The concept of $\sigma$-morphism as a probability measure on the set of effects

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Abstract

In this paper, we introduce the concepts of effects and observable as generalizations of event and random variable, respectively. Also, we introduce the concept of $\sigma$-morphism and we investigate some results on $\sigma$-morphism as a probability measure on the set of effects.

Key Words : $\sigma$-morphism, fuzzy probability

1. 서 론

The imprecision in probability theory comes from our incomplete knowledge of the system but the random variables (measurements) still have precise values. But, in fuzzy theory, we also have an imprecision in our measurements, and so random variables must be replaced by fuzzy random variables and events by fuzzy events. In this sense, S. Gudder introduced the concepts of effects (fuzzy events), observable (fuzzy random variables) and their distribution. Also, he introduced the concept of $\sigma$-morphism on the set of effects. In this paper, we have some results on $\sigma$-morphism as a probability measure on the set of effects.

For general fuzzy theoretical background, we refer to L. A. Zadeh [5].

2. Preliminaries

Let $\Omega$ be a non-empty set. Let $\mathcal{F}$ be a $\sigma$-field of subsets of $\Omega$. That is, a non-empty class of subsets of $\Omega$, which is closed under countable union and complementation. The basic structure is a measurable space $(\Omega, \mathcal{F})$ where $\Omega$ is a sample space consisting of outcomes and $\mathcal{F}$ is a $\sigma$-field of events in $\Omega$ corresponding to some probabilistic experiment. If $\mu$ is a probability measure on $(\Omega, \mathcal{F})$, then $\mu(A)$ is interpreted as the probability that the event $A$ occurs. A measurable function $f: \Omega \to R$ is called a random variable. The expectation of $f$ is defined by $E[f] = \int_R d\mu$.

Denoting the Borel $\sigma$-algebra on the real line $R$ by $\mathcal{B}(R)$, the distribution of $f$ is the probability measure $\mu_f$ on $(R, \mathcal{B}(R))$ given by $\mu_f(B) = \mu(f^{-1}(B))$. We interpret $\mu_f(B)$ as the probability that $f$ has a value in the set $B$.

A random variable $f: \Omega \to [0,1]$ is called an effect or fuzzy event. Thus, an effect is just a measurable fuzzy subset of $\Omega$. The set of effects is denoted by $\mathcal{E} = \mathcal{E}(\Omega, \mathcal{F})$. If $\mu$ is a probability measure on $(\Omega, \mathcal{F})$ and $f \in \mathcal{E}$, we define the probability of $f$ to be its expectation $E[f] = \int_R d\mu$. If $(f_i)$ is an increasing sequence in $\mathcal{E}$, then by the monotone convergence theorem, $E[\lim f_i] = \lim E[f_i]$ so $E$ is countably additive. Stated in another
Definition 2.1 Let $\mathcal{B}$ be a $\sigma$-field of $\Lambda$. An observable is a map $X: \mathcal{B} \rightarrow \mathcal{E}(\Omega, \mathcal{J})$ such that $X(\Lambda) = 1_\mathcal{J}$ and if $B_i \in \mathcal{B}_i$ $(i = 1, 2, 3 \ldots)$ are mutually disjoint, then $X(\bigcup B_i) = \sum X(B_i)$ where the convergence of the summation is pointwise.

Example 2.2 If $f: (\Lambda, \mathcal{B}) \rightarrow (\Omega, \mathcal{J})$ is a measurable function, the corresponding sharp observable $X_f: \mathcal{J} \rightarrow \mathcal{E}(\Lambda, \mathcal{B})$ is given by $X_f(B) = I_{f^{-1}(B)}$.

Definition 2.3 A state on $\mathcal{E}(\Omega, \mathcal{J})$ is a map $s: \mathcal{E}(\Omega, \mathcal{J}) \rightarrow [0, 1]$ that satisfies $s(1_\mathcal{J}) = 1$ and if $(f_i)$ is a sequence in $\mathcal{E}$ such that $\sum f_i \in \mathcal{E}(\Omega, \mathcal{J})$, then $s(\sum f_i) = \sum s(f_i)$.

Definition 2.4 $X: \mathcal{E}(\Omega, \mathcal{J}) \rightarrow \mathcal{E}(\Lambda, \mathcal{B})$ is a $\sigma$-morphism if $X(1_\mathcal{J}) = 1_\mathcal{B}$ and if $(f_i)$ is a sequence in $\mathcal{E}$ such that $\sum f_i \in \mathcal{E}(\Omega, \mathcal{J})$, then $X(\sum f_i) = \sum X(f_i)$.

Example 2.5 Let $\Omega = [0, 1]$ and $\Lambda = [1, 2]$. Let $\mathcal{J}$ and $\mathcal{B}$ be $\sigma$-fields of $\Omega$ and $\Lambda$, respectively. Define $X: \mathcal{E}(\Omega, \mathcal{J}) \rightarrow \mathcal{E}(\Lambda, \mathcal{B})$ by $X(f)(x) = f(x) - 1$.

Then $X$ is a $\sigma$-morphism. In fact, $X(1_\mathcal{J}) = 1_\mathcal{B} = 1_x$ and $X(f_i) = f_i - 1$.

Example 2.6 Let $\Omega = \Lambda = [0, 1]$. Let $\mathcal{J}$ and $\mathcal{B}$ be $\sigma$-fields of $\Omega$ and $\Lambda$, respectively. Define $X: \mathcal{E}(\Omega, \mathcal{J}) \rightarrow \mathcal{E}(\Lambda, \mathcal{B})$ by $X(f)(x) = \frac{1}{2}(f(x) + f(1 - x))$.

Then $X$ is a $\sigma$-morphism. In fact, $X(1_\mathcal{J}) = \frac{1}{2}(1_x + 1_{1-x}) = 1_x$ and $X(f_i) = \frac{1}{2}(f_i(x) + f_i(1 - x))$.

3. Basic properties

Theorem 3.1 (1) We have the followings.
1. If $X: \mathcal{E}(\Omega, \mathcal{J}) \rightarrow \mathcal{E}(\Lambda, \mathcal{B})$ is a $\sigma$-morphism, then $X(\lambda f) = \lambda X(f)$ for every $\lambda \in [0, 1]$ and $f \in \mathcal{E}(\Omega, \mathcal{J})$.
2. If $s: \mathcal{E}(\Omega, \mathcal{J}) \rightarrow [0, 1]$ is a state, then there exists a unique probability measure $\mu$ on $(\Omega, \mathcal{J})$ such that $s(f) = \int f d\mu$ for every $f \in \mathcal{E}(\Omega, \mathcal{J})$.

The next result shows that there exists a natural one-to-one correspondence between observables and $\sigma$-morphisms.

Theorem 3.2 (1) If $X: \mathcal{J} \rightarrow \mathcal{E}(\Lambda, \mathcal{B})$ is an observable, then $X$ has a unique extension to a $\sigma$-morphism $\tilde{X}: \mathcal{E}(\Omega, \mathcal{J}) \rightarrow \mathcal{E}(\Lambda, \mathcal{B})$. If $Y: \mathcal{E}(\Omega, \mathcal{J}) \rightarrow \mathcal{E}(\Lambda, \mathcal{B})$ is a $\sigma$-morphism, then $Y1_\mathcal{J}$ is an observable.

If $f: \Lambda \rightarrow \Omega$ is a measurable function, the corresponding sharp observable $X_f: \mathcal{J} \rightarrow \mathcal{E}(\Lambda, \mathcal{B})$ is given by $X_f(B) = I_{f^{-1}(B)}$. The next result shows that $X_f: \mathcal{J} \rightarrow \mathcal{E}(\Lambda, \mathcal{B})$ has a simple form.

Corollary 3.3 (1) If $f: \Lambda \rightarrow \Omega$ is a measurable function, then $X_f(g) = g \circ f$ for every $g \in \mathcal{E}(\Omega, \mathcal{J})$, where $X_f$ is an extension of $X_f$ in Example 2.2.

Theorem 3.4 If $\tilde{X}: \mathcal{E}(\Omega, \mathcal{J}) \rightarrow \mathcal{E}(\Lambda, \mathcal{B})$ is a $\sigma$-morphism, then
1. $X(0_\mathcal{J}) = 0_\Lambda$.
2. $\tilde{X}(\sum f_i) = \sum \tilde{X}(f_i)$.
3. If $f - g \in \mathcal{E}(\Omega, \mathcal{J})$, then $\tilde{X}(f - g) = \tilde{X}(f) - \tilde{X}(g)$. In particular, $\tilde{X}(1_\mathcal{J} - g) = 1_\Lambda - \tilde{X}(g)$.
4. If $f \leq g$, then $\tilde{X}(f) \leq \tilde{X}(g)$.
5. $\tilde{X}(f + g - fg) = \tilde{X}(f) + \tilde{X}(g) - \tilde{X}(fg)$.

Proof. (1) Let $f_1 = 1_{\mathcal{J}}$, $f_2 = 0_\mathcal{J}$ $(i \geq 2)$. Since $\sum f_i = \tilde{X}(1_\mathcal{J}) = 1_\Lambda$ and $\sum \tilde{X}(f_i) = \tilde{X}(f_1) + \sum \tilde{X}(f_i)$.
Proof.

\[ \text{Theorem 4.1} \]
\[ \mathcal{X} \left( \sum_{i=1}^{n} f_i \right) = \sum_{i=1}^{n} \mathcal{X} \left( f_i \right) \]
\[ \mathcal{X} \left( \sum_{i=1}^{n} f_i \right) = \sum_{i=1}^{n} \mathcal{X} \left( f_i \right) \]
\[ \mathcal{X} \left( \sum_{i=1}^{n} f_i \right) = \sum_{i=1}^{n} \mathcal{X} \left( f_i \right) \]

Let \( f_i = 0_i \) \((i \geq n + 1)\), then \( \sum_{i=1}^{n} f_i \). Thus
\[ \mathcal{X} \left( \sum_{i=1}^{n} f_i \right) = \mathcal{X} \left( \sum_{i=1}^{n} f_i \right) \]
\[ \mathcal{X} \left( \sum_{i=1}^{n} f_i \right) = \mathcal{X} \left( \sum_{i=1}^{n} f_i \right) \]
\[ \mathcal{X} \left( \sum_{i=1}^{n} f_i \right) = \mathcal{X} \left( \sum_{i=1}^{n} f_i \right) \]

(3) Since 
\[ \mathcal{X} (f) = \mathcal{X} (f - g + g) = \mathcal{X} (f) + \mathcal{X} (g) \], we have
\[ \mathcal{X} (f - g) = \mathcal{X} (f) - \mathcal{X} (g) \).

(4) Since \( \mathcal{X} (g) = \mathcal{X} (g - f) + \mathcal{X} (f) \), \( \mathcal{X} (g - f) \geq 0 \).

(5) It is trivial.

\[ \text{Theorem 3.5} \]
If \( f \colon (A_1, \mathcal{B}_1) \rightarrow (A_2, \mathcal{B}_2) \) and \( g \colon (A_2, \mathcal{B}_2) \rightarrow (A_3, \mathcal{B}_3) \) are measurable functions, then
\[ \mathcal{X}_{g \cdot f} = \mathcal{X}_f \cdot \mathcal{X}_g. \]

Proof. Note that \( \mathcal{X}_f : \mathcal{E} (A_2, \mathcal{B}_2) \rightarrow \mathcal{E} (A_1, \mathcal{B}_1) \), \( \mathcal{X}_g : \mathcal{E} (A_3, \mathcal{B}_3) \rightarrow \mathcal{E} (A_2, \mathcal{B}_2) \) and \( \mathcal{X}_{g \cdot f} : \mathcal{E} (A_3, \mathcal{B}_3) \rightarrow \mathcal{E} (A_1, \mathcal{B}_1) \). Since
\[ \mathcal{X}_{g \cdot f} (h) (\omega) = h \cdot (g \cdot f) (\omega) \]
\[ = (h \cdot g) (\omega) \]
\[ = \mathcal{X}_g (h) (\omega) \]
\[ = \mathcal{X}_f \cdot \mathcal{X}_g (h) (\omega) \]

Hence \( \mathcal{X}_{g \cdot f} = \mathcal{X}_f \cdot \mathcal{X}_g \).

4. Main results

\[ \text{Theorem 4.1} \]
Let \( f : A_2 \rightarrow A_1 \) be a measurable function and \( \mu_i : (A_i, \mathcal{B}_i) \rightarrow [0, 1] \) be a probability measure \((i = 1, 2)\). If \( \mu_1 = \mu_2 \), then \( \mu_2 \cdot \mathcal{X}_f = \mu_1 \).

Proof. Let \( g = \sum_{i=1}^{n} c_i I_{R_i} \) be a simple function in \( \mathcal{E} (A_1, \mathcal{B}_1) \), then by Corollary 3.3,
\[ \mu_2 \cdot \mathcal{X}_f (g) = \int \mathcal{X}_f (g) \, d\mu_2 \]
\[ = \int (g \cdot f) \, d\mu_2 \]
\[ = \int \sum_{i=1}^{n} (c_i I_{R_i} \cdot f) \, d\mu_2 \]
\[ = \int \sum_{i=1}^{n} c_i I_{R_i} \cdot d\mu_2. \]

And, by the definition of expectation and distribution,
\[ \int \sum_{i=1}^{n} c_i I_{R_i} \cdot d\mu_2 = \int \sum_{i=1}^{n} c_i \mu_2 (f^{-1} (R_i)) \]
\[ = \int \sum_{i=1}^{n} c_i \mu (R_i) \]
\[ = \int \mu_1 (g) \]

Hence, \( \mu_2 \cdot \mathcal{X}_f (g) = \mu_1 (g) \).

Now for an arbitrary \( g \in \mathcal{E} (A_1, \mathcal{B}_1) \), there exists an increasing sequence of simple functions \( g_n \in \mathcal{E} (A_1, \mathcal{B}_1) \) such that \( \lim_{n \to \infty} g_n = g \). Then by Corollary 3.3,
\[ \mu_2 \cdot \mathcal{X}_f (g) = \int \mathcal{X}_f (g) \, d\mu_2 \]
\[ = \int \mathcal{X}_f \left( \lim_{n \to \infty} g_n \right) \, d\mu_2 \]
\[ = \int \lim_{n \to \infty} g_n \, f) \, d\mu_2. \]

By the monotone convergence theorem and the continuity of probability,
\[ \int \lim_{n \to \infty} g_n \, f) \, d\mu_2 = \lim_{n \to \infty} \int g_n \, f) \, d\mu_2 \]
\[ = \lim_{n \to \infty} \mu_2 \cdot \mathcal{X}_f (g_n) \]
\[ = \lim_{n \to \infty} \mu_1 (g_n) \]
\[ = \mu_1 (g). \]

Therefore, \( \mu_2 \cdot \mathcal{X}_f = \mu_1 \).

\[ \text{Theorem 4.2} \]
Let \( \mathcal{X} : \mathcal{E} (\Omega, \mathcal{F}) \rightarrow \mathcal{E} (A, \mathcal{B}) \) be a \( \sigma \)-morphism. If \( \left( g_n \right) \) is an increasing sequence in \( \mathcal{E} (\Omega, \mathcal{F}) \) with \( \lim_{n \to \infty} g_n = g \) then \( \lim_{n \to \infty} \mathcal{X} (g_n) = \mathcal{X} (g) \) in \( \mathcal{E} (A, \mathcal{B}) \).

Proof. Let \( f_1 = g_1 \) and \( f_n = g_n - g_{n-1} \) \((n \geq 2)\). Then \( f_n \in \mathcal{E} (\Omega, \mathcal{F}) \) for all \( n \) and \( g_n = \sum_{i=1}^{n} f_i \).

Since \( g = \sum_{i=1}^{\infty} f_i \), we have
\[ \mathcal{X} (g) = \mathcal{X} \left( \sum_{i=1}^{\infty} f_i \right) \]
\[ = \sum_{i=1}^{\infty} \mathcal{X} (f_i) \]
\[ = \lim_{n \to \infty} \sum_{i=1}^{n} \mathcal{X} (f_i) \]
\[ = \lim_{n \to \infty} \mathcal{X} \left( \sum_{i=1}^{n} f_i \right) \]
\[ = \lim_{n \to \infty} \mathcal{X} (g_n). \]
Hence \( \lim_{n \to \infty} \bar{X}(g_n) = \bar{X}(g) \).

**Corollary 4.3** Let \( \bar{X} : \mathcal{E}(\Omega, \mathcal{F}) \to \mathcal{E}(\Lambda, \mathcal{B}) \) be a \( \sigma \)-morphism. If \( (g_n) \) is a decreasing sequence in \( \mathcal{E}(\Omega, \mathcal{F}) \) with \( \lim_{n \to \infty} g_n = g \), then \( \lim_{n \to \infty} \bar{X}(g_n) = \bar{X}(g) \) in \( \mathcal{E}(\Lambda, \mathcal{B}) \).

**Theorem 4.4** Let \( \bar{X} : \mathcal{E}(\Omega, \mathcal{F}) \to \mathcal{E}(\Lambda, \mathcal{B}) \) be a \( \sigma \)-morphism. If \( (g_n) \) is sequence in \( \mathcal{E}(\Omega, \mathcal{F}) \) with \( \lim_{n \to \infty} g_n = g \), then \( \lim_{n \to \infty} \bar{X}(g_n) = \bar{X}(g) \) in \( \mathcal{E}(\Lambda, \mathcal{B}) \).

**Proof:** First, we prove that
\[
\bar{X} \left( \lim_{n \to \infty} g_n \right) \leq \lim_{n \to \infty} \bar{X}(g_n) \\
\leq \lim_{n \to \infty} \bar{X}(g_n) \\
= \bar{X} \left( \lim_{n \to \infty} g_n \right).
\]

Let \( f_n = \inf_{i \geq n} g_i \). Since \( (f_n) \) is an increasing sequence in \( \mathcal{E}(\Omega, \mathcal{F}) \), by Theorem 4.2, we have
\[
\bar{X} \left( \lim_{n \to \infty} g_n \right) = \bar{X} \left( \sup_{n \geq 1} \inf_{i \geq n} g_i \right) \\
= \bar{X} \left( \sup_{n \geq 1} f_n \right) \\
= \bar{X} \left( \lim_{n \to \infty} f_n \right) \\
= \lim_{n \to \infty} \bar{X}(f_n).
\]

Let \( n \in \mathbb{N} \). Then, for each \( n \leq i \), \( f_n \leq g_i \), we have \( \bar{X}(f_n) \leq \bar{X}(g_i) \) and hence \( \bar{X}(f_n) \leq \inf_{i \geq n} \bar{X}(g_i) \).

Therefore
\[
\sup_{n \geq 1} \bar{X}(f_n) \leq \sup_{n \geq 1} \inf_{i \geq n} \bar{X}(g_i) = \lim_{n \to \infty} \bar{X}(g_n).
\]

But, since \( \lim_{n \to \infty} \bar{X}(f_n) = \sup_{n \geq 1} \bar{X}(f_n) \),
\[
\bar{X} \left( \lim_{n \to \infty} g_n \right) \leq \lim_{n \to \infty} \bar{X}(g_n).
\]

Similarly, \( \lim_{n \to \infty} \bar{X}(g_n) \leq \bar{X} \left( \lim_{n \to \infty} g_n \right) \).

For \( g_n \in \mathcal{E}(\Omega, \mathcal{F}) \), since
\[
\lim_{n \to \infty} \bar{X}(g_n) \leq \bar{X} \left( \lim_{n \to \infty} g_n \right) \\
= \bar{X} \left( \lim_{n \to \infty} g_n \right) \\
= \bar{X} \left( \lim_{n \to \infty} g_n \right) \\
\leq \lim_{n \to \infty} \bar{X}(g_n),
\]
we have
\[
\lim_{n \to \infty} \bar{X}(g_n) = \lim_{n \to \infty} \bar{X}(g_n) = \lim_{n \to \infty} \bar{X}(g_n) \]

\[
\lim_{n \to \infty} \bar{X}(g_n) = \bar{X} \left( \lim_{n \to \infty} g_n \right) = \bar{X}(g).
\]

Hence \( \lim_{n \to \infty} \bar{X}(g_n) = \bar{X}(g) \).

**References**


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