Intuitionistic Fuzzy Generalized Topological Spaces

On Intuitionistic Fuzzy Generalized Topological Spaces

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Abstract
In this paper, we introduce the concepts of intuitionistic fuzzy generalized topological spaces and intuitionistic gradation of generalized openness. We also introduce the concepts of IFG-mapping, weak IFG-mapping and IFG-open mapping, and obtain some characterizations for such mappings.

Key Words: intuitionistic gradation of generalized openness, intuitionistic fuzzy generalized topological spaces, IFG-mapping, weak IFG-mapping.

1. Introduction
In 1992 [2, 4], Chattopadyay et al. introduced the concept of fuzzy topology redefined by a gradation of openness and investigated some fundamental properties. In [6], Ramadan called the fuzzy topology “a smooth topology” and studied several topological properties in the fuzzy topological space. Atanassov [1] introduced the concept of intuitionistic fuzzy set which is a generalization of fuzzy set in Zadeh’s sense [7]. In [5], Mondal and Sambanta introduced and investigated the concept of intuitionistic gradations of openness which is defined by Chattopadyay. In this paper, we introduce the concept of intuitionistic gradation of generalized openness which is a generalization of the concepts of intuitionistic gradations of openness and gradations of basic properties for intuitionistic fuzzy generalized topological spaces defined by the given intuitionistic gradation of generalized openness. We also introduce the concepts of IFG-mapping, weak IFG-mapping and IFG-open mapping and obtain some characterizations for such mappings.

2. Preliminaries
Let I be the unit interval [0, 1] of the real line. A member A of I^X is called a fuzzy set of X. For any A ∈ I^X, A^c denotes the complement 1_X - A. By 0_X and 1_X we denote constant maps on X with value 0 and 1, respectively.

Definition 2.1 ([2,4]). Let X be a non-empty set and τ:I^X → I be a mapping satisfying the following conditions:
(1) τ(0_X)=τ(1_X)=1;
(2) ∀ A, B ∈ I^X, τ(A ∩ B) ≥ τ(A) ∧ τ(B);
(3) For every subfamily \{A_i: i ∈ J\} ⊆ I^X, τ(∪_{i ∈ J} A_i) ≥ \bigwedge_{i ∈ J} τ(A_i).

Then the mapping τ:I^X → I is called a fuzzy topology (or gradation of openness) on X. We call the ordered pair (X,τ) a fuzzy topological space. The value τ(A) is called the degree of openness of A.

Definition 2.2 ([1]). Let X be a nonempty set. An intuitionistic fuzzy set A is an ordered pair A=(< μ_A(x), γ_A(x) > : x ∈ X) (simply, A=(μ_A, γ_A)) where the functions μ_A: X → I and γ_A: X → I denote the degree of membership and the degree of non-membership, respectively, and 0 ≤ μ_A(x) + γ_A(x) ≤ 1 for each x ∈ X.

Definition 2.3 ([5]). An intuitionistic gradation of openness (briefly IGO) of fuzzy subsets of a set X is an ordered pair (τ, τ') of functions τ, τ':I^X → I such that
Let $f$ be a map from a set $X$ to a set $Y$. Let $A=(\mu_A, \gamma_A)$ be an intuitionistic fuzzy set of $X$ and $B=(\mu_B, \gamma_B)$ an intuitionistic fuzzy set of $Y$. Then:

1. The image of $A$ under $f$, denoted by $f(A)$ is an intuitionistic fuzzy set in $Y$ defined by
   $$f(A)= ( f(\mu_A), 1_Y- f(1_X- \gamma_A)) .$$
2. The inverse image of $B$ under $f$, denoted by $f^{-1}(B)$ is an intuitionistic fuzzy set in $X$ defined by
   $$f^{-1}(B)= ( f^{-1}(\mu_B), f^{-1}(\gamma_B)) .$$

**3. Intuitionistic fuzzy generalized topological spaces**

**Definition 3.1** An intuitionistic gradation of generalized openness (briefly IGGO) of fuzzy subsets of a set $X$ is an ordered pair $(\tau, \bar{\tau})$ of functions $\tau, \bar{\tau}: I^X \rightarrow I$ such that

- $(\text{IGGO1})$ $\tau(A)+\bar{\tau}(A) \leq 1$ for all $A \subseteq I^X$;
- $(\text{IGGO2})$ $\tau(0_X)=\bar{\tau}(0_X)=1$;
- $(\text{IGGO3})$ For every subfamily $\{A_i : i \in J\} \subseteq I^X$, $\tau(\bigcup_{i \in J} A_i) \geq \bigwedge_{i \in J} \tau(A_i)$ and $\bar{\tau}(\bigcup_{i \in J} A_i) \leq \bigvee_{i \in J} \bar{\tau}(A_i)$.

Then the triplet $(X, \tau, \bar{\tau})$ is called an intuitionistic fuzzy generalized topological space (briefly IFGTS) on $X$. $\tau$ and $\bar{\tau}$ may be interpreted as gradation of generalized openness and gradation of generalized nonopenness, respectively.

Obviously we get the following implications:

gradation of openness $\Rightarrow$ intuitionistic gradation of openness
gradation of openness $\Rightarrow$ intuitionistic gradation of super-openness
gradation of generalized openness

**Definition 3.2** Let $X$ be a nonempty set and two functions $\psi, \psi^*: I^X \rightarrow I$ be satisfying

- $(\text{IGC1})$ $\psi(A)+\psi^*(A) \leq 1$ for all $A \subseteq I^X$;
- $(\text{IGC2})$ $\psi(0_X)=\psi^*(0_X)=0$;
- $(\text{IGC3})$ For every subfamily $\{A_i : i \in J\} \subseteq I^X$, $\psi(\bigcap_{i \in J} A_i) \geq \bigwedge_{i \in J} \psi(A_i)$ and $\psi^*(\bigcap_{i \in J} A_i) \leq \bigvee_{i \in J} \psi^*(A_i)$.

Then the ordered pair $(\psi, \psi^*)$ is called an intuitionistic fuzzy supra-topological space (briefly IFSTS) on $X$. $\psi$ and $\psi^*$ may be interpreted as gradation of supra-openness and gradation of supra-nonopenness, respectively.

**Definition 2.4** (5]). Let $X$ be a nonempty set and two functions $\psi, \psi^*: I^X \rightarrow I$ be satisfying

- $(\text{IGC1})$ $\psi(A)+\psi^*(A) \leq 1$ for all $A \subseteq I^X$;
- $(\text{IGC2})$ $\psi(0_X)=\psi^*(0_X)=0$;
- $(\text{IGC3})$ For every subfamily $\{A_i : i \in J\} \subseteq I^X$, $\psi(\bigcap_{i \in J} A_i) \geq \bigwedge_{i \in J} \psi(A_i)$ and $\psi^*(\bigcap_{i \in J} A_i) \leq \bigvee_{i \in J} \psi^*(A_i)$.

Then the ordered pair $(\psi, \psi^*)$ is called an intuitionistic gradation of closedness (briefly IGC) on $X$. $\psi$ and $\psi^*$ may be interpreted as gradation of closedness and gradation of nonclosedness, respectively.

**Definition 2.5** ([3]). An intuitionistic gradation of supra-openness (briefly IGGO) of fuzzy subsets of a set $X$ is an ordered pair $(\tau, \bar{\tau})$ of functions $\tau, \bar{\tau}: I^X \rightarrow I$ such that

- $(\text{IGGO1})$ $\tau(A)+\bar{\tau}(A) \leq 1$ for all $A \subseteq I^X$;
- $(\text{IGGO2})$ $\tau(0_X)=\bar{\tau}(0_X)=1$;
- $(\text{IGGO3})$ For every subfamily $\{A_i : i \in J\} \subseteq I^X$, $\tau(\bigcup_{i \in J} A_i) \geq \bigwedge_{i \in J} \tau(A_i)$ and $\bar{\tau}(\bigcup_{i \in J} A_i) \leq \bigvee_{i \in J} \bar{\tau}(A_i)$.

Then the triplet $(X, \tau, \bar{\tau})$ is called an intuitionistic fuzzy supra-topological space (briefly IFGTS) on $X$. $\tau$ and $\bar{\tau}$ may be interpreted as gradation of generalized openness and gradation of generalized nonopenness, respectively.
**Theorem 3.3** (1) If \((τ, τ')\) is an IGGO on \(X\), then the ordered pair \((ψ, ψ')\), defined by \(ψ(A) = τ(A')\) and \(ψ'(A) = τ'(A')\) is an intuitionistic gradation of generalized closedness on \(X\).

(2) If \((ψ, ψ')\) is an intuitionistic gradation of generalized closedness on SXG, then the ordered pair \((τ, τ')\), defined by \(τ(A) = ψ(A')\) and \(τ'(A) = ψ'(A')\) is an intuitionistic gradation of generalized openness on \(X\).

Proof. (1) From Definition 3.7, we have

\[
ψ(A) + ψ'(A) = τ(A') + τ'(A') ≤ 1, \quad ψ(A) + ψ'(A) ≤ 1 \text{ for all } A ∈ 𝐼.\
\]

From \(ψ(1_X)=τ(1_X')=τ(0_X)=1\) and \(ψ'(1_X)=τ'(1_X')=τ'(0_X)=0\), we have the condition (IGGC2).

For every subfamily \(\{A_i : i ∈ J\} \subseteq 𝐼\),

\[
ψ(\bigcap_{i ∈ J} A_i) = τ(\bigcap_{i ∈ J} A_i')
= τ(\bigcup_{i ∈ J} A_i')
≥ \bigwedge_{i ∈ J} τ(A_i')
= \bigwedge_{i ∈ J} ψ(A_i)
\]

and

\[
ψ'(\bigcap_{i ∈ J} A_i) = τ'(\bigcap_{i ∈ J} A_i')
= τ'(\bigcup_{i ∈ J} A_i')
≤ \bigvee_{i ∈ J} τ'(A_i')
= \bigvee_{i ∈ J} ψ'(A_i)
\]

Thus we have (IGGC3).

(2) It is similar to (1).

Henceforward, the ordered pair \((ψ, ψ')\) is an intuitionistic gradation of generalized closedness defined by \(ψ(A) = τ(A')\) and \(ψ'(A) = τ'(A')\) on an IFGTS \((X, τ, τ')\) unless explicitly stated.

**Remark 3.4** Let \(\{(τ_i, τ'_i) : i ∈ J\}\) be a family of IGGO’s on \(X\). Then the intersection \(\bigcap_{i ∈ J} (τ_i, τ'_i) = \bigwedge_{i ∈ J} τ_i(A) = \bigwedge_{i ∈ J} τ'_i(A)\) is an IGGO on \(X\), where \(\bigwedge_{i ∈ J} τ_i(A) = \bigwedge_{i ∈ J} τ'_i(A)\), \(\bigvee_{i ∈ J} τ_i(A) = \bigvee_{i ∈ J} τ'_i(A)\).

**Definition 3.5** Given a set \(X\) and IFGT’s \((τ_i, τ'_i)\) and \((τ_j, τ'_j)\) on \(X\). We say that \((τ_j, τ'_j)\) finer than \((τ_i, τ'_i)\) (denoted by \((τ_i, τ'_i) ≤ (τ_j, τ'_j)\)) if \(τ_i(A) ≤ τ_j(A)\) and \(τ'_i(A) ≥ τ'_j(A)\) for every \(A ∈ 𝐼\).

**Theorem 3.6** Let \((τ, τ')\) denote an IFGT on \(X\) such that \(τ(A) ≥ r\) and \(τ'_i(A) ≤ s\) for every \(A ∈ 𝐼\). Then if \(r ≤ l\) and \(s ≥ m\), then \((τ, τ') ≤ (τ, τ')\).

Proof. The proof is straightforward.

**Definition 3.7** Let \((X, τ, τ')\) be an IFGTS and \(A ∈ 𝐼\). Then the closure of \(A\), denoted by \(\overline{A}\), is defined by \(\overline{A} = \bigcap \{K ∈ 𝐼 : τ(K) > 0 \text{ and } τ'(K) ≤ τ(A), A ⊆ K\}\) and the interior of \(A\), denoted by \(A^o\), is defined by \(A^o = \bigcup \{K ∈ 𝐼 : τ(K) > 0 \text{ and } τ'(K) ≤ τ(A), K ⊆ A\}\).

**Theorem 3.8** Let \((X, τ, τ')\) be an IFGTS and \(A, B ∈ 𝐼\). Then

(1) \(ψ'(\overline{A}) ≤ ψ'(A)\);
(2) \(τ'(A^o) ≤ τ'(A)\);
(3) \(A ⊆ B \rightarrow ψ'(B) ≤ ψ'(A) \rightarrow \overline{A} ⊆ \overline{B}\);
(4) \(A ⊆ B \rightarrow τ'(A) ≤ τ'(B) \rightarrow A^o ⊆ B^o\).

Proof. (1) From Definition 3.1 and Definition 3.7, it follows \(ψ'(A) = ψ'(\bigcap \{K ∈ 𝐼 : τ(K) > 0 \text{ and } τ'(K) ≤ τ'(A)\}) ≤ \bigvee \{ψ'(K) : τ(K) > 0 \text{ and } τ'(K) ≤ τ'(A)\}\).

(2) It is similar to (1).

(3) We have the following:
\(\overline{A} = \bigcap \{K ∈ 𝐼 : τ(K) > 0 \text{ and } τ'(K) ≤ τ'(A)\}, A ⊆ K\)
\(\bigcap \{K ∈ 𝐼 : τ(K) > 0 \text{ and } τ'(K) ≤ τ'(B)\}, B ⊆ K\).

Hence \(\overline{A} ⊆ \overline{B}\).

(4) The proof is similar to (3).

**Theorem 3.9** Let \((X, τ, τ')\) be an IFGTS and \(A ∈ 𝐼\). Then

(1) \((\overline{A})^o = ((A^o)^o)^o\);
(2) \(\overline{A} = (\overline{A})^o\);
(3) \(A^o = (A^o)^o\).

Proof. (1) From Definition 3.7, we have
\(\overline{A}^o = \bigcap \{K ∈ 𝐼 : τ(K) > 0 \text{ and } τ'(K) ≤ τ'(A)\}, A ⊆ K\)^o
\(= \bigcup \{K ∈ 𝐼 : τ(K) > 0 \text{ and } τ'(K) ≤ τ'(A)\}, K ⊆ A\)^o
\(= \bigcup \{U ∈ 𝐼 : \overline{U} ⊆ A^o, τ(U) > 0 \text{ and } τ'(U) ≤ τ'(A)\}\).

(2), (3) and (4) are easily obtained from (1).

**Theorem 3.10** Let \((X, τ, τ')\) be an IFGTS and \(A, B ∈ 𝐼\). Then

(1) \(A ⊆ \overline{A}^o, A^o ⊆ \overline{A}\);
(2) \(A ⊆ (\overline{A})^o, (A^o)^o ⊆ A^o\).
4. IFG-mappings and Weak IFG-mappings

Definition 4.1 Let \((X, \tau_1, \tau'_1)\) and \((Y, \tau_2, \tau'_2)\) be two IFGTS’s. \(f: X \to Y\) is called

(1) an IFG-mapping if \(\tau_1(f^{-1}(A)) \geq \tau_2(A)\) and \(\tau'_1(f^{-1}(A)) \leq \tau'_2(A)\) for every \(A \in P^Y\);

(2) a weak IFG-mapping if \(\tau_1(A) > 0 \Rightarrow \tau_1(f^{-1}(A)) > 0\) and \(\tau'_1(f^{-1}(A)) \leq \tau'_2(A)\) for every \(A \in P^Y\).

It is obvious that every IFG-map is a weak IFG-map from the above definition. But the converse is not always true as shown in the next example.

Example 4.2 Let \(X = I\) and and let \(N\) denote the set of all natural numbers. For \(n \in N\), we consider a fuzzy set \(A_n\) as he following:

\[ A_n(x) = \frac{n-1}{n} x \quad \text{for} \quad x \in X. \]

Define \(\tau_n, \tau'_n: I \to I\) by

\[ \tau_1(0) = 1, \quad \tau'_1(0) = 0; \]

\[ \tau_1(A) = \frac{1}{n+2}, \quad \tau'_1(A) = \frac{1}{n+2} \quad \text{for each} \quad n \in N; \]

\[ \tau_1(A) = 0, \quad \tau'_1(A) = 1 \quad \text{for all other fuzzy set} \quad A \in X. \]

Then \(\tau_1, \tau'_1\) and \(\tau_2, \tau'_2\) are two intuitionistic gradations of generalized openness on \(X\). Consider the identity mapping \(f: (X, \tau_1, \tau'_1) \to (Y, \tau_2, \tau'_2)\). Then obviously \(f\) is a weak IFG-mapping. But since for each fuzzy set \(A_n, \tau_1(\tau'_1(A_n)) > \tau_2(f^{-1}(A_n))\), \(f\) is not an IFG-mapping.

Theorem 4.3 Let \((X, \tau_1, \tau'_1)\) and \((Y, \tau_2, \tau'_2)\) be two IFGTS’s. Then a mapping \(f: X \to Y\) is a weak IFG-mapping iff for every \(A \in P^Y, \psi_2(A) > 0 \Rightarrow \psi_1(f^{-1}(A)) > 0 \) and \(\psi'_2(f^{-1}(A)) \leq \psi'_1(A)\).

Proof. Suppose \(f\) is a weak IFG-mapping and let \(\psi(A) > 0\) for \(A \in P^Y\) then \(\psi(A^c) = \tau_2(\tau'_2(A^c)) > 0\). Since \(f\) is a weak IFG-mapping, it follows \(\tau_1(f^{-1}(A^c)) > 0\) and \(\tau'_1(f^{-1}(A^c)) \leq \tau'_2(A^c)\). Thus we get \(\psi(f^{-1}(A)) > 0\) and \(\psi'_1(f^{-1}(A)) \leq \psi'_2(A)\).

The converse is obvious.

Theorem 4.4 Let \((X, \tau_1, \tau'_1)\) and \((Y, \tau_2, \tau'_2)\) be two IFGTS’s. Then \(f: X \to Y\) is an IFG-mapping iff for every \(A \in P^Y, \psi_2(A) \leq \psi_1(f^{-1}(A))\) and \(\psi'_2(f^{-1}(A)) \leq \psi'_1(A)\).

Proof. The proof is similar to that of Theorem 4.3.

Theorem 4.5 Let \((X, \tau_1, \tau'_1)\) and \((Y, \tau_2, \tau'_2)\) be two IFGTS’s. If \(f: X \to Y\) is a weak IFG-mapping, then we have

(1) \(f(A) \subseteq f(A)^c\) for every \(A \in P^Y\);

(2) \(f^{-1}(A) \subseteq f^{-1}(A)^c\) for every \(A \in P^Y\);

(3) \(f^{-1}(A^c) \subseteq (f^{-1}(A))^c\) for every \(A \in P^Y\).
Proof. (1) Let $A \in F^X$; then by Definition 3.7 and Theorem 4.3, we have

$$f^{-1}(\overline{f(A)}) \subseteq f^{-1}([1 \cap \{ U \in F^X : \psi_2(U) > 0 \} \land \psi_2(U) \leq \psi_2(f(A)), f(A) \subseteq U])$$

$$\subseteq \cap \{ f^{-1}(U) \cap F^X : \psi_1(f^{-1}(U)) > 0 \land \psi_1(f^{-1}(U)) \leq \psi_1(f(A)), A \subseteq f^{-1}(U) \}.$$

From $\psi_1(f^{-1}(U)) > 0$ and Theorem 3.11, it follows $A \subseteq f^{-1}(f(A)) = f^{-1}(U)$, and so $\cap \{ f^{-1}(U) \cap F^X : \psi_1(f^{-1}(U)) > 0 \land \psi_1(f^{-1}(U)) \leq \psi_1(f(A)), A \subseteq f^{-1}(U) \} \subseteq A$.

This implies $f(\overline{A}) \subseteq f(A)$.

(2) It follows from (1).

(3) It obtains by (2) and Theorem 3.9.

Corollary 4.6 Let $(X, \tau_1, \tau_2)$ and $(Y, \tau_2, \tau_3)$ be two IFGT’s. If $f: X \to Y$ is an IFG-mapping, then we have

1. $f(\overline{A}) \subseteq f(A)$ for every $A \in F^X$.
2. $f^{-1}(\overline{A}) \subseteq f^{-1}(A)$ for every $A \in F^X$.
3. $f^{-1}(A^o) \subseteq (f^{-1}(A))^o$ for every $A \in F^X$.

Definition 4.7 Let $(X, \tau_1, \tau_2)$ and $(Y, \tau_2, \tau_3)$ be two IFGT’s: $f: X \to Y$ is called an IFG-open mapping (resp., IFG-closed mapping) iff $\tau_2(A) \subseteq \tau_1(f(A))$ (resp., $\tau_2(f(A)) \subseteq \tau_1(A)$) for every $A \in F^X$.

Theorem 4.8 Let $(X, \tau_1, \tau_2)$ and $(Y, \tau_2, \tau_3)$ be two IFGT’s: If a mapping $f: X \to Y$ is IFG-open, then $f(A^o) \subseteq (f(A))^o$ for every $A \in F^X$.

Proof. For every $A \in F^X$, we have

$$f(A^o) = f(\cup \{ U \in F^X : \tau_2(U) > 0 \} \land \tau_2(U) \leq \tau_2(A), U \subseteq A)$$

$$= f(\cup \{ U \in F^X : \tau_2(U) > 0 \} \land \tau_2(U) \leq \tau_2(f(A)), f(U) \subseteq f(A))$$

$$\subseteq f(\cup \{ U \in F^X : \tau_2(f(U)) > 0 \} \land \tau_2(f(U)) \leq \tau_2(U), f(U) \subseteq f(A)).$$

From $\tau_2(f(U)) > 0$ and Theorem 3.11, it follows $f(U)^o = f(U) \subseteq (f(A))^o$.

Thus we get $f(A^o) \subseteq (f(A))^o$.

Theorem 4.9 Let $(X, \tau_1, \tau_2)$ and $(Y, \tau_2, \tau_3)$ be two IFGT’s. If $f: X \to Y$ is an injective IFG-closed mapping, then $f(\overline{A}) \subseteq f(A)$ for every $A \in F^X$.

Proof. Let $A \in F^X$; then since $f$ is an injective IFG-closed mapping, we have

$$f(\overline{A}) = f(\cap \{ U \in F^X : \psi_1(U) > 0 \} \land \psi_1(U) \leq \psi_1(f(A)), f(A) \subseteq U)$$

$$= \cap \{ f(U) \in F^X : \psi_1(f(U)) > 0 \} \land \psi_1(f(U)) \leq \psi_1(f(A)), f(A) \subseteq f(U)$$

$$\subseteq \cap \{ f(U) \in F^X : \psi_2(f(U)) > 0 \} \land \psi_2(f(U)) \leq \psi_2(f(A)), f(A) \subseteq f(U)).$$

Thus from $\psi_2(f(U)) > 0$ and Theorem 3.11, we have $f(\overline{A}) \subseteq f(A)$.

References


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