Interval-Valued Fuzzy m-semicontinuous 함수의 특성 연구

Characterizations For Interval-Valued Fuzzy m-semicontinuous Mappings On Interval-Valued Fuzzy Minimal Spaces

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요 약

In [5], we introduced the concepts of IVF m-semiopen sets and IVF m-semicontinuous mappings on interval-valued fuzzy minimal spaces. In this paper, we investigate some properties of IVF m-semiopen sets and characterizations for the IVF m-semicontinuous mapping.

Key Words: interval-valued fuzzy minimal spaces, IVF m-semiopen sets, IVF m-semicontinuous

1. Introduction and Preliminaries

Zadeh [7] introduced the concept of fuzzy set and several researchers were concerned about the generalizations of the concepts of fuzzy sets, intuitionistic fuzzy sets [1] and interval-valued fuzzy sets [3]. Alimohammady and Roohi [2] introduced fuzzy minimal structures and fuzzy minimal spaces and some results are given. In [4], Min introduced the concepts of IVF minimal structures and IVF m-continuous mappings which are generalizations of IVF topologies and IVF continuous mappings [6], respectively. In [5], Min et al. introduced the concepts of IVF m-semiopen sets and IVF m-semicontinuous mappings on interval-valued fuzzy minimal spaces. We investigated basic properties of IVF m-semiopen sets and IVF m-semicontinuous mappings. In this paper, we investigate characterizations for the IVF m-semicontinuous mapping and some properties of IVF m-semiopen sets.

Let \( D[0,1] \) be the set of all closed subintervals of the interval \([0,1]\). The elements of \( D[0,1] \) are generally denoted by capital letters \( M, N, \ldots \) and note that \( M= [M^L, M^U] \), where \( M^L \) and \( M^U \) are the lower and the upper end points respectively. We also note that

\[
(1) \quad (\forall M, N \in D[0,1])
\]

\[
(2) \quad (\forall M, N \in D[0,1])
\]

\[
M=-N \iff M^L=N^U, \quad M^U=N^L.
\]

For each \( M \in D[0,1] \), the complement of \( M \), denoted by \( M^c \), is defined by \( M^c=[1-M^U, 1-M^L] \).

Let \( X \) be a nonempty set. A mapping \( A: X \rightarrow D[0,1] \) is called an interval-valued fuzzy set (simply, IVF set) in \( X \). For each \( x \in X \), \( A(x) \) is a closed interval whose lower and upper end points are denoted by \( A(x)^L \) and \( A(x)^U \), respectively. For any \( [a,b] \in D[0,1] \), the IVF set whose value is the interval \([a,b]\) for all \( x \in X \) is denoted by \([a,b]_x \). We denote \( \emptyset \) and \( \tilde{1} \) as follows: \( \emptyset = [0,0], \tilde{1} = [1,1] \). In particular, for any \( c \in [a,b] \), the IVF set whose value is \( c(x)=[c,c] \) for all \( x \in X \) is denoted by simply \( c \). For a point \( p \in X \) and for \( [a,b] \in D[0,1] \) with \( b=0 \), the IVF set which takes the value \([a,b]\) at \( p \) and \( \tilde{1} \) elsewhere in \( X \) is called an interval-valued fuzzy point (simply, IVF point) and is denoted by \([a,b]_p \). In particular, if \( b=a \), then it is also denoted by \( a_p \). We denote the set of all IVF sets in \( X \) by IVF\( (X) \). An IVF point \( M_x \), where \( M \in D[0,1] \), is said to belong to an IVF set \( A \) in \( X \), denoted by \( M_x \in A \), if \( A(x)^L \geq M^L \) and \( A(x)^U \geq M^U \). In [6], it has been shown that \( A = \bigcup \{ M_x : M_x \in A \} \).

For every \( A, B \in IVF(X) \), we define

\[
A = B \iff (\forall x \in X)(A(x)^L = B(x)^L, A(x)^U = B(x)^U).
\]
A ⊆ B ⇔ (∀ x ∈ X) (A(x)^θ ⊆ B(x)^θ), A(x)^θ ⊆ B(x)^θ).

The complement A' of A is defined by

\[ A'(x)^θ = 1 - A(x)^θ \]

and

\[ \overline{A'(x)^θ} = 1 - \overline{A(x)^θ} \]

for all x ∈ X.

For a family of IVF sets \( \{A_i; i ∈ J\} \) where J is an index set, the union

\[ G = \bigcup_{i ∈ J} A_i \]

and the intersection

\[ F = \bigcap_{i ∈ J} A_i \]

are defined by

\[ G(x)^θ = \sup_{i ∈ J} [A_i(x)]^θ, \]

\[ F(x)^θ = \inf_{i ∈ J} [A_i(x)]^θ, \]

respectively, for all x ∈ X.

Let \( f : X → Y \) be a mapping and let A be an IVF set in X. Then the image of A under f, denoted by \( f(A) \), is defined as follows

\[ f(A)(y)^θ = \left\{ \begin{array}{ll}
\sup_{z ∈ f^{-1}(y)} [A(z)]^θ, & \text{if } f^{-1}(y) \neq \emptyset, \\
0, & \text{otherwise},
\end{array} \right. \]

\[ f(A)(y)^θ = \left\{ \begin{array}{ll}
\inf_{z ∈ f^{-1}(y)} [A(z)]^θ, & \text{if } f^{-1}(y) \neq \emptyset, \\
0, & \text{otherwise},
\end{array} \right. \]

for all \( y ∈ Y \).

Let B be an IVF set in Y. Then the inverse image of B under f, denoted by \( f^{-1}(B) \), is defined as follows

\[ f^{-1}(B(x)^θ) = [B(f(x))]^θ, \quad f^{-1}(B(x))^θ = [B(f(x))]^θ \]

for all \( x ∈ X \).

Definition 1.1 ([6]). A family \( τ \) of IVF sets in X is called an interval-valued fuzzy topology on X if it satisfies the following properties:

1. \( \emptyset, \overline{X} ∈ τ \).
2. \( A, B ∈ τ \Rightarrow A ∩ B ∈ τ \).
3. For \( i ∈ J \), \( A_i ∈ τ \Rightarrow \bigcup_{i ∈ J} A_i ∈ τ \).

Every member of τ is called an IVF open set. An IVF set A is called an IVF closed set if the complement of A is an IVF open set. And the pair \((X, τ)\) is called an interval-valued fuzzy topological space.

Definition 1.2 ([4]). A family M of interval-valued fuzzy sets in X is called an interval-valued fuzzy minimal structure on X if

\( \emptyset, \overline{X} ∈ M \).

In this case, \((X, M)\) is called an interval-valued fuzzy minimal space (simply, IVF minimal space). Every member of M is called an IVF m-open set. An IVF set A is called an IVF m-closed set if the complement of A (simply, \( A^c \)) is an IVF m-open set.

Let \((X, M)\) be an IVF minimal space and \( A ∈ \text{IVF}(X) \). The IVF minimal-closure and the IVF minimal-interior of A [6], denoted by \( mC(A) \) and \( mI(A) \), respectively, are defined as

\[ mC(A) = \cap \{ B ∈ \text{IVF}(X); B^c ⊆ A \}, \]

\[ mI(A) = \cup \{ B ∈ \text{IVF}(X); B ⊆ M \text{ and } B ⊆ A \}. \]

Theorem 1.3 ([4]). Let \((X, M)\) be an IVF minimal space and \( A, B ∈ \text{IVF}(X) \). Then the following properties hold:

1. \( mI(A) ⊆ A \) and if A is an IVF m-open set, then \( mI(A) = A \).
2. \( A ⊆ mC(A) \) and if A is an IVF m-closed set, then \( mI(A) = A \).
3. If \( A ⊆ B \), then \( mI(A) ⊆ mI(B) \) and \( mC(A) ⊆ mC(B) \).
4. \( mI(A) ∩ mI(B) ⊆ mI(A ∩ B) \) and \( mC(A) ∪ mC(B) = mC(A ∪ B) \).
5. \( mR(mI(A)) = mI(A) \) and \( mC(mC(A)) = mC(A) \).
6. \( I - mC(A) = mI(\overline{A}) \) and \( I - mI(A) = mC(\overline{A}) \).

2. Main Results

Definition 2.1 ([5]). Let \((X, M)\) be an IVF minimal space and \( A ∈ \text{IVF}(X) \). Then an IVF set A is called an IVF m-semiopen set if the complement of A is an IVF m-open set. And the pair \((X, τ)\) is called an interval-valued fuzzy topological space.

Definition 2.2 ([5]). Let \((X, M)\) be an IVF minimal space and \( A ∈ \text{IVF}(X) \). Then

1. \( smC(A) ⊆ A ⊆ smC(A) \).
2. If \( A ⊆ B \), then \( smC(A) ⊆ smC(B) \).
3. \( A \) is IVF m-semiopen iff \( smC(A) = A \) and \( F \) is IVF m-semiclosed iff \( smC(F) = F \).
4. \( smR(smC(A)) = smC(A) \) and \( smC(smC(A)) = smC(A) \).
5. \( smC(\overline{A}) = \overline{smC(A)} \) and \( smC(\overline{A}) = \overline{smC(A)} \).

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Lemma 2.3 Let \((X, M_\mathcal{A})\) be an IVF minimal space and \(A \subseteq \text{IVF}(X)\). Then
\[
\begin{align*}
(1) \ & m \text{R}(m \text{C}(A)) \subseteq m \text{R}(m \text{C}(m \text{S}(A))) \subseteq m \text{S}(A), \\
(2) \ & m \text{S}(A) \subseteq m \text{C}(m \text{S}(m \text{R}(A))) \subseteq m \text{C}(m \text{R}(A)), \\
(3) \ & m \text{R}(m \text{S}(A)) = m \text{R}(m \text{S}(m \text{C}(A))), \\
(4) \ & m \text{S}(m \text{R}(A)) = m \text{C}(m \text{S}(m \text{R}(A))).
\end{align*}
\]
Proof. (1) For \(A \subseteq \text{IVF}(X)\), since \(A \subseteq m \text{S}(A)\) and \(m \text{C}(A) \subseteq m \text{S}(A)\), we have \(m \text{I}(m \text{C}(A)) \subseteq m \text{R}(m \text{C}(m \text{S}(A))) \subseteq m \text{S}(A)\).
(2) It is similar to the proof of (1).
(3) From (1) and Theorem 2.2, it follows \(m \text{R}(m \text{S}(A)) \subseteq m \text{S}(m \text{C}(A)) \subseteq m \text{C}(m \text{R}(A))\).
(4) It is similar to the proof of (3).

Definition 2.4 ([5]). Let \((X, M_\mathcal{A})\) and \((Y, M_\mathcal{B})\) be two IVF minimal spaces. Then \(f : X \to Y\) is said to be interval-valued \(m\) -semicontinuous (simply, IVF \(m\) -semicontinuous) if for each IVF point \(M_x\) and each IVF \(m\) -open set \(V\) containing \(f(M_x)\), there exists an IVF \(m\) -semiopen set \(U\) containing \(M_x\) such that \(f(U) \subseteq V\).

Theorem 2.5 Let \(f : X \to Y\) be a function on IVF minimal spaces \((X, M_\mathcal{A})\) and \((Y, M_\mathcal{B})\). Then \(f\) is IVF \(m\) -semicontinuous if and only if for each IVF \(m\) -open set \(V\) in \(Y\), \(f^{-1}(V) \subseteq m \text{C}(m \text{R}(f^{-1}(V)))\).

Proof. Let \(V\) be an IVF \(m\) -open set in \(Y\) and \(M_y \subseteq f^{-1}(V)\). By IVF \(m\) -semicontinuity of \(f\), there exists an IVF \(m\) -semiopen set \(U\) containing \(M_y\) such that \(f(U) \subseteq V\). Thus from IVF \(m\) -semiopenness and Theorem 1.3 (3), it follows
\[
M_y \subseteq U \subseteq m \text{C}(m \text{R}(U)) \subseteq m \text{C}(m \text{R}(f^{-1}(V))).
\]
Hence we have \(f^{-1}(V) \subseteq m \text{C}(m \text{R}(f^{-1}(V)))\).

For the converse, let \(M_y\) be an IVF point of \(X\) and \(V\) an IVF \(m\) -open set in \(Y\) containing \(f(M_y)\). Then by hypothesis, we have \(M_y \subseteq f^{-1}(V) \subseteq m \text{C}(m \text{R}(f^{-1}(V)))\).

Corollary 2.6. Let \(f : X \to Y\) be a mapping on IVF minimal spaces \((X, M_\mathcal{A})\) and \((Y, M_\mathcal{B})\). Then the following statements are equivalent:

(1) \(f\) is IVF \(m\) -semicontinuous.
(2) For each IVF \(m\) -open set \(V\) in \(Y\), \(f^{-1}(V)\) is IVF \(m\) -semiopen.
(3) For each IVF \(m\) -closed set \(V\) in \(Y\), \(f^{-1}(V)\) is IVF \(m\) -semiclosed.

Proof. It follows from Definition 2.1 and Theorem 2.5.

Corollary 2.7. Let \(f : X \to Y\) be a mapping on IVF minimal spaces \((X, M_\mathcal{A})\) and \((Y, M_\mathcal{B})\). Then \(f\) is IVF \(m\) -semicontinuous if and only if for each IVF \(m\) -open set \(V\) in \(Y\), \(f^{-1}(V) \subseteq m \text{C}(m \text{R}(f^{-1}(V)))\).

Proof. It follows from Theorem 2.5 and Lemma 2.3.

Theorem 2.8 Let \(f : X \to Y\) be a function on IVF minimal spaces \((X, M_\mathcal{A})\) and \((Y, M_\mathcal{B})\). Then \(f\) is IVF \(m\) -semicontinuous if and only if \(f^{-1}(m \text{R}(B)) \subseteq m \text{C}(m \text{I}(f^{-1}(B)))\) for \(B \subseteq \text{IVF}(Y)\).

Proof. For each \(M_x \subseteq f^{-1}(m \text{R}(B))\), since \(f(M_x) \subseteq m \text{I}(B)\), there exists an IVF \(m\) -open set \(V\) such that \(f(M_x) \subseteq V \subseteq B\). From IVF \(m\) -semicontinuity of \(f\), there exists an IVF \(m\) -semiopen set \(U\) containing \(M_x\) such that \(f(U) \subseteq V\). This implies \(M_x \subseteq U \subseteq f^{-1}(V) \subseteq f^{-1}(B)\).

Thus \(M_x \subseteq U \subseteq m \text{C}(m \text{R}(U)) \subseteq m \text{C}(m \text{R}(f^{-1}(B)))\).

Hence \(f^{-1}(m \text{R}(B)) \subseteq m \text{C}(m \text{R}(f^{-1}(B)))\).

Corollary 2.9. Let \(f : X \to Y\) be a mapping on IVF minimal spaces \((X, M_\mathcal{A})\) and \((Y, M_\mathcal{B})\). Then \(f\) is IVF \(m\) -semicontinuous if and only if \(f^{-1}(m \text{R}(B)) \subseteq m \text{C}(m \text{C}(f^{-1}(B)))\) for \(B \subseteq \text{IVF}(Y)\).

Proof. It follows from Theorem 2.8 and Lemma 2.3.

Theorem 2.10 Let \(f : X \to Y\) be a function on IVF minimal spaces \((X, M_\mathcal{A})\) and \((Y, M_\mathcal{B})\). Then the following statements are equivalent:

(1) \(f\) is IVF \(m\) -semicontinuous.
(2) \(m \text{R}(m \text{C}(f^{-1}(B))) \subseteq f^{-1}(m \text{C}(f(B)))\) for \(B \subseteq \text{IVF}(Y)\).
(3) \(m \text{R}(m \text{C}(f(A))) \subseteq m \text{C}(f(A))\) for \(A \subseteq \text{IVF}(X)\).

Proof. (1) \(\Rightarrow\) (2) Let \(A \subseteq \text{IVF}(X)\). Then from Theorem 1.3 and Theorem 2.8, it follows
\[
f^{-1}(m \text{R}(B)) = f^{-1}(m \text{R}(f^{-1}(B))) = f^{-1}(m \text{I}(f^{-1}(B))) \subseteq m \text{C}(m \text{R}(f^{-1}(B))).
\]

(2) \(\Rightarrow\) (3) For \(A \subseteq \text{IVF}(X)\), by (2), we have \(m \text{R}(m \text{C}(f(A))) \subseteq m \text{C}(f^{-1}(f(A))) \subseteq f^{-1}(m \text{C}(f(A)))\).

This implies \(m \text{R}(m \text{C}(f(A))) \subseteq m \text{C}(f(A))\).

(3) \(\Rightarrow\) (1) Let \(F\) an IVF \(m\) -closed set in \(Y\). Then
by (3), \( f(m\text{sm}(f^{-1}(F))) \subseteq mC(f^{-1}(F)) \subseteq mC(F) = F \). This implies \( f^{-1}(F) \) is an IVF \( m \)-semiclosed set. Hence by Corollary 2.6, \( f \) is IVF \( m \)-semicontinuous.

**Corollary 2.11.** Let \( f : X \to Y \) be a function on IVF minimal spaces \((X,M_X)\) and \((Y,M_Y)\). Then the following statements are equivalent:

1. \( f \) is IVF \( m \)-semicontinuous.
2. \( m\text{sm}(f^{-1}(B)) \subseteq f^{-1}(mC(B)) \) for \( B \in \text{IVF}(Y) \).
3. \( f(m\text{sm}(C(A))) \subseteq mC(f(A)) \) for \( A \in \text{IVF}(X) \).

**Proof.** It follows from Theorem 2.10 and Lemma 2.3.

**References**


