.3.** If \( H \) and \( K \) are intuitionistic fuzzy \( G \)-equivalence relations on a set \( X \), then \( H \cap K \) is so on \( X \).

**Proof.** It is clear that \( H \cap K \) is intuitionistic fuzzy \( G \)-reflexive and intuitionistic fuzzy symmetric. Let \( x, y \in X \).
Then:
\[
\mu_{H \cap K}(x, y) = \mu_H(x, y) \land \mu_K(x, y) \\
\geq \mu_{H \cap K}(x, y) = \mu_{H \cap K}(x, y)
\]
(Since \(H \) and \(K \) are intuitionistic fuzzy transitive)
\[
(\forall_{z_1 \in X} [\mu_H(x, z_1) \land \mu_H(z_1, y)]) \\
\land (\forall_{z_2 \in X} [\mu_K(x, z_2) \land \mu_K(z_2, y)]) \\
\geq (\forall_{z_1 \in X} [\mu_H(x, z_1) \land \mu_H(z_1, y)]) \\
\land (\forall_{z_2 \in X} [\mu_K(x, z_2) \land \mu_K(z_2, y)]) \\
= \mu_{H \cap K}(x, z_1) \land \mu_{H \cap K}(z_1, y) \\
= \mu_{H \cap K}(x, y) \land \mu_{H \cap K}(y, x)
\]
and
\[
\nu_{H \cap K}(x, y) = \nu_H(x, y) \lor \nu_K(x, y) \\
\leq \nu_{H \cap K}(x, y) \lor \nu_{H \cap K}(y, x)
\]
\[
(\forall_{z_1 \in X} [\nu_H(x, z_1) \lor \nu_H(z_1, y)]) \\
\lor (\forall_{z_2 \in X} [\nu_K(x, z_2) \lor \nu_K(z_2, y)]) \\
\leq (\forall_{z_1 \in X} [\nu_H(x, z_1) \lor \nu_H(z_1, y)]) \\
\lor (\forall_{z_2 \in X} [\nu_K(x, z_2) \lor \nu_K(z_2, y)]) \\
= \nu_{H \cap K}(x, z_1) \lor \nu_{H \cap K}(z_1, y) \\
= \nu_{H \cap K}(x, y) \lor \nu_{H \cap K}(y, x).
\]
Thus \(H \cap K \) is intuitionistic fuzzy transitive. Hence \(H \cap K \) is an intuitionistic fuzzy G-equivalence relation on \(X \). \(\square\)

If \(H \) and \(K \) are intuitionistic fuzzy G-reflexive relation on a set \(X \), then \(K \circ H \) may not be intuitionistic fuzzy G-reflexive.

**Example 3.4.** Let \(X = \{a, b\} \). Let \(H \) and \(K \) be the intuitionistic fuzzy relations defined as follows:
\[
H(a, a) = (1, 0), \quad H(b, b) = (\frac{1}{4}, \frac{1}{2})
\]
\[
H(a, b) = (\frac{1}{4}, \frac{3}{4}), \quad H(b, a) = (\frac{1}{3}, \frac{2}{3})
\]
and
\[
K(a, a) = (1, 0), \quad K(b, b) = (\frac{1}{2}, \frac{1}{4})
\]
\[
K(a, b) = (\frac{1}{3}, \frac{1}{2}), \quad K(b, a) = (\frac{1}{2}, \frac{3}{4}).
\]
Then clearly, \(H \) and \(K \) are both intuitionistic fuzzy G-reflexive on \(X \). But
\[
\mu_{K \circ H}(a, b) = \frac{1}{2} > \frac{1}{3} = \mu_{K \circ H}(b, b)
\]
and
\[
\nu_{K \circ H}(a, b) = \frac{1}{4} < \frac{2}{3} = \nu_{K \circ H}(b, b).
\]
So \(K \circ H \) is not intuitionistic fuzzy G-reflexive on \(X \).

**Proposition 3.5.** Let \(H \) and \(K \) be intuitionistic fuzzy G-reflexive relations on a set \(X \) such that \(\mu_{H \circ K}(x, y) \lor \mu_{K \circ H}(y, x) \leq \delta_1(H) \land \delta_1(K) \) and \(\nu_{H \circ K}(x, y) \lor \nu_{K \circ H}(y, x) \geq \delta_2(H) \lor \delta_2(K) \) for any \(x, y \in X \) with \(x \neq y \). Then \(K \circ H \) is intuitionistic fuzzy G-reflexive on \(X \) with \(\delta_1(H \circ K) = \delta_1(H) \land \delta_1(K) \) and \(\delta_2(H \circ K) = \delta_2(H) \lor \delta_2(K) \).

**Proof.** Let \(x \in X \). Since \(H \) and \(K \) are intuitionistic fuzzy G-reflexive, by Proposition 3.2(1), \(\mu_{K \circ H}(x, x) = \mu_H(x, x) \lor \mu_K(x, x) > 0 \) and \(\nu_{K \circ H}(x, x) = \nu_H(x, x) \lor \nu_K(x, x) < 1 \). Thus
\[
\delta_1(K \circ H) = (\forall_{t \in X} \mu_{K \circ H}(t, t)) \\
= (\forall_{t \in X} \mu_H(t, t) \land \mu_K(t, t)) \\
= (\forall_{t \in X} \mu_H(t, t)) \land (\forall_{t \in X} \mu_K(t, t)) \\
= \delta_1(H) \land \delta_1(K)
\]
and
\[
\delta_2(K \circ H) = (\forall_{t \in X} \nu_{K \circ H}(t, t)) \\
= (\forall_{t \in X} \nu_H(t, t)) \lor (\forall_{t \in X} \nu_K(t, t)) \\
= \delta_2(H) \lor \delta_2(K).
\]
Now let \(x, y \in X \) with \(x \neq y \), and let \(t \in X \) with \(x \neq t \neq y \). Since \(H \) and \(K \) are intuitionistic fuzzy G-reflexive, \(\mu_H(x, t) \land \mu_K(t, y) \leq \delta_1(H) \land \delta_1(K) \) and \(\nu_H(x, t) \lor \nu_K(t, y) \geq \delta_2(H) \lor \delta_2(K) \). Also, by the hypothesis,
\[
\mu_H(x, y) \land \mu_K(x, y) \leq \mu_K(x, y) \\
\leq \mu_H(x, y) \lor \mu_K(x, y) \\
\leq \delta_1(H) \land \delta_1(K),
\]
\[
\nu_H(x, y) \lor \nu_K(x, y) \geq \nu_K(x, y) \\
\geq \nu_H(x, y) \lor \nu_K(x, y) \\
\geq \delta_2(H) \lor \delta_2(K),
\]
and
\[
\mu_H(x, y) \land \mu_K(y, x) \leq \mu_H(x, y) \\
\leq \mu_H(x, y) \lor \mu_K(x, y) \\
\leq \delta_1(H) \land \delta_1(K),
\]
\[
\nu_H(x, y) \lor \nu_K(y, x) \geq \nu_K(y, x) \\
\geq \nu_H(x, y) \lor \nu_K(y, x) \\
\geq \delta_2(H) \lor \delta_2(K).
\]
So \(\mu_{K \circ H}(x, y) \leq \delta_1(H \circ K) = \delta_1(H) \land \delta_1(K) \) and \(\nu_{K \circ H}(x, y) \geq \delta_2(H \circ K) = \delta_2(H) \lor \delta_2(K) \). Hence \(K \circ H \) is intuitionistic fuzzy G-reflexive with \(\delta_1(K \circ H) = \delta_1(H \circ K) = \delta_1(H) \land \delta_1(K) \) and \(\delta_2(K \circ H) = \delta_2(H) \lor \delta_2(K) \). \(\square\)

The following is the immediate result of Proposition 3.5.

**Corollary 3.5.** Let \(H \) and \(K \) be intuitionistic fuzzy G-reflexive relation on a set \(X \) with \(\delta_1(H) = \delta_1(K) \) and \(\delta_2(H) = \delta_2(K) \). Then \(K \circ H \) is intuitionistic fuzzy G-reflexive with \(\delta_1(K \circ H) = \delta_1(H \circ K) = \delta_1(H) \land \delta_1(K) \) and \(\delta_2(K \circ H) = \delta_2(H) \lor \delta_2(K) \).

**Proposition 3.6.** Let \(H \) and \(K \) be intuitionistic fuzzy symmetric relations on a set \(X \). Then \(K \circ H \) is intuitionistic fuzzy symmetric if and only if \(K \circ H = H \circ K \).

**Proof.** (\(\Rightarrow\)): Suppose \(K \circ H \) is intuitionistic fuzzy symmetric and let \(x, y \in X \). Then \(\mu_{K \circ H}(x, y) = \mu_{H \circ K}(x, y)\)
The following is the immediate result of Corollary 3.5 and Proposition 3.6.

**Corollary 3.6.** Let $H$ and $K$ be intuitionistic fuzzy $G$-equivalence relations on a set $X$ with $\delta_1(H) = \delta_1(K)$ and $\delta_2(H) = \delta_2(K)$ such that $K \circ H = H \circ K$. Then $K \circ H$ is an intuitionistic fuzzy $G$-equivalence relation on $X$.

**4. Intuitionistic fuzzy $G$-congruences on a groupoid**

**Definition 4.1[15].** An IFR $R$ on a groupoid $S$ is said to be:

1. **intuitionistic fuzzy left compatible** if $\mu_R(x, y) \leq \mu_R(x, z) \land \nu_R(x, y) \geq \nu_R(x, z)$, for any $x, y, z \in S$.
2. **intuitionistic fuzzy right compatible** if $\mu_R(x, y) \leq \mu_R(x, z) \land \nu_R(x, y) \geq \nu_R(x, z)$, for any $x, y, z \in S$.
3. **intuitionistic fuzzy compatible** if $\mu_R(x, y) \land \mu_R(z, t) \leq \mu_R(x, z) \land \nu_R(z, t) \geq \nu_R(x, z)$, for any $x, y, z, t \in S$.

**Proposition 4.2.** If $H$ and $K$ are intuitionistic fuzzy compatible relations on a groupoid $S$, then $H \cap K$ is intuitionistic fuzzy compatible on $S$.

**Proof.** Let $x, y, a, b \in S$. Then

\[
\nu_{H \cap K}(xa, yb) = \nu_H(xa, yb) \land \nu_K(xa, yb) \\
\geq \nu_H(xa, yb) \land \nu_H(a, b) \\
\land \nu_K(xa, yb) \land \nu_K(a, b) \\
= \mu_{H \cap K}(x, y) \land \mu_{H \cap K}(a, b) \\
\geq \nu_{H \cap K}(x, y) \land \nu_{H \cap K}(a, b)
\]

and

\[
\nu_{H \cap K}(xa, yb) = \nu_H(xa, yb) \lor \nu_K(xa, yb) \\
\leq \nu_H(xa, yb) \lor \nu_H(a, b) \\
\lor \nu_K(xa, yb) \lor \nu_K(a, b) \\
= \nu_{H \cap K}(x, y) \lor \nu_{H \cap K}(a, b)
\]

Hence $H \cap K$ is intuitionistic fuzzy compatible on $S$.

**Proposition 4.3.** If $H$ and $K$ are intuitionistic fuzzy compatible relations on a groupoid $S$, then $K \circ H$ is intuitionistic fuzzy compatible on $S$.

**Proof.** Let $x, y, a, b \in S$. Then

\[
\nu_{K \circ H}(xa, yb) = \nu_K(xa, yb) \land \nu_H(xa, yb) \\
\geq \nu_K(xa, yb) \land \nu_K(a, b) \\
\land \nu_H(xa, yb) \land \nu_H(a, b) \\
= \mu_{K \circ H}(x, y) \land \mu_{K \circ H}(a, b) \\
\geq \nu_{K \circ H}(x, y) \land \nu_{K \circ H}(a, b)
\]

and

\[
\nu_{K \circ H}(xa, yb) = \nu_K(xa, yb) \lor \nu_H(xa, yb) \\
\leq \nu_K(xa, yb) \lor \nu_K(a, b) \\
\lor \nu_H(xa, yb) \lor \nu_H(a, b) \\
= \nu_{K \circ H}(x, y) \lor \nu_{K \circ H}(a, b)
\]

Hence $K \circ H$ is intuitionistic fuzzy compatible on $S$.

**Definition 4.4.** Let $R$ be an intuitionistic fuzzy relation on a groupoid $S$. Then $R$ is called an intuitionistic fuzzy $G$-congruence on $S$ if $R \in IFEC_G(S)$ and $R$ is intuitionistic fuzzy compatible.

We will denote the set of all intuitionistic fuzzy $G$-congruences on $S$ as $IFCG_G(S)$.
Example 4.4. Let $K = \{e, a, b, c\}$ be the Klein 4-group, where $e$ is the identity. Let $R = (\mu_R, \nu_R)$ be the intuitionistic fuzzy relation on $K$ defined as follows:
\[
R(x, y) = \left(\frac{1}{2}, \frac{1}{2}\right) \text{ for any } x, y \in K \text{ with } x \neq y, \\
R(a, a) = R(b, b) = \left(\frac{1}{2}, \frac{3}{2}\right), \\
R(c, c) = \left(\frac{1}{2}, \frac{1}{2}\right), \\
R(e, c) = (1, 0).
\]
Then we can see that $R$ is an intuitionistic fuzzy $G$-congruence on $K$.

The following is the immediate result of Propositions 4.2 and 3.3.

Proposition 4.5. If $H$ and $K$ are intuitionistic fuzzy $G$-congruences on a groupoid $S$, then $H \cap K$ is an intuitionistic fuzzy $G$-congruence on $S$.

The following is the immediate result of Corollary 3.6 and Proposition 4.3.

Proposition 4.6. Let $H$ and $K$ are intuitionistic fuzzy $G$-congruences on a groupoid $S$ with $\delta_1(H) = \delta_1(K)$ and $\delta_2(H) = \delta_2(K)$ such that $K \circ H = H \circ K$. Then $K \cap H$ is an intuitionistic fuzzy $G$-congruence on $S$ with $\delta_1((K \cap H)) = \delta_1(K)$ and $\delta_2((K \cap H)) = \delta_2(K)$.

Definition 4.7. Let $R$ be an intuitionistic fuzzy relation on a groupoid $S$.

1. $R$ is said to be right conformable if for any $a, b, c \in S$, $\mu_R(a, c) \geq \mu_R(a, b)$ and $\nu_R(a, c) \leq \nu_R(a, b)$ imply $\mu_R(ac, bc) \geq \mu_R(a, b)$ and $\nu_R(ac, bc) \leq \nu_R(a, b)$.

2. $R$ is said to be left conformable if for any $a, b, c \in S$, $\mu_R(c, c) \geq \mu_R(a, b)$ and $\nu_R(c, c) \leq \nu_R(a, b)$ imply $\mu_R(ac, ca) \geq \mu_R(a, b)$ and $\nu_R(ac, ca) \leq \nu_R(a, b)$.

3. $R$ is called an intuitionistic fuzzy right [resp. left] $G$-congruence if
   (i) it is an intuitionistic fuzzy $G$-equivalence relation,
   (ii) it is intuitionistic fuzzy right [resp. left] conformable.

The following is the immediate result of Definitions 4.1 and 4.7.

Proposition 4.8. Let $R$ be an intuitionistic fuzzy relation on a groupoid $S$.

1. If $R$ is intuitionistic fuzzy right [resp. left] compatible, then it is intuitionistic fuzzy right [resp. left] conformable.

2. If $R$ is both intuitionistic fuzzy reflexive and intuitionistic fuzzy right [resp. left] compatible, then it is intuitionistic fuzzy right [resp. left] compatible.

Proposition 4.9. (1) If $R$ is an intuitionistic fuzzy compatible relation on a groupoid $S$, then it is both intuitionistic fuzzy right and left conformable.

2. Let $R$ be an intuitionistic fuzzy $G$-preorder on a groupoid $S$. If $R$ is both intuitionistic fuzzy right and left conformable, then it is intuitionistic fuzzy compatible.

Proof. (1) Suppose $\mu_R(a, c) \geq \mu_R(a, b)$ and $\nu_R(a, c) \leq \nu_R(a, b)$ for any $a, b, c \in S$. Since $R$ is intuitionistic fuzzy compatible, $\mu_R(ac, bc) \geq \mu_R(a, b) \land \mu_R(c, c) = \mu_R(a, b)$ and $\nu_R(ac, bc) \leq \nu_R(a, b) \lor \nu_R(c, c) = \nu_R(a, b)$. Thus $R$ is intuitionistic fuzzy right conformable. Similarly, $R$ is intuitionistic fuzzy left conformable. Hence $R$ is both intuitionistic fuzzy right and left conformable.

(2) Let $a, b, c \in S$. Since $R$ is intuitionistic fuzzy transitive,
\[
\begin{align*}
\mu_R(ac, bd) & \geq \mu_R(ac, bd) \\
& = \bigvee_{t \in S}[\mu_R(ac, t) \land \mu_R(t, bd)] \\
& \geq \mu_R(ac, bc) \land \mu_R(bc, bd) \\
& \geq \mu_R(ac, bd) \land \mu_R(bc, bd) \\
& \geq \mu_R(ac, bc) \land \nu_R(bc, bd). \\
\end{align*}
\]
and
\[
\begin{align*}
\nu_R(ac, bd) & \leq \nu_R(ac, bd) \\
& = \bigwedge_{t \in S}[\nu_R(ac, t) \lor \nu_R(t, bd)] \\
& \leq \nu_R(ac, bc) \lor \nu_R(bc, bd). \\
\end{align*}
\]
Case (i): Suppose $a \neq b$ and $c \neq d$. Since $R$ is intuitionistic fuzzy $G$-reflexive,
\[
\mu_R(c, c) \geq \mu_R(a, b), \quad \nu_R(c, c) \leq \nu_R(a, b)
\]
and
\[
\mu_R(b, b) \geq \mu_R(c, d), \quad \nu_R(b, b) \leq \nu_R(c, d).
\]
Since $R$ is both intuitionistic fuzzy right and left conformable,
\[
\begin{align*}
\mu_R(ac, bc) & \geq \mu_R(a, b), \quad \nu_R(ac, bc) \leq \nu_R(a, b) \\
& \geq \mu_R(ac, ca) \land \mu_R(ca, cb) \land \nu_R(ac, ca) \land \nu_R(ca, cb)
\end{align*}
\]
and
\[
\begin{align*}
\nu_R(ac, bc) & \leq \nu_R(ac, bc) \land \nu_R(ac, ca) \land \nu_R(ca, cb) \\
& \leq \nu_R(ac, ca) \lor \nu_R(ca, cb)
\end{align*}
\]
Case (ii): Suppose $a \neq b$ and $c = d$. Since $R$ is intuitionistic fuzzy $G$-reflexive, $\mu_R(c, c) \geq \mu_R(a, b)$ and $\nu_R(c, c) \leq \nu_R(a, b)$. Since $R$ is intuitionistic fuzzy right conformable,
\[
\begin{align*}
\mu_R(ac, bd) & = \mu_R(ac, bc) \geq \mu_R(a, b) \\
& = \mu_R(a, b) \land \mu_R(c, c) \\
& = \mu_R(a, b) \land \mu_R(c, c)
\end{align*}
\]
and
\[
\begin{align*}
\nu_R(ac, bd) & = \nu_R(ac, bc) \leq \nu_R(a, b) \\
& = \nu_R(a, b) \lor \nu_R(c, c) \\
& = \nu_R(a, b) \lor \nu_R(c, c).
\end{align*}
\]
Case (iii): Suppose $a = b$ and $c \neq d$. By the similar arguments of Case (ii), we have the same result as Case (ii).
Case (iv): Suppose $a = b$ and $c = d$. If $\mu_R(a, a) \geq \mu_R(c, c)$ and $\nu_R(a, a) \leq \nu_R(c, c)$, then, by intuitionistic fuzzy left conformability, we obtain $\mu_R(ac, ac) \geq \mu_R(c, c)$ and $\nu_R(ac, ac) \leq \nu_R(c, c)$. If $\mu_R(c, c) \geq \mu_R(a, a)$ and $\nu_R(c, c) \leq \nu_R(a, a)$, then, by intuitionistic fuzzy...
right conformability, we obtain \( \mu_R(ac, ac) \geq \mu_R(a, a) \) and \( \nu_R(ac, ac) \geq \nu_R(a, a) \). So

\[
\mu_R(ac, bd) = \mu_R(ac, ac) \\
\geq \mu_R(a, a) \wedge \mu_R(c, c) \\
= \mu_R(a, b) \wedge \mu_R(c, d)
\]

and

\[
\nu_R(ac, bd) = \nu_R(ac, ac) \\
\leq \nu_R(a, a) \vee \nu_R(c, c) \\
= \nu_R(a, b) \vee \nu_R(c, d)
\]

Hence \( R \) is intuitionistic fuzzy compatible. This completes the proof.

\[\Box\]

**Corollary 4.9.** Let \( R \) be an intuitionistic fuzzy relation on a groupoid \( S \). Then \( R \) is an intuitionistic fuzzy \( G \)-congruence on \( S \) if and only if it is both an intuitionistic fuzzy right and left \( G \)-congruence on \( S \).

**5. Intuitionistic fuzzy \( G \)-congruences on a group**

**Proposition 5.1.** Let \( R \) be an intuitionistic fuzzy \( G \)-reflexive relation on a group \( K \).

1. If \( R \) is intuitionistic fuzzy right conformable,
   then \( R(ac, bc) = R(a, b) \), whenever \( a \neq b, c \in K \).

2. If \( R \) is intuitionistic fuzzy left conformable,
   then \( R(ca, cb) = R(a, b) \), whenever \( a \neq b, c \in K \).

3. If \( R \) is both intuitionistic fuzzy right and left conformable,
   then \( R(cad, cdb) = R(ad, bd) = R(ca, cb) = R(a, b) \), whenever \( a \neq b, c, d \in K \).

**Proof.** (1) Let \( a \neq b, c \in K \). Since \( R \) is intuitionistic fuzzy \( G \)-reflexive,

\[
\mu_R(c, c) \geq \mu_R(a, b), \quad \nu_R(c, c) \leq \nu_R(a, b)
\]

and

\[
\mu_R(c^{-1}, c^{-1}) \geq \mu_R(ac, bc), \\
\nu_R(c^{-1}, c^{-1}) \leq \nu_R(ac, bc).
\]

Since \( R \) is intuitionistic fuzzy right conformable,

\[
\mu_R(a, b) = \mu_R(ac, bc) \geq \mu_R(ac, bc) \geq \mu_R(a, b)
\]

and

\[
\nu_R(a, b) = \nu_R(ac, bc) \leq \nu_R(ac, bc) \leq \nu_R(a, b).
\]

Thus \( \mu_R(a, b) \geq \mu_R(ac, bc) \) and \( \nu_R(a, b) \leq \nu_R(ac, bc) \). Hence \( R(ac, bc) = R(a, b) \).

The proofs of (2) and (3) are similar. \[\Box\]

**Corollary 5.1.** (1) If \( R \) is an intuitionistic fuzzy \( G \)-congruence on a group \( K \), then \( R(cad, cdb) = R(ad, bd) = R(ca, cb) = R(a, b) \), whenever \( a \neq b, c, d \in K \).

2. If \( R \) is an intuitionistic fuzzy \( G \)-reflexive and symmetric relation on a group \( K \), which is both intuitionistic fuzzy right and left conformable, then \( R(a^{-1}, b^{-1}) = R(a, b) \), whenever \( a \neq b \in K \).

**Proof.** (1) By Proposition 4.9(1), \( R \) is both intuitionistic fuzzy right and left conformable. Hence, by Proposition 5.1(3), we get the result.

(2) Let \( a \neq b \in K \). Then, by Proposition 5.1(3),

\[
R(a, b) = R(b^{-1}a^{-1}, b^{-1}a^{-1}) = R(b^{-1}, a^{-1}) = R(a^{-1}, b^{-1}).
\]

(3) It follows from Proposition 4.9(1) and Corollary 5.1(2).

**Example 5.2.** Consider the intuitionistic fuzzy relation \( R \) given in Example 4.4. Then \( R \) is intuitionistic fuzzy \( G \)-reflexive and right conformable on \( K \) with \( R(ab, ab) \neq R(a, a) \).

**Definition 5.3[9].** Let \( (S, \cdot) \) be a groupoid and let \( A \in IFS(S) \). Then \( A \) is called an intuitionistic fuzzy subgroupoid (in short, IFGP) of \( S \) if for any \( x, y \in S \),

\[
\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)
\]

and

\[
\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y).
\]

We will denote the set of all IFGP of a groupoid \( S \) as IFGP(S). Then it is clear that \( 0_\infty \) and \( 1_\infty \in IFS(S) \).

**Definition 5.4[10].** Let \( K \) be a group and let \( A \in IFS(K) \). Then \( A \) is called an intuitionistic fuzzy subgroup (in short, IFG) of \( K \) if \( A(x^{-1}) \geq A(x) \), i.e.,

\[
\mu_A(x^{-1}) \geq \mu_A(x) \quad \text{and} \quad \nu_A(x^{-1}) \leq \nu_A(x), \quad \text{for each} \ x \in K.
\]

We will denote the set of all IFG of \( K \) as IFG(K).

**Result 5.1[10, Proposition 2.6].** Let \( K \) be a group and let \( A \in IFS(K) \). Then \( A(x^{-1}) = A(x) \) and \( \mu_A(x) \leq \mu_A(e) \), \( \nu_A(x) \geq \nu_A(e) \) for each \( x \in K \), where \( e \) is the identity element of \( K \).

**Definition 5.5[10].** Let \( K \) be a group and let \( A \in IFS(K) \). Then \( A \) is said to be normal if \( A(xy) = A(yx) \) for any \( x, y \in K \).

We will denote the family of all intuitionistic fuzzy normal subgroups of a group \( K \) as IFNG(K). In particular, we will denote the set \( \{N \in IFS(G) : N(e) = (1, 0)\} \) as IFNG(K).

**Definition 5.6[15].** An IFER \( R \) on a groupoid \( S \) is called an:

1. intuitionistic fuzzy left congruence (in short, IFLC)
if it is intuitionistic fuzzy left compatible.
(2) intuitionistic fuzzy right congruence (in short, IFRC) if it is intuitionistic fuzzy right compatible.
(3) intuitionistic fuzzy congruence (in short, IFC) if it is intuitionistic fuzzy compatible.

We will denote the set of all IFCs [resp. IFLC’s and IFRC’s] on a groupoid $S$ as IFC(S) [resp. IFLC(S) and IFRC(S)].

Result 5.5.16, Lemma 5.6. Let $K$ be a group and let $A \in IFN(K)$. We define the complex mapping $R_A = (\mu_{R_A}, \nu_{R_A}) : K \times K \rightarrow I \times I$ as follows: For each $(a, b) \in K \times K$,$R_A(a, b) = A(ab^{-1})$.
Then $R_A \in IFC(G)$.

The following is the modification of Result 5.B.

Lemma 5.7. Let $K$ be a group and let $A$ be an intuitionistic fuzzy nonempty normal subgroup of $K$. We define a complex mapping $R_A = (\mu_{R_A}, \nu_{R_A}) : K \times K \rightarrow I \times I$ as follows: For any $a, b \in K$,$R_A(a, b) = A(ab^{-1})$.
Then $R_A \in IFC(G)$.

Proof. Let $a \neq b \in K$. Then clearly $\mu_{R_A}(a, a) = \mu_A(aa^{-1}) = \mu_A(e) > 0$ and $\nu_{R_A}(a, a) = \nu_A(aa^{-1}) = \nu_A(e) < 1$. On the other hand,$\delta_1(R_A) = \bigwedge_{t \in K} \mu_{R_A}(t, t) = \mu_A(e) \geq \mu_A(ab^{-1}) = \mu_{R_A}(a, b)$ and$\delta_2(R_A) = \bigvee_{t \in K} \nu_{R_A}(t, t) = \nu_A(e) \leq \nu_A(ab^{-1}) = \nu_{R_A}(a, b)$.
Thus $R_A$ is intuitionistic fuzzy $G$-reflexive. By Result 5.B, $R_A$ is intuitionistic fuzzy symmetric and transitive. Also, by Result 5.B, $R_A$ is intuitionistic fuzzy compatible. Hence $R_A \in IFC_G(K)$.

Result 5.5.15, Proposition 2.18. Let $K$ be a group and let $R \in IFC(G)$. We define the complex mapping $A_R = (\mu_{A_R}, \nu_{A_R}) : K \rightarrow I \times I$ as follows: For each $a \in K$,$A_R(a) = R(a, e) = R(e, a)$.
Then $A_R \in IFN(K)$.

The following is the modification of Result 5.C.

Lemma 5.8. Let $K$ be a group and let $R \in IFC_G(K)$. We define a complex mapping $A_R = (\mu_{A_R}, \nu_{A_R}) : K \rightarrow I \times I$ as follows: For each $x \in K$,$A_R(x) = R(x, e)$.
Then $A_R$ is an intuitionistic fuzzy nonempty normal subgroup of $K$.

Proof. Let $x, y \in K$. Since $R$ is intuitionistic fuzzy compatible,$\mu_{A_R}(xy) = \mu_R(xy, e) = \mu_R(x, e) \wedge \mu_R(y, e)$
$\geq \mu_R(x, e) \wedge \mu_R(y, e) = \mu_{A_R}(x) \wedge \mu_{A_R}(y)$
and$\nu_{A_R}(xy) = \nu_R(xy, e) = \nu_R(x, e)$
$\leq \nu_R(x, e) \vee \nu_R(y, e) = \nu_{A_R}(x) \vee \nu_{A_R}(y)$.
Thus, $A_R \in IFN(K)$. Now let $xy \neq e$. Then, by Corollary 5.1(3),$A_R(x^{-1}) = R(x^{-1}, e) = R(x, e) = A_R(x)$.
So $A_R \in IFN(G)$. Now let $xy = e$. Then, by Corollary 5.1(1),$A_R(xy) = R(xy, e) = R(x^{-1}xy, x^{-1}ex) = R(x, e) = A_R(x)$.
Thus $A_R \in IFN(G)$. Moreover, $\mu_{A_R}(e) = \mu_R(e, e) > 0$ and $\nu_{A_R}(e) = \nu_R(e, e) < 1$. So $A_R \neq 0$. Hence $A_R$ is an intuitionistic fuzzy nonempty normal subgroup of $K$.

Let $K$ be any group. We define a relation $\sim$ on $IFC_G(K)$ as follows: For any $P, Q \in IFC_G(K), P \sim Q$ if and only if $P(x, y) = Q(x, y)$ for all $x \neq y \in K$ and $P(e, e) = Q(e, e)$. Then clearly, $\sim$ is an equivalence relation on $IFC_G(K)$, which partitions $IFC_G(K)$ into disjoint equivalence classes $[R], R \in IFC_G(K)$. Let $IFC_G(K)/\sim$ be the family of all these equivalence classes and let $IFN^+(K)$ denote the set of all intuitionistic fuzzy nonempty normal subgroups of $K$.

Proposition 5.9. Let $K$ be a group. Then, under the product $[P][Q] = [P \cap Q], P, Q \in IFC_G(K), IFC_G(K)/\sim$ is a commutative monoid of idempotents.

Proof. We note that, by Proposition 4.5, $IFC_G(K)$ is closed under the formation of finite intersections. For any $P, P_1, Q_1 \in IFC_G(K)$, suppose $P \sim P_1$ and $Q \sim Q_1$. Let $x \neq y \in K$. Then$\mu_{P \cap Q}(x, y) = \mu_p(x, y) \wedge \mu_q(x, y)$
$= \mu_{P_1}(x, y) \wedge \mu_{Q_1}(x, y)$
and$\nu_{P \cap Q}(x, y) = \nu_p(x, y) \vee \nu_q(x, y)$
$\leq \nu_{P_1}(x, y) \vee \nu_{Q_1}(x, y)$.
Thus $(P \cap Q)(x, y) = (P_1 \cap Q_1)(x, y)$. Also, $(P \cap Q)(e, e) = (P_1 \cap Q_1)(e, e)$. Thus $P \cap Q \sim P_1 \cap Q_1$. So the product on $IFC_G(K)/\sim$ is well-defined. We can easily see that $(IFC_G(K)/\sim, \cdot)$ forms a commutative semi-group of idempotents. Now we define a complex mapping $e = (\mu_e, \nu_e) : K \times K \rightarrow I \times I$ as follows: For any $x, y \in K, e(x, y) = (1, 0)$. Then clearly $e \in IFC_G(K)$.
Moreover,$[R][e] = [R] = [e][R]$ for each $R \in IFC_G(K)$.
Hence $[e]$ is the identity element of $IFC_G(K)/\sim$. This completes the proof. □

**Proposition 5.10.** Let $K$ be any group. Then $(IFN^*(K), \cap)$ forms a commutative monoid of idempotents.

**Proof.** We can easily prove that $(IFN^*(K), \cap)$ forms a commutative semigroup of idempotents. It is clear that $1_\sim \in IFN^*(K)$. Moreover, $1_\sim \cap A = A = A \cap 1_\sim$ for each $A \in IFN^*(K)$. So $1_\sim$ is the identity element of $IFN^*(K)$. □

**Theorem 5.11.** Let $K$ be any group. Then $IFC_G(K)$ is isomorphic to $IFN^*(K)$.

**Proof.** Consider the mapping $\Psi : IFC_G(K) \to IFN^*(K)$ defined by $\Psi([R]) = A_R$ for each $R \in IFC_G(K)$. Then we can easily see that $\Psi$ is well-defined. Let $A \in IFN^*(K)$. Then, by Lemma 5.7, $R_A \in IFC_G(K)$. Let $x \in K$. Then $A_{R_A}(x) = R_A(x, e) = A(xe^{-1}) = A(x)$. Thus $A_{R_A} = A$. So $\Psi([R_A]) = A$. Hence $\Psi$ is surjective.

For any $P, Q \in IFC_G(K)$, suppose $\Psi([P]) = \Psi([Q])$. Then $A_P = A_Q$. Thus $P(x, e) = Q(x, e)$ for each $x \in K$. Let $x \neq y \in K$. Then, by Corollary 5.1(1),

$$P(x, y) = P(xy^{-1}, e) = Q(xy^{-1}, e) = Q(x, y).$$

Thus $P \sim Q$, i.e., $[P] = [Q]$. So $\Psi$ is injective. Now let $P, Q \in IFC_G(K)$ and let $x \in K$. Then

$$\mu_{A_{P \cap Q}}(x) = \mu_{P \cap Q}(x, e) = \mu_P(x, e) \land \mu_Q(x, e) = \mu_{A_P \cap A_Q}(x)$$

and

$$\nu_{A_{P \cap Q}}(x) = \nu_{P \cap Q}(x, e) = \nu_P(x, e) \lor \nu_Q(x, e) = \nu_{A_P \cap A_Q}(x).$$

Thus $A_{P \cap Q} = A_P \cap A_Q$. So

$$\Psi([P \cap Q]) = \Psi([P \cap Q]) = \Psi([P \cap Q]) = A_{P \cap Q} = A_P \cap A_Q = \Psi([P]) \cap \Psi([Q]).$$

Moreover, $\Psi([e]) = A_e = 1_\sim$. Hence $\Psi$ is a monoid homomorphism. Therefore $IFC_G(K)/\sim$ and $IFN^*(K)$ are isomorphic under $\Psi$. □

**Proposition 5.12.** Let $K$ be a group. If $P, Q \in IFC_G(K)$ such that $\delta_1(P) = \delta_1(Q)$ and $\delta_2(P) = \delta_2(Q)$, then $P \circ Q = Q \circ P$ and $P \circ Q \in IFC_G(K)$ such that $\delta_1(P \circ Q) = \delta_1(Q)$ and $\delta_2(P \circ Q) = \delta_2(Q)$.

**Proof.** Let $x \neq y \in K$. Then

$$\mu_{Q \circ P}(x, y) = \bigvee_{t \in E} [\mu_P(x, t) \land \mu_Q(t, y)]$$

and

$$\nu_{Q \circ P}(x, y) = \bigwedge_{t \in E} [\nu_P(x, t) \lor \nu_Q(t, y)].$$

Thus, by (5.1) and (5.2),

$$\mu_{P \circ Q}(x, y) = \mu_{Q \circ P}(x, y) = \mu_{Q \circ P}(x, y)$$

and

$$\nu_{P \circ Q}(x, y) = \nu_{Q \circ P}(x, y) = \nu_{Q \circ P}(x, y).$$

Hence $P \circ Q = Q \circ P$. Moreover, by Proposition 4.6, $P \circ Q \in IFC_G(K)$ such that $\delta_1(P \circ Q) = \delta_1(Q)$ and $\delta_2(P \circ Q) = \delta_2(Q)$. This completes the proof. □
6. The \((\lambda, \mu)\)-partition of \(\text{IFC}_G(K)\)

**Definition 6.1[9].** Let \((X, \cdot)\) be a groupoid and let \(A, B \in \text{IFS}(X)\). Then the intuitionistic fuzzy product of \(A\) and \(B\), \(A \circ B\), is defined as follows: For each \(x \in X\),

\[
(A \circ B)(x) = \begin{cases} 
\{y \in X | [\mu_A(y) \land \mu_B(z)] \lor \nu_A(y) \lor \nu_B(z)\}, \\
(0, 1) \text{ if } x \text{ is not expressible as } x = yz.
\end{cases}
\]

**Result 6.A[9, Proposition 2.3].** Let \((X, \cdot)\) be a groupoid. If 
\(\cdot\) is an associative [resp. commutative], then so is 
\(\cdot\) in \(\text{IFS}(X)\).

**Result 6.B[10, Proposition 2.4].** Let \(A\) be an IPG of a group \(G\). Then \(A \circ A = A\).

**Result 6.C[10, Proposition 3.2].** Let \(K\) be a group, let \(A \in \text{IFS}(K)\) and let \(B \in \text{IFNG}(K)\). Then \(A \circ B = B \circ A\).

**Result 6.D[10, Proposition 3.4].** Let \(K\) be a group and let \(A, B \in \text{IFNG}(K)\). Then \(A \circ B \in \text{IFNG}(K)\).

For a group \(K\) and for each \((\lambda, \mu) \in (0, 1] \times (0, 1]\) with \(\lambda + \mu \leq 1\), let

\[\text{IFC}_{G, (\lambda, \mu)}(K) = \{R \in \text{IFC}_G(K) | \delta_1(R) = \lambda\}
\]

and

\[\text{IFNG}_{(\lambda, \mu)}(K) = \{R \in \text{IFS}(K) | \delta_2(R) = \mu\}\]

We define a relation \(\sim\) on \(\text{IFC}_{G, (\lambda, \mu)}(K)\) as follows: For any \(P, Q \in \text{IFC}_{G, (\lambda, \mu)}(K)\),

\[P \sim Q\] if and only if \(P(x, y) = Q(x, y)\) whenever \(x \neq y \in K\) and \(P(e, e) = Q(e, e)\).

Then we can easily prove that \(\sim\) is an equivalence relation on \(\text{IFC}_{G, (\lambda, \mu)}(K)\). For each \(R \in \text{IFC}_{G, (\lambda, \mu)}(K)\), let \(\text{IFC}_{G, (\lambda, \mu)}(K)/\sim R \in \text{IFC}_{G, (\lambda, \mu)}(K)\)

Finally, let \(\text{IFNG}_{(\lambda, \mu)}(K) = \{A \in \text{IFNG}(K) | \mu_A(x) \leq \lambda \leq \nu_A(e) \land \nu_A(e) \leq \mu \leq \nu_A(x) \neq x \in K\} \) if \(K \neq (e)\) and \(\text{IFNG}_{(\lambda, \mu)}(K) = ((e)_{(\lambda, \mu)})\) if \(K = (e)\).

**Proposition 6.2.** Let \(K\) be a group and let \((\lambda, \mu) \in (0, 1] \times (0, 1]\) with \(\lambda + \mu \leq 1\). Then \(\text{IFC}_{G, (\lambda, \mu)}(K)\) is a commutative semigroup of idempotents.

**Proof.** By Proposition 5.12, \((\text{IFC}_{G, (\lambda, \mu)}(K), \circ)\) is a commutative semigroup. Moreover, by Result 6.A it is clear that \(\circ\) is associative. On the other hand, by Proposition 3.2(2), each member of \(\text{IFC}_{G, (\lambda, \mu)}(K)\) is an idempotent. Hence \((\text{IFC}_{G, (\lambda, \mu)}(K), \circ)\) is a commutative semigroup of idempotents.

**Lemma 6.3.** Let \(K\) be a group and let \((\lambda, \mu) \in (0, 1] \times (0, 1]\) with \(\lambda + \mu \leq 1\). We define a binary relation 
\(\ast\) on \(\text{IFC}_{G, (\lambda, \mu)}(K)\) as follows: For any \(P, Q \in \text{IFC}_{G, (\lambda, \mu)}(K)\),

\[P \ast Q = Q \ast P\]

Then \((\text{IFC}_{G, (\lambda, \mu)}(K), \ast\), \(\sim\)) is a commutative monoid of idempotents.

**Proof.** Suppose \(K = (e)\). Then \(\text{IFC}_{G, (\lambda, \mu)}(K)/\sim = \{R(e, e) = \{R\}\}\), where \(R(e, e) = (\lambda, \mu)\). Thus the lemma is trivially true in this case. Suppose \(K \neq (e)\).

We are obliged to prove that \(\ast\) is well-defined. Let \(P, P_1, Q, Q_1 \in \text{IFC}_{G, (\lambda, \mu)}(K)\) such that \(P \sim P_1\) and \(Q \sim Q_1\). Let \(x \neq y \in K\). Then

\[
\mu_{Q \circ P, (x, y)} = \bigwedge_{t \in K} [\mu_{P(x, t)} \land \mu_{Q(t, y)}] \\
= (\bigwedge_{x \neq y} [\neg (\mu_{P(x, t)} \land \mu_{Q(t, y)})].
\]

and

\[
\nu_{Q \circ P, (x, y)} = \bigwedge_{t \in K} \nu_{P(x, t)} \lor \nu_{Q(t, y)}] \\
= (\bigwedge_{x \neq y} [\neg (\nu_{P(x, t)} \lor \nu_{Q(t, y)})]
\]

Finally, \(\text{IFNG}_{(\lambda, \mu)}(K) = \{A \in \text{IFS}(K) | \mu_A(x) \leq \lambda \leq \nu_A(e) \land \nu_A(e) \leq \mu \leq \nu_A(x) \neq x \in K\} \) if \(K \neq (e)\) and \(\text{IFNG}_{(\lambda, \mu)}(K) = ((e)_{(\lambda, \mu)})\) if \(K = (e)\).

Thus \(Q \circ P \sim Q_1 \circ P_1\). So \(\ast\) is well-defined. Moreover, by Proposition 6.2, we can see that \((\text{IFC}_{G, (\lambda, \mu)}(K), \ast, \sim)\) is a commutative semigroup of idempotents.

Now we define a complex mapping \(E : K \times K \rightarrow I \times I\) as follows: For any \(x, y \in K\),

\[
E(x, y) = \begin{cases} 
(1, 0) & \text{if } x = y = e, \\
(\lambda, \mu) & \text{if } x = y \neq e, \\
(0, 1) & \text{if } x \neq y.
\end{cases}
\]

Then we can routinely prove that \(E \in \text{IFC}_{G, (\lambda, \mu)}(K)\) and that \(E \circ R = R \circ E \sim R\) for each \(R \in \text{IFC}_{G, (\lambda, \mu)}(K)\).

Thus

\[
[E]_{(\lambda, \mu)} \ast [R]_{(\lambda, \mu)} = [E \circ R]_{(\lambda, \mu)} = [R]_{(\lambda, \mu)} \ast [E]_{(\lambda, \mu)}
\]
So $[E]_{(\lambda, \mu)}$ is the identity element of $IFC_{G, (\lambda, \mu)}(K)/\sim$. This completes the proof. \hfill \Box

**Proposition 6.4.** Let $K$ be a group and let $(\lambda, \mu) \in (0, 1) \times (0, 1)$ with $\lambda + \mu \leq 1$. Then $(IFNG_{(\lambda, \mu)}, \circ)$ is a commutative monoid of idempotents.

**Proof.** Let $A, B \in IFNG_{(\lambda, \mu)}(K)$. Then, by Result 6.D, $A \circ B \in IFNG(K)$. Let $x \neq e \in K$. Since $A, B \in IFNG_{(\lambda, \mu)}(K)$,

$$\mu_A(x) \leq \lambda \leq \mu_A(e), \quad \nu_A(e) \leq \nu_A(x),$$
and

$$\mu_B(x) \leq \lambda \leq \mu_B(e), \quad \nu_B(e) \leq \nu_B(x).$$

Then $\mu_A(x^t) \wedge \mu_B(t) \leq \lambda$ and $\nu_A(x^t) \vee \nu_B(t) \geq \mu$. Thus for each $t \in K$,

$$\mu_{A \circ B}(x) = \bigvee_{t \in K} \left[ \mu_A(x^t) \wedge \mu_B(t) \right] \leq \lambda$$

and

$$\nu_{A \circ B}(x) = \bigwedge_{t \in K} \left[ \nu_A(x^t) \vee \nu_B(t) \right] \geq \mu.$$  

Moreover, $\mu_{A \circ B}(e) = \mu_A(e) \wedge \mu_B(e) \geq \lambda$ and $\nu_{A \circ B}(e) = \nu_A(e) \vee \nu_B(e) \leq \mu$. So $A \circ B \in IFNG_{(\lambda, \mu)}(K)$. On the other hand, by Results 6.C and 6.B, $A \circ B = B \circ A$ and $A \circ A = A$. Furthermore, by Result 6.A, $\circ$ is associative. Hence $(IFNG_{(\lambda, \mu)}(K), \circ)$ is a commutative semigroup of idempotents. Finally, consider the intuitionistic fuzzy point $e(1, 0)$ of $K$. Then clearly, $e(1, 0)$ is the identity element of $IFNG_{(\lambda, \mu)}(K)$. This completes the proof. \hfill \Box

**Proposition 6.5.** Let $P$ and $Q$ be intuitionistic fuzzy $G$-reflexive relations on a group $K$ such that $\delta_1(P) = \delta_1(Q)$ and $\delta_2(P) = \delta_2(Q)$. If $P$ is intuitionistic fuzzy right conformal, then $A_{Q \circ P} = A_P \circ A_Q$.

**Proof.** Let $x \in K$. Then

$$\mu_{A_P \circ A_Q}(x) = \bigvee_{t \in K} \left[ \mu_{A_P}(x^t) \wedge \mu_{A_Q}(t) \right] = \bigvee_{t \in K} \left[ \mu_{A_P}(x^t) \wedge \mu_{A_Q}(t) \right] = \bigvee_{t \in K} \left[ \nu_{A_P}(x^t) \vee \nu_{A_Q}(t) \right] = \bigwedge_{t \in K} \left[ \mu_{A_P}(x^t) \wedge \mu_{A_Q}(t) \right],$$

and

$$\nu_{A_P \circ A_Q}(x) = \bigwedge_{t \in K} \left[ \nu_{A_P}(x^t) \vee \nu_{A_Q}(t) \right].$$

Hence $A_{Q \circ P} = A_P \circ A_Q$. \hfill \Box

**Corollary 6.5.** Let $K$ be a group and let $(\lambda, \mu) \in (0, 1) \times (0, 1)$ with $\lambda + \mu \leq 1$. If $P, Q \in IFC_{G, (\lambda, \mu)}(K)$, then $A_{P \circ Q} = A_P \circ A_Q \in IFNG_{(\lambda, \mu)}(K)$.

**Proof.** Let $P, Q \in IFC_{G, (\lambda, \mu)}(K)$. Then, by Proposition 5.12, $Q \circ P = P \circ Q \in IFC_{G, (\lambda, \mu)}(K)$, Thus, by Lemma 5.8, $A_{P \circ Q} \in IFNG_{(\lambda, \mu)}(K)$. Hence, by Propositions 5.12 and 6.5, $A_{P \circ Q} = A_P \circ A_Q$. \hfill \Box

**Theorem 6.6.** Let $K$ be a group and let $(\lambda, \mu) \in (0, 1) \times (0, 1)$ with $\lambda + \mu \leq 1$. Then $(IFC_{G, (\lambda, \mu)}(K)/\sim, \circ)$ and $(IFNG_{(\lambda, \mu)}(K), \circ)$ are isomorphic.

**Proof.** Suppose $K = \{e\}$. Then $IFC_{G, (\lambda, \mu)}(K)/\sim$ and $IFNG_{(\lambda, \mu)}(K)$ are trivially isomorphic, since both are singletons. Suppose $K \neq \{e\}$. We define a mapping $\Psi : IFC_{G, (\lambda, \mu)}(K) \to IFNG_{(\lambda, \mu)}(K)$ by $\Psi([R]_{(\lambda, \mu)}) = A_B$ for each $R \in IFC_{G, (\lambda, \mu)}(K)$. Then, as in the proof of Theorem 5.11, we can show that $\Psi$ is a well-defined injection. Let $[E]_{(\lambda, \mu)}$ be the class which occurs in the proof of Lemma 6.3. Then clearly $[E]_{(\lambda, \mu)}$ is the identity element of $IFC_{G, (\lambda, \mu)}(K)$. Moreover, we can easily see that $\Psi([E]_{(\lambda, \mu)}) = A_B = e(1, 0)$. Now let $P, Q \in IFC_{G, (\lambda, \mu)}(K)$.

Then

$$\Psi([P]_{(\lambda, \mu)} \circ [Q]_{(\lambda, \mu)}) = \Psi([P]_{(\lambda, \mu)}) \circ \Psi([Q]_{(\lambda, \mu)}) = \Psi([P]_{(\lambda, \mu)}) \circ \Psi([Q]_{(\lambda, \mu)}).$$

So $\Psi$ is a monoid homomorphism. Let $A \in IFNG_{(\lambda, \mu)}(K)$. Then, by Lemma 5.7, $R_A \in IFC_{G}(K)$. We define a complex mapping $P = (\mu_P, \nu_P) : K \times K \to I \times I$ as follows: For any $x, y \in K$,

$$P(x, y) = \begin{cases} R_A(x, y) & \text{if } x \neq y, \\ (\lambda, \mu) & \text{if } x = y \neq e, \\ e(1, 0) & \text{if } x = y = e, \end{cases}$$

and

$$\mu_P(e, e) = \mu_{R_A}(e, e) = \mu_A(e) \geq \lambda,$$

$$\nu_P(e, e) = \nu_{R_A}(e, e) = \nu_A(e) \leq \mu.$$  

Then clearly $P \in IFR(K)$. Moreover, $P$ is intuitionistic fuzzy $G$-reflexive and symmetric. Now let $x \neq y \in K$.

Then

$$\mu_{P \circ P}(x, y) = \bigvee_{t \in K} \left[ \mu_{R_A}(x, t) \wedge \mu_{R_A}(t, y) \right] \wedge \mu_{R_A}(x, y)$$

and

$$\nu_{P \circ P}(x, y) = \bigwedge_{t \in K} \left[ \nu_{R_A}(x, t) \vee \nu_{R_A}(t, y) \right].$$
\[ \geq \nu_{R_A}(x, y) \]
\[ = \nu_P(x, y). \]
Thus \( P \preceq P \subset P \). So \( P \) is intuitionistic fuzzy transitive. Hence \( P \in IFC_{G}(K) \).

Now we show that \( P \) is intuitionistic fuzzy right conformable. For any \( a, b, c \in K \). Suppose \( \mu_P(c, c) \geq \mu_P(a, b) \) and \( \nu_P(c, c) \leq \nu_P(a, b) \).

Case (i): Suppose \( a \neq b \). Since \( R_A \) is intuitionistic fuzzy \( G \)-reflexive, \( \mu_{R_A}(c, c) \geq \mu_{R_A}(a, b) \) and \( \nu_{R_A}(c, c) \leq \nu_{R_A}(a, b) \). Also, by Proposition 4.9(1), \( R_A \) is intuitionistic fuzzy right conformable. Thus
\[
\mu_P(ac, bc) = \mu_{R_A}(ac, bc) \geq \mu_{R_A}(a, b) = \mu_P(a, b)
\]
and
\[
\nu_P(ac, bc) = \nu_{R_A}(ac, bc) \leq \nu_{R_A}(a, b) = \nu_P(a, b).
\]
Case (ii): Suppose \( a = b \). If \( c = e \), then \( P(ac, ac) = P(a, e) = P(a, b) \). If \( c \neq e \), then
\[
\mu_P(ac, ac) = \mu_P(ac, ac) \geq \nu_P(c, c) \geq \mu_P(a, b)
\]
and
\[
\nu_P(ac, ac) = \nu_P(ac, ac) \leq \mu_P(e, e) \leq \nu_P(a, b).
\]

So, in all, \( P \) is intuitionistic fuzzy conformable. By the similar arguments, we can see that \( P \) is intuitionistic fuzzy conformable. Hence, by Proposition 4.9(2), \( P \in IFC_{G(\lambda, \mu)}(K) \). Let \( x \in K \). Then
\[
A_P(x) = P(x, e) = R_A(x, e) = A(x).
\]
Thus \( \Psi(P(\lambda, \mu)) = A_P = A \). So \( \Psi \) is surjective. Hence \( \Psi \) is a monoid isomorphism. Therefore \( IFC_{G(\lambda, \mu)}(K) \) and \( IFC_{G(\lambda, \mu)}(K) \) are isomorphic monoids under \( \Psi \).

This completes the proof. \( \Box \)

**Corollary 6.6.** Let \( K \) be a group. Then the semi-group \( IFC_{G(\lambda, \mu)}(K) \) is isomorphic to the semigroup \( IFC_{NG(\lambda, \mu)}(K) \).

**Proof.** It is clear that \( R_{[1, 0]} = \{ R \} \) for each \( R \in IFC_{G(\lambda, \mu)}(K) \). Then \( IFC_{G(\lambda, \mu)}(K) \) can be identified with \( IFC_{G(\lambda, \mu)}(K) \) as semigroups. Moreover, \( IFC_{G(\lambda, \mu)}(K) = IFC(\lambda, \mu) \) and \( IFC_{G(\lambda, \mu)}(K) = IFC(\lambda, \mu) \). \( \Box \)

**References**


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