Intuitionistic Vague Groups*

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1. Introduction

In 1971, Rosenfeld\textsuperscript{12} defined the fuzzy subgroup of a group \((G, \cdot)\) as a fuzzy set in \(G\) satisfying the conditions, by using the concept of fuzzy sets introduced by Zadeh\textsuperscript{13}. Many authors \cite{3, 5, 9, 11} have mainly investigated various algebraic notions based on his approach. In fact, in Rosenfeld’s work, only the subset are fuzzy, but the group operation is crisp. Recently, Demirci\textsuperscript{7} introduced the concept of fuzzy equalities and fuzzy mappings. By using them, he provided a good tool for fuzzyifying the group operation on a crisp set\textsuperscript{6}.

In 1989, Biswas\textsuperscript{3} introduced the intuitionistic fuzzy set as the generalization of fuzzy sets. In 1990, Demirci\textsuperscript{7} introduced the concept of fuzzy equalities and fuzzy mappings. By using them, he provided a good tool for fuzzyifying the group operation on a crisp set\textsuperscript{6}.

In 1989, Biswas\textsuperscript{3} introduced the intuitionistic fuzzy subgroup of a group \((G, \cdot)\) as an intuitionistic fuzzy set of \(G\) satisfying some conditions. Also many authors\textsuperscript{2,9} have worked to present the intuitionistic fuzzy setting of various algebraic concept based on their approach.

In this paper, by taking the group operation on a crisp set as an intuitionistic fuzzy mapping in the sense of \cite{10}, we establish the group structure on a crisp set and study the validity of the classical results in this setting.

2. Preliminaries

We will list some concept and one result needed in the later sections.

For sets \(X, Y\) and \(Z\), \(f = (f_1, f_2) : X \rightarrow Y \times Z\) is called a complex mapping if \(f_1 : X \rightarrow Y\) and \(f_2 : X \rightarrow Z\) are mappings. Throughout this paper, we will denote the unit interval \([0, 1]\) as \(I\) and \(X, Y, Z\), etc., are nonempty crisp sets.

Definition 2.1\textsuperscript{[1,6]}. A complex mapping \(A = (\mu_A, \nu_A) : X \rightarrow I \times I\) is called an intuitionistic fuzzy set (in short, IFS) in \(X\) if \(\mu_A(x) + \nu_A(x) \leq 1\) for each \(x \in X\), where the mappings \(\mu_A : X \rightarrow I\) and \(\nu_A : X \rightarrow I\) denote the degree of membership (namely \(\mu_A(x)\)) and the degree of non-membership (namely \(\nu_A(x)\)) of each \(x \in X\) to \(A\), respectively. In particular, \(0_\sim\) and \(1_\sim\) denote the intuitionistic fuzzy empty set and the intuitionistic fuzzy whole set in a set \(X\) defined by \(0_\sim(x) = (0, 1)\) and \(1_\sim(x) = (1, 0)\) for each \(x \in X\), respectively.

We will denote the set of all IFSs in \(X\) as IFS\((X)\).

Definition 2.2\textsuperscript{[1,9]}. Let \(A = (\mu_A, \nu_A)\) and \(B = (\mu_B, \nu_B)\) be IFSs in \(X\) and let \(\{A_\alpha\}_{\alpha \in \Gamma} \subset \text{IFS}(X)\).

Then
\begin{enumerate}
\item \(A \subset B\) if and only if \(\mu_A \leq \mu_B\) and \(\nu_A \leq \nu_B\).
\item \(A = B\) if and only if \(A \subset B\) and \(B \subset A\).
\item \(A^c = (\nu_A, \mu_A)\).
\item \(A \cap B = (\mu_A \land \mu_B, \nu_A \lor \nu_B)\).
\item \(A \cup B = (\mu_A \lor \mu_B, \nu_A \land \nu_B)\).
\item \(\bigwedge_{\alpha \in \Gamma} A_\alpha = (\bigwedge_{\alpha \in \Gamma} \mu_{A_\alpha}, \bigvee_{\alpha \in \Gamma} \nu_{A_\alpha})\).
\item \(\bigvee_{\alpha \in \Gamma} A_\alpha = (\bigvee_{\alpha \in \Gamma} \mu_{A_\alpha}, \bigwedge_{\alpha \in \Gamma} \nu_{A_\alpha})\).
\item \(\mathcal{A} = (\mu_A, 1 - \mu_A)\), \(<> A = (1 - \nu_A, \nu_A)\).
\end{enumerate}

Definition 2.3\textsuperscript{[4]}. \(R\) is called an intuitionistic fuzzy relation from \(X\) to \(Y\) (or on \(X \times Y\)) if \(R \in \text{IFS}(X \times Y)\).

In particular, if \(R \in \text{IFS}(X \times X)\) then \(R\) is called an intuitionistic fuzzy relation on \(X\).

We will denote the set of all intuitionistic fuzzy relation on \(X\) as IFR\((X)\).
Definition 2.4[10]. Let $IE_X=(\mu_{IE_X}, \nu_{IE_X}) \in IFR(X)$. Then $IE_X$ is called an intuitionistic fuzzy equality on $X$ if it satisfies the following conditions:

(i.e.1) $IE_X(x, y) = (1, 0) \Leftrightarrow x = y, \forall x, y \in X$,

(i.e.2) $IE_X(x, y) = IE_X(y, x), \forall x, y \in X$,

(i.e.3) $\mu_{IE_X}(x, y) \land \mu_{IE_X}(y, z) \leq \mu_{IE_X}(x, z)$ and

$\nu_{IE_X}(x, y) \lor \nu_{IE_X}(y, z) \geq \nu_{IE_X}(x, z), \forall x, y, z \in X$.

We will denote the set of all intuitionistic fuzzy equalities on $X$ as $IE(X)$. Let $IE \in IE(X)$ and let $a, b \in X$. Then $\mu_{IE}(a, b)$ [resp. $\nu_{IE}(a, b)$] is interpreted as the value of the grade of “$a$ and $b$ are equal” [resp. the grade of “$a$ and $b$ are nearly equal”].

Definition 2.4'[7]. Let $X$ be a nonempty set and let $E_X$ be a fuzzy relation on $X$. Then $E_X$ is called a fuzzy equality on $X$ if it satisfies the following conditions:

(e.1) $E_X(x, y) = 1 \Leftrightarrow x = y, \forall x, y \in X$,

(e.2) $E_X(x, y) = E_X(y, x), \forall x, y \in X$,

(e.3) $E_X(x, z) \geq E_X(x, y) \land E_X(y, z), \forall x, y, z \in X$.

Let $E$ be a fuzzy equality on $X$ and let $a, b \in X$. Then we interpret the value $E(a, b)$ as the grade of “$a$ and $b$ are nearly equal”. We will denote the set of all fuzzy equalities on $X$ as $E(X)$.

Remark 2.4. (a) If $E_X \in E(X)$, then $(E_X, E_X^T) \in IE(X)$.

(b) If $IE_X \in IE(X)$, then $[\ ]IE_X, \not\subseteq IE_X \in IE(X)$. Moreover, $\mu_{IE_X}, \nu_{IE_X} \in E(X)$.

Definition 2.5[10]. Let $IE_X$ and $IE_Y$ be two intuitionistic fuzzy equalities on $X$ and $Y$, respectively and let $f \in IFS(X \times Y)$. Then $f$ is called an intuitionistic fuzzy mapping from $X$ to $Y$ with respect to (in short, w.r.t.) $IE_X \in IE(X)$ and $IE_Y \in IE(Y)$, denoted by $f : X \rightarrow Y$, if it satisfies the following condition:

(f.1) $\forall x \in X, \exists y \in Y$ such that $\mu(f(x, y)) > 0$ and $\nu(f(x, y)) < 1$.

(f.2) $\forall x, y \in X, \forall z, w \in Y$,

$\mu(f(x, z) \land \mu(f(y, w)) \land \mu_{IE_Y}(z, w) \leq \mu_{IE_X}(x, y)$ and

$\nu(f(x, z) \lor \nu(f(y, w)) \lor \nu_{IE_X}(x, y) \geq \nu_{IE_Y}(z, w)$.

Definition 2.5'[7]. Let $f$ be a fuzzy relation from $X$ to $Y$, i.e., $R \in I^X \times Y$. Let $E_X$ and $E_Y$ be fuzzy equalities on $X$ and $Y$, respectively. Then $f$ is called a fuzzy mapping from $X$ to $Y$ w.r.t. $E_X$ and $E_Y$, denoted by $f : X \rightarrow Y$, if it satisfies the following conditions:

(f.1) $\forall x \in X, \exists y \in Y$ such that $f(x, y) > 0$,
bijective, strong bijective] fuzzy mapping w.r.t. fuzzy
equalities $E_X$ and $E_Y$ on $X$ and $Y$, respectively, then
$(f, f^c): X \rightarrow Y$ is a strong [surjective, strong
surjective, injective, bijective, strong bijective] intuitionistic
fuzzy mapping w.r.t. $(E_X, E_X^c) \in IE(X)$ and $(E_Y, E_Y^c) \in IE(Y)$.

(b) Let $f = (\mu_f, \nu_f): X \rightarrow Y$ be a strong [surjective,
strong surjective, injective, bijective, strong bijective] intuitionistic
fuzzy mapping w.r.t. $IE_X \in IE(X)$ and $IE_Y \in IE(Y)$. Then $<> f$ and $]] f$ are a strong [surjective,
strong surjective, injective, bijective, strong bijective] intuitionistic
fuzzy mapping w.r.t. intuitionistic fuzzy equalities $<> IE_X$ and $<> IE_Y$, and $[ ] IE_X$ and $[ ] IE_Y$ on $X$ and $Y$, respectively.

(c) Let $f = (\mu_f, \nu_f): X \rightarrow Y$ be a strong [surjective,
strong surjective, injective, bijective, strong bijective] intuitionistic
fuzzy mapping w.r.t. $IE_X \in IE(X)$ and $IE_Y \in IE(Y)$. Then $\mu_f$ and $\nu_f^c$ are a strong [surjective,
strong surjective, injective, bijective, strong bijective] fuzzy mapping w.r.t. intuitionistic fuzzy equalities
$\mu_{IE_X}$ and $\mu_{IE_Y}$, and $\nu_{IE_X}^c$ and $\nu_{IE_Y}^c$ on $X$ and $Y$, respectively.

Result 2.C[10, Proposition 3.7]. Let $\Delta_X$ be the
intuitionistic fuzzy relation on a set $X$ defined by : For
each $(x, y) \in X \times X$,
$$
\Delta_X(x, y) = \begin{cases}
(1, 0), & \text{if } x = y, \\
(0, 1), & \text{if } x \neq y.
\end{cases}
$$
Then $\Delta_X$ is a strong and strong bijective intuitionistic
fuzzy equality $IE_X$ on $X$. In fact, $\Delta_X$ is an intuitionistic fuzzy
equality on $X$. In this case, $\Delta_X$ is called an
identity intuitionistic fuzzy mapping on $X$.

3. Definition of intuitionistic vague
groups and their properties.

Definition 3.1. (i) A strong intuitionistic fuzzy
mapping $f : X \times X \rightarrow X$ w.r.t. $IE_{X \times X}$
$\in IE(X \times X)$ and $IE_X \in IE(X)$ is called an
intuitionistic vague operation
on $X$ w.r.t. $IE_{X \times X}$ and $IE_X$.

(ii) An intuitionistic vague binary operation $f$ on $X$
w.r.t. $IE_{X \times X}$ and $IE_X$ is said to be intuitionistic
transitive of first order if
(IT.1) $\mu_f(a, b, c) \land \mu_{IE_X}(c, d) \leq \mu_f(a, b, d)$
and
$$
\nu_f(a, b, c) \lor \nu_{IE_X}(c, d) \geq \nu_f(a, b, d), \forall a, b, c, d \in X.
$$

(iii) An intuitionistic vague binary operation $f$
[w.r.t. $IE_{X \times X}$ and $IE_X$ is said to be intuitionistic
transitive of second order if
(IT.2) $\mu_f(a, b, c) \land \mu_{IE_X}(b, d) \leq \mu_f(a, b, d)$
and
$$
\nu_f(a, b, c) \lor \nu_{IE_X}(b, d) \geq \nu_f(a, b, d), \forall a, b, c, d \in X.
$$

It can easily be seen that every crisp mapping $f : X \times X \rightarrow X$ is an intuitionistic vague binary operation on $X$ w.r.t. $\Delta_{X \times X}$ and $\Delta_X$, and it is transitive of both first order and second order.

Definition 3.1'[6]. (i) A strong fuzzy mapping $f : X \times X \rightarrow X$ w.r.t. $E_{X \times X} \in E(X \times X)$ and $E_X \in E(X)$ is called a vague binary operation on $X$ w.r.t. $E_{X \times X}$ and $E_X$.

(ii) A vague binary operation $f$ on $X$ w.r.t. $E_{X \times X}$ and $E_X$ is said to be transitive of first order if
(T.1) $f(a, b, c) \land E_X(c, d) \leq f(a, b, d)$.

and

(iii) An intuitionistic vague binary operation $f$ on $X$
w.r.t. $E_{X \times X}$ and $E_X$ is said to be transitive of second order if
(T.2) $f(a, b, c) \land E_X(b, d) \leq f(a, b, d)$.

Remark 3.2. (a) If $f$ is a vague binary operation on $X$ w.r.t. $E_{X \times X} \in E(X \times X)$ and $E_X \in E(X)$, then $(\mu_f, \nu_f)$ is an intuitionistic vague binary operation on $X$ w.r.t. $(E_{X \times X}, E_{X \times X}^c) \in IE(X \times X)$ and $(E_X, E_X^c) \in IE(X)$.

(b) If a vague binary operation $f$ on $X$ w.r.t. $E_{X \times X} \in E(X \times X)$ and $E_X \in E(X)$ is transitive of first [resp. second] order, then an intuitionistic
vague binary operation $(\mu_f, \mu_f^c)$ on $X$ w.r.t. $(E_{X \times X}, E_{X \times X}^c) \in IE(X \times X)$ and $(E_X, E_X^c) \in IE(X)$ is intuitionistic transitive of first[resp. second] order.

(c) If $f$ is an intuitionistic vague binary operation on $X$ w.r.t. $IE_{X \times X} \in IE(X \times X)$ and $IE_X \in IE(X)$, then $[f$ resp. $<> f$] is an intuitionistic vague binary
operation on $X$ w.r.t. $[IE_{X \times X}, IE_{X \times X}^c] \in IE(X \times X)$ and $[IE_X, IE_X^c] \in IE(X)$] resp. $<> IE_{X \times X} \in IE(X \times X)$ and $<> IE_X \in IE(X)]$. Moreover, $\mu_f$ [resp. $\nu_f^c$] is a
vague binary operation on $X$ w.r.t. $\mu_{IE_{X \times X}} \in IE(X \times X)$ and $\nu_{IE_{X \times X}} \in IE(X \times X)$ and $\nu_{IE_X} \in IE(X)]$. Moreover, $\mu_f$ and $\nu_f^c$ are
transitive of first [resp. second] order, respectively. Moreover, $\mu_f$ and $\nu_f^c$ are transitive of first [resp. second] order, respectively.

Let $G$ be a nonempty crisp set.

Definition 3.3. Let $\circ$ be an intuitionistic vague binary
operation on $G$ w.r.t. $IE_{G \times G} \in IE(G \times G)$ and $IE_G \in IE(G)$.

(i) $(G, \circ)$ is called an intuitionistic vague semigroup
if it satisfies the following condition:

\[(IVG, 1) \forall a, b, c, d, m, q, w \in G,\]

\[\mu_\circ(b, c, d) \land \mu_\circ(a, d, m) \land \mu_\circ(a, b, q) \land \mu_\circ(q, c, w) \leq \mu_{IE_G}(m, w)\]

and

\[\nu_\circ(b, c, d) \lor \nu_\circ(a, d, m) \lor \nu_\circ(a, b, q) \lor \nu_\circ(q, c, w) \geq \nu_{IE_G}(m, w).\]

(ii) An intuitionistic vague semigroup \((G, \circ)\) is called an intuitionistic vague monoid if it satisfies the following condition:

\[(IVG, 2) \exists \text{ an (two-sided) identity element } e \in G \text{ such that } \mu_\circ(e, a, a) \land \mu_\circ(a, e, a) = 1\]

and

\[\nu_\circ(e, a, a) \lor \nu_\circ(a, e, a) = 0, \forall a \in G.\]

(iii) An intuitionistic vague monoid \((G, \circ)\) is called an intuitionistic vague group if it satisfies the following condition:

\[(IVG, 3) \forall a \in G, \exists \text{ an (two-sided) inverse element } a^{-1} \in G \text{ such that } \mu_\circ(a^{-1}, a, e) \land \mu_\circ(a, a^{-1}, e) = 1\]

and

\[\nu_\circ(a^{-1}, a, e) \lor \nu_\circ(a, a^{-1}, e) = 0.\]

(iv) An intuitionistic vague semigroup \((G, \circ)\) is said to be abelian (commutative) if it satisfies the condition:

\[(IVG, 4) \forall a, b, m, w \in G,\]

\[\mu_\circ(a, b, m) \land \mu_\circ(b, a, w) \leq \mu_{IE_G}(m, w)\]

and

\[\nu_\circ(a, b, m) \lor \nu_\circ(b, a, w) \geq \mu_{IE_G}(m, w).\]

**Definition 3.3’ [6].** Let \(\circ\) be a vague binary operation on \(G\) w.r.t. \(E_G \times G \in E(G \times G)\) and \(E_G \in E(G)\).

(i) \((G, \circ)\) is called a vague semigroup if it satisfies the following condition:

\[(VG, 1) \forall a, b, c, d, m, q, w \in G,\]

\[\circ(b, c, d) \land \circ(a, d, m) \land \circ(a, b, q) \land \circ(q, c, w) \leq \circ(m, w)\]

(ii) A vague semigroup \((G, \circ)\) is called a vague monoid if it satisfies the following condition:

\[(VG, 2) \exists \text{ an (two-sided) identity element } e \in G \text{ such that } \circ(e, a, a) \land \circ(a, e, a) = 1, \forall a \in G.\]

(iii) A vague monoid \((G, \circ)\) is called a vague group if it satisfies the following condition:

\[(VG, 3) \forall a \in G, \exists \text{ an (two-sided) inverse element } a^{-1} \in G \text{ such that } \circ(a^{-1}, a, e) \land \circ(a, a^{-1}, e) = 1.\]

(iv) A vague semigroup \((G, \circ)\) is said to be abelian (commutative) if it satisfies the condition:

\[(VG, 4) \forall a, b, m, w \in G,\]

\[\circ(a, b, m) \land \circ(b, a, w) \leq E_G(m, w).\]

**Remark 3.4.** (a) If \((G, \circ)\) is a vague semigroup [resp. abelian semigroup, monoid and group] w.r.t. \(E_G \times G \in E(G \times G)\) and \(E_G \in E(G)\), then \((G, \mu_\circ, \mu^\circ)\) is an intuitionistic vague semigroup [resp. abelian semigroup, monoid and group] w.r.t. \((E_G \times G, E_G \times G) \in IE(G \times G)\) and \((E_G, E^\circ_G) \in IE(G)\).

(b) Let \(\circ\) be an intuitionistic vague binary operation on \(G\) w.r.t. \(IE_G \times G \in IE(G \times G)\) and \(IE_G \in IE(G)\). If \((G, \circ)\) is an intuitionistic vague semigroup [resp. abelian semigroup, monoid and group], then \((G, [\circ])\) and \((G, \langle \circ \rangle)\) are intuitionistic vague semigroup [resp. abelian semigroups, monoids and groups] w.r.t. \([IE_G \times G] \in IE(G \times G)\), \([IE_G] \in IE(G)\) and \(\langle IE_G \times G \rangle \in IE(G \times G)\), \(\langle IE_G \rangle \in IE(G)\), respectively. Moreover, \((G, \mu_\circ)\) and \((G, \nu^\circ)\) are vague semigroups [resp. abelian semigroups, monoids and groups] w.r.t. \(\mu_{IE_G \times G} \in E(G \times G)\), \(\mu_{IE_G} \in E(G)\) and \(\nu_{IE_G \times G} \in E(G \times G)\), \(\nu_{IE_G} \in E(G)\), respectively.

Let \(\circ\) be an intuitionistic vague binary operation on \(G\) w.r.t. \(\Delta_{G \times G} \land \Delta_G\) such that \(\circ(G \times G \times G) \subset [0, 1] \times [0, 1]\). Then an intuitionistic vague group \((G, \circ)\) one-to-one way corresponds to a group in the classical sense. In this case, an intuitionistic vague group is simply called a crisp group. For a given classical group \((G, \cdot)\), an infinite number of nontrivial intuitionistic vague group can be defined on \(G\).

**Example 3.5.** Let \((G, \cdot)\) be a classical group, let \(\alpha, \beta, \theta, \lambda, \mu, \gamma\) be fixed real number such that \(0 < \theta \leq \alpha \leq \beta < 1\) and \(0 < \lambda \leq \mu \leq \gamma < 1\), where \(\theta + \gamma \leq 1, \alpha + \mu \leq 1\) and \(\beta + \lambda \leq 1\). Let \(IE_G\) and \(IE_{G \times G}\) be intuitionistic fuzzy equalities on \(G\) and \(G \times G\) defined as follows, respectively:

\[IE_G(x, y) = \{ (1, 0), \text{ if } x = y, \}
\[\quad (\beta, \lambda), \text{ if } x \neq y, \}
\]
and

\[IE_{G \times G}((x, y), (z, w)) = \{ (1, 0), \text{ if } (x, y) = (z, w), \}
\[\quad (\alpha, \mu), \text{ if } (x, y) \neq (z, w). \}
\]

We define the intuitionistic fuzzy relation \(*\) on \(G \times G \times G\) as follows:

\[* (x, y, z) = \{ (1, 0), \text{ if } z = x \cdot y, \}
\[\quad (\theta, \gamma), \text{ if } z \neq x \cdot y. \}
\]
Then we can easily see that \((G, \cdot)\) is an intuitionistic vague semigroup. Furthermore, the element \(e\) of \((G, \cdot)\) and the inverse element \(a^{-1}\) of \(a\) in \((G, \cdot)\) are the identity element of \((G, \circ)\) and the inverse element of \(a\) in \((G, \circ)\), respectively. Thus \((G, \circ)\) is an intuitionistic vague group. If \((G, \cdot)\) is abelian, so is \((G, \circ)\). It should also be noticed that \((G, \cdot)\) is neither intuitionistic transitive of first order nor intuitionistic transitive of second order for \(\theta < \beta\) and \(\gamma > \lambda\), and that when \((\theta, \gamma) = (\alpha, \mu) = (\beta, \lambda)\), * is both intuitionistic trans-
(b) If \( \circ \) is intuitionistic transitive if first order, and \( f \) is intuitionistic vague injective and surjective, then the mapping \( f^{-1} : G' \to G \) is an intuitionistic vague homomorphism.

**Proof.** (a) The proof is the analogue of the classical case in [9].

(b) Suppose \( \circ \) is intuitionistic transitive of first order, and \( f \) is an intuitionistic vague injective and surjective. Let \( u, v, w \in G' \). Since \( f \) is surjective and \( \circ \) is strong, \( \exists a, b, c \in G \) such that \( a = f^{-1}(u) \), \( b = f^{-1}(v) \) and \( \circ(a, b, c) = (1, 0) \). Since \( f \) is an intuitionistic vague homomorphism,

\[
\circ'(f(a), f^{-1}(a), e_{G'}) = \circ'(f(a), f^{-1}(b), e_{G'}) = (1, 0)
\]

Since \( f \) is an intuitionistic vague homomorphism, by Proposition 4.8(b) and the hypothesis,

\[
\circ'(f(a), f^{-1}(b), f(c)) = \circ'(f(a), f^{-1}(b), f(c)) = (1, 0).
\]

Thus

\[
\mu_{G'}([f(a), f^{-1}(b), e_{G'}]) = \mu_{G'}([f(a), f^{-1}(b), f(c)]) = 1 \leq \mu_{IE_{G'}},(f(c), e_{G'})
\]

and

\[
\nu_{G'}([f(a), f^{-1}(b), e_{G'}]) = \nu_{G'}([f(a), f^{-1}(b), f(c)]) = 0 \geq \nu_{IE_{G'}},(f(c), e_{G'})
\]

So \( IE_{G'}(f(c), e_{G'}) = (1, 0) \), i.e., \( f(c) = e_{G'} \). Hence

\[
c \in Ker_{IV} f. \]

Therefore, by Theorem 4.3, \( Ker_{IV} f \) is an intuitionistic vague subgroup of \( G \).

(b) Suppose \( H \) is an intuitionistic vague subgroup of \( G \). For any \( a, b \in f(H) \) and each \( c \in G' \). Suppose \( \circ'(a, b^{-1}, c) = (1, 0) \). \( \circ \) is an intuitionistic fuzzy mapping, \( \exists \mu, \nu \in H \) and \( w \in G \) such that \( f(u) = a \) \( f(v) = b \) and \( \circ(v, w, a) = (1, 0) \). Since \( H \) is an intuitionistic vague subgroup of \( G \), by Theorem 4.3, \( w \in H \). Then \( f(w) = f(h) \). Since \( f \) is an intuitionistic vague homomorphism, by Proposition 4.8(b),

\[
\circ'(f(u), f^{-1}(v), f(w)) = \circ'(f(u), f^{-1}(v), f(w)) = \circ'(a, b^{-1}, f(w)) = (1, 0)
\]

Thus

\[
\mu_{G'}([f(a), f^{-1}(b), f(c)]) = \mu_{IE_{G'}},(f(c), f(w)) = 1 \leq \mu_{IE_{G'}},(f(c), f(w))
\]

and

\[
\nu_{G'}([f(a), f^{-1}(b), f(c)]) = \nu_{IE_{G'}},(f(c), f(w)) = 0 \geq \nu_{IE_{G'}},(f(c), f(w)).
\]

Thus \( IE_{G'}(f(c), f(w)) = (1, 0) \), i.e., \( f(c) = f(w) \). Since \( w \in H \), \( c \in f(h) \). Hence, by Theorem 4.3, \( f(h) \) is an intuitionistic vague subgroup of \( G' \).

(c) It can be proved in a similar manner to the proof of (b).

For a mapping \( f : X \to Y \), let \( Imf = \{ f(a) \in Y : a \in X \} \). Then the following is the immediate result of Proposition 4.14(b).

**Corollary 4.14.** Let \((G, \circ)\) and \((G', \circ')\) be two intuitionistic vague groups w.r.t. \( IE_{G \times G} \in IE(G \times G) \), \( IE_{G} \in IE(G) \) and \( IE_{G' \times G'} \in IE(G' \times G') \) and \( IE_{G'} \in IE(G') \), respectively, and let \( f : G \to G' \) be an intuitionistic vague homomorphism. Then \( Imf \) is an intuitionistic vague subgroup of \( G' \).

**References**


$IE_G \in IE(G)$, and let $H$ be a nonempty crisp subset of $G$. Let $\{H_\alpha\}_{\alpha \in \Gamma}$ be the family of all intuitionistic vague subgroups of $G$ containing $H$. Then $\bigcap_{\alpha \in \Gamma} H_\alpha$ is an intuitionistic vague subgroup of $G$. In this case, $\bigcap_{\alpha \in \Gamma} H_\alpha$ is called the intuitionistic vague subgroup of $G$ generated by $H$, and it is denoted by $< H >$.

**Definition 4.5.** Let $(G, \circ)$ and $(G', \circ')$ be two intuitionistic vague groups. Then a mapping (in the classical sense) $f : G \rightarrow G'$ is called an intuitionistic vague homomorphism if it satisfies the following conditions:

$\mu(a, b, c) \leq \mu(f(a), f(b), f(c))$

and

$\nu(a, b, c) \geq \nu(f(a), f(b), f(c))$, $\forall a, b, c \in G$.

**Definition 4.5'[6].** Let $(G, \mu)$ and $(G', \mu')$ be two vague groups. Then a mapping $f : G \rightarrow G'$ is called a vague homomorphism if

$\mu(a, b, c) \leq \mu'(f(a), f(b), f(c))$, $\forall a, b, c \in G$.

**Remark 4.5.** (a) If $f : (G, \mu) \rightarrow (G', \mu')$ is a vague homomorphism, then $f : (G, [\mu]) \rightarrow (G', [\mu'])$ resp. $f : (G, \mu(\overline{a})) \rightarrow (G', \mu'(\overline{a}))$ is an intuitionistic vague homomorphism.

(b) If $f : (G, \circ) \rightarrow (G', \circ')$ is an intuitive vague homomorphism, then $f : (G, [\circ]) \rightarrow (G', [\circ'])$ resp. $f : (G, \circ(\overline{a})) \rightarrow (G', \circ'(\overline{a}))$ is an intuitionistic vague homomorphism. Moreover, $f : (G, \mu) \rightarrow (G', \mu')$ [resp. $f : (G, \mu_\alpha) \rightarrow (G', \mu'_\alpha)$] is a vague homomorphism.

**Proposition 4.6.** Let $(G, \circ)$ and $(G', \circ')$ be two intuitionistic vague groups w.r.t. $IE_G \times G \in IE(G \times G)$, $IE_G \in IE(G)$ and $IE_G \times G' \in IE(G' \times G')$, $IE_G' \in IE(G')$, and let $f : G \rightarrow G'$ be an intuitionistic vague homomorphism:

(a) If $E_G$ and $E_G'$ are identities of $(G, \circ)$ and $(G', \circ')$, respectively, then $f(E_G) = E_{G'}$.

(b) For each $a \in G$, $f^{-1}(a) = f(a^{-1})$.

**proof.** (a) Let $E_G$ and $E_G'$ be identities of $(G, \circ)$ and $(G', \circ')$, respectively, and let $a \in G$. Then $\circ(a, E_G, a) = (1, 0)$. Since $f : G \rightarrow G'$ is an intuitionistic vague homomorphism,

$\circ(f(a), f(E_G), f(a)) = (1, 0)$.

On the other hand,

$\circ(f(a), e_{G'}, f(a)) = (1, 0)$.

Thus, by Proposition 3.8,

$\mu(f(a), f(E_G), f(a)) \leq \mu(f(a), e_{G'}, f(a))$

and

$\nu(f(a), f(E_G), f(a)) \geq \nu(f(a), e_{G'}, f(a))$

$= 1 \leq \mu_{IE_{G'}}(f(E_G), e_{G'})$, $\forall a \in G$.

So $IE_{G'}(f(E_G), e_{G'}) = (1, 0)$. Hence $f(E_G) = e_{G'}$.

(b) Let $a \in G$. Then clearly $\circ(a, a^{-1}, E_G) = (1, 0)$. Since $f : G \rightarrow G'$ is an intuitionistic vague homomorphism, by (a),

$\circ'(f(a), f(a^{-1}), f(E_G)) = \circ'(f(a), f(a^{-1}), e_{G'}) = (1, 0)$.

Thus, by Proposition 3.8,

$\mu_{IE_{G'}}(f(a), f(a^{-1}), e_{G'}) \leq \mu_{IE_{G'}}(f(a), f(a^{-1}), e_{G'})$

and

$\nu_{IE_{G'}}(f(a), f(a^{-1}), e_{G'}) \geq \nu_{IE_{G'}}(f(a), f(a^{-1}), e_{G'})$

$= 1 \leq \mu_{IE_{G'}}(f(a^{-1}), f(a^{-1}))$.

So $IE_{G'}(f(a^{-1}), f(a^{-1})) = (1, 0)$. Hence $f(a^{-1}) = f^{-1}(a)$.

**Definition 4.7.** Let $(G, \circ)$ and $(G', \circ')$ be two intuitionistic vague groups w.r.t. $IE_G \times G \in IE(G \times G)$, $IE_G \in IE(G)$ and $IE_G \times G' \in IE(G' \times G')$, $IE_G' \in IE(G')$, respectively, and let $f : G \rightarrow G'$ be an intuitionistic vague homomorphism. Then crisp set $\{a \in X : f(a) = E_{G'}\}$ is called an intuitionistic vague kernel of $f$, and it is denoted by $\ker_{IV} f$.

**Definition 4.8.** Let $IE_X \in IE(X)$ and let $IE_Y \in IE(Y)$. Then a mapping $g : X \rightarrow Y$ is said to be intuitionistic vague injective w.r.t. $IE_X$ and $IE_Y$ if

$\mu_{IE_X}(g(a), g(b)) \leq \mu_{IE_Y}(a, b)$

and

$\nu_{IE_X}(g(a), g(b)) \leq \nu_{IE_Y}(a, b)$, $\forall a, b \in X$.

It is clear that an intuitionistic vague injective mapping is injective in the classical sense.

**Definition 4.8'[6].** A mapping $g : X \rightarrow Y$ is said to be vague injective w.r.t. $IE_X \in IE(X)$ and $IE_Y \in IE(Y)$ if $E_Y(g(a), g(b)) \leq E_{X}(a, b)$, $\forall a, b \in X$.

**Remark 4.8.** (a) If $g : X \rightarrow Y$ is vague injective w.r.t. $IE_X \in IE(X)$ and $IE_Y \in IE(Y)$, then $g : X \rightarrow Y$ is intuitionistic vague injective w.r.t. $(E_X, E_{X'}) \in IE(X)$ and $(E_Y, E_Y') \in IE(Y)$.

(b) If $g : X \rightarrow Y$ is intuitionistic vague injective w.r.t. $IE_X \in IE(X)$ and $IE_Y \in IE(Y)$, and $IE_{X'} \in IE(X)$ and $IE_{Y'} \in IE(Y')$, respectively, then $g : X \rightarrow Y$ is intuitionistic vague injective w.r.t. $IE_X \in IE(X)$ and $IE_Y \in IE(Y)$.

Thus, $g : X \rightarrow Y$ is vague injective w.r.t. $IE_X \in IE(X)$ and $IE_Y \in IE(Y)$.

**Proposition 4.9.** Let $(G, \circ)$ and $(G', \circ')$ be two intuitionistic vague groups w.r.t. $IE_G \times G \in IE(G \times G)$, $IE_G \in IE(G)$ and $IE_G \times G' \in IE(G' \times G')$, $IE_G' \in IE(G')$, respectively, and let $f : G \rightarrow G'$ be an intuitionistic vague homomorphism. Let $E_G$ be the identity of $(G, \circ)$.

(a) $f$ is injective if and only if $\ker_{IV} f = \{E_G\}$.

**proof.** (a) Let $a \in G$. Then clearly $\circ(a, a^{-1}, E_G) = (1, 0)$. Since $f : G \rightarrow G'$ is an intuitionistic vague homomorphism, by (a),

$\circ'(f(a), f(a^{-1}), f(E_G)) = \circ'(f(a), f(a^{-1}), e_{G'}) = (1, 0)$.

Thus, by Proposition 3.8,

$\mu_{IE_{G'}}(f(a), f(a^{-1}), e_{G'}) \leq \mu_{IE_{G'}}(f(a), f(a^{-1}), e_{G'})$

and

$\nu_{IE_{G'}}(f(a), f(a^{-1}), e_{G'}) \geq \nu_{IE_{G'}}(f(a), f(a^{-1}), e_{G'})$

$= 1 \leq \mu_{IE_{G'}}(f(a^{-1}), f(a^{-1}))$.

So $IE_{G'}(f(a^{-1}), f(a^{-1})) = (1, 0)$. Hence $f(a^{-1}) = f^{-1}(a)$.

(b) Let $a \in G$. Then clearly $\circ(a, a^{-1}, E_G) = (1, 0)$. Since $f : G \rightarrow G'$ is an intuitionistic vague homomorphism, by (a),

$\circ'(f(a), f(a^{-1}), f(E_G)) = \circ'(f(a), f(a^{-1}), e_{G'}) = (1, 0)$.

Thus, by Proposition 3.8,

$\mu_{IE_{G'}}(f(a), f(a^{-1}), e_{G'}) \leq \mu_{IE_{G'}}(f(a), f(a^{-1}), e_{G'})$

and

$\nu_{IE_{G'}}(f(a), f(a^{-1}), e_{G'}) \geq \nu_{IE_{G'}}(f(a), f(a^{-1}), e_{G'})$

$= 1 \leq \mu_{IE_{G'}}(f(a^{-1}), f(a^{-1}))$.

So $IE_{G'}(f(a^{-1}), f(a^{-1})) = (1, 0)$. Hence $f(a^{-1}) = f^{-1}(a)$.
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\[ \nu_v(a^{-1}, a, e) \lor \nu_v(b, e, b) \lor \nu_v(b, a^{-1}, y) \lor \nu_v(y, a, v) = 1 \geq \mu_{IE_G}(b, v). \]

Thus \( IE_G(b, v) = (1, 0) \), i.e. \( b = v \). So \( \nu(y, a, b) = (1, 0) \).

(\( \Leftarrow \)) Suppose the necessary conditions hold. Let \( m \in G \) be fixed and let \( a \in G \). Then \( \exists e^* \in G \) such that
\[ \nu(e^*, m, m) = \nu(m, x, a) = (1, 0). \]
Since \( e^* \) is an intuitionistic fuzzy mapping, for \( a \in G \),
\[ \exists u \in G \text{ such that } \nu(e^*, a, u) = (1, 0). \]
Thus
\[ \nu_v(m, x, a) \lor \nu_v(e^*, a, u) \lor \nu_v(e^*, m, m) \lor \nu_v(m, x, a) = 0 \geq \mu_{IE_G}(u, a). \]
So \( IE_G(u, a) = (1, 0) \), i.e., \( u = a \). Hence \( \nu(e^*, a, a) = (1, 0) \), i.e., \( e^* \) is a left identity of \((G, \circ)\).

On the other hand, by the hypothesis, for each \( a \in G \),
\[ \exists x \in G \text{ such that } \nu(a, a, e) = (1, 0). \]
Thus \( w \) is a left inverse of \( a \). So, the required result is immediately obtained from Theorem 3.10. This completes the proof.

\[ \Box \]

4. Intuitionistic vague subgroups and intuitionistic vague homomorphisms

For a given intuitionistic fuzzy equality \( IE_X \) on \( X \) and for a crisp subset \( H \) of \( X \), the restriction of the complex mapping \( IE_X \) on \( X \) and for a crisp subset \( H \) of \( X \), the restriction of the complex mapping \( IE_X \) on \( H \times H \), denoted by \( IE_H^X \), is clearly an intuitionistic fuzzy equality on \( H \). For a given intuitionistic vague binary operation \( f \) on \( X \), we say that a crisp subset \( B \) of \( X \) is intuitionistic vague closed under \( f \) if it satisfies the following condition :

\[ (IVGC) \quad f(a, b, c) = (1, 0) \quad \Rightarrow \quad c \in B \forall a, b, c \in X. \]

For given intuitionistic vague operation \( f \) on \( X \) w.r.t. \( IE_X \times X \in IE(X \times X) \) and \( IE_X \in IE(X) \), if a crisp subset \( H \) of \( X \) is intuitionistic vague closed under \( f \), then it is easily seen that \( f|_{H \times H} \) is an intuitionistic vague operation on \( H \) and \( f|_{H \times H} \) preserves the transitive properties of \( f \).

Definition 4.1. Let \( (G, \circ) \) be an intuitionistic vague group w.r.t. \( IE_G \times G \in IE(G \times G) \) and \( IE_G \in IE(G) \), and let \( H \) be a nonempty crisp subset of \( G \) that is intuitionistic vague closed under \( \circ \). Then \( H \) is called an intuitionistic vague subgroup of \( G \) if \( (H, \circ|_{H \times H}) \) is itself an intuitionistic vague group.

For a given fuzzy equality \( E_X \) on \( X \) and for a crisp subset \( H \) of \( X \), the restriction of the complex mapping \( E_X \) on \( H \times H \), denoted by \( E_H^X \), is clearly a fuzzy equal-
ity on \( H \). For a given vague binary operation \( f \) on \( X \), we say that a crisp subset \( B \) of \( X \) is vague closed under \( f \) if it satisfies the following condition:

\[(\text{VGC}) \; f(a, b, c) = (1, 0) \Rightarrow c \in B, \forall a, b \in B, \forall c \in X.\]

For given vague operation \( f \) on \( X \) w.r.t. \( E_X \times X \in E(X \times X) \) and \( E_X \in E(X) \), if a crisp subset \( H \) of \( X \) is vague closed under \( f \), then it is easily seen that \( f|_{H \times H \times H} \) is a vague operation on \( H \) and \( f|_{H \times H \times H} \) preserves the transitive properties of \( f \).

**Definition 4.1** [6]. Let \((G, \circ)\) be a vague group w.r.t. \( IEG \times G \in IE(G \times G) \) and \( IEG \in IE(G) \), and let \( H \) be a nonempty crisp subset of \( G \) that is vague closed under \( \circ \). Then \( H \) is called a vague subgroup of \( G \) if \((H, \circ|_{H \times H \times H}) \) is itself a vague group.

**Remark 4.2.** (a) Let \((G, \circ)\) be a vague group w.r.t. \( E_G \times G \in IE(G \times G) \) and \( E_G \in IE(G) \), and let \( H \) be a nonempty crisp subset of \( G \). If \((H, \circ|_{H \times H \times H}) \) is an intuitionistic vague subgroup of \( G \), then \((H, (\mu, \nu)|_{H \times H \times H}) \) is an intuitionistic vague subgroup of \( G \) w.r.t. \( (\mu_{EG}, \nu_{EG}) = \mu \times \nu \in IE(G \times G) \).

(b) Let \((G, \circ)\) be an intuitionistic vague group w.r.t. \( IEG \times G \in IE(G \times G) \) and \( IEG \in IE(G) \), and let \( H \) be a nonempty crisp subset of \( G \). If \((H, \circ|_{H \times H \times H}) \) is an intuitionistic vague subgroup of \( G \), then \((H, [\circ|_{H \times H \times H}) \) resp. \((H, \circ|_{H \times H \times H}) \) is an intuitionistic vague subgroup of the intuitionistic vague group \((G, [\circ]) \) w.r.t. \([\circ]|_{H \times H \times H} \) is an intuitionistic vague subgroup of \( G \).

**Theorem 4.3.** Let \((G, \circ)\) be an intuitionistic vague subgroup w.r.t. \( IEG \times G \in IE(G \times G) \) and \( IEG \in IE(G) \), and let \( H \) be a nonempty crisp subset of \( G \). Then \( H \) is an intuitionistic vague subgroup of \( G \) if and only if \((\forall a, b \in H)(\circ|_{H \times H \times H})\).

**Proof** \((\Rightarrow)\): Suppose \( H \) is an intuitionistic vague subgroup of \( G \). Let \( e_G \) be an identity of \((G, \circ)\) w.r.t. \((H, \circ|_{H \times H \times H})\). Then, for \( a \in H \), it is clear that

\[\circ|_{H \times H \times H}(a, e_G, a) = a, \forall a \in H\]

By Proposition 3.8,

\[
\mu_0(a, e_G, a) \land \mu_0(a, e_G, a) = 1 \land \mu_{IEG}(a, e_G) \quad \text{and} \\
\nu_0(a, e_G, a) \lor \nu_0(a, e_G, a) = 0 \lor \nu_{IEG}(a, e_G).
\]

Thus \( IEG(e_H, e_G) = (1, 0) \), i.e., \( e_H = e_G \). For \( b \in H \), let \( b^{-1} \) be the inverse of \( b \) in \((H, \circ|_{H \times H \times H}) \). Then, it is clear that \( \circ|_{H \times H \times H}(b, b^{-1}, e_H) = (1, 0) \).

Thus

\[
\mu_0(b, b^{-1}, e_H) \land \mu_0(b, b^{-1}, e_H) = 1 \land \mu_{IEG}(b^{-1}, b^{-1}) \quad \text{and} \\
\nu_0(b, b^{-1}, e_H) \lor \nu_0(b, b^{-1}, e_H) = 0 \lor \nu_{IEG}(b^{-1}, b^{-1}).
\]

Now, for \( a, b \in H \), suppose \( a, b^{-1} \in H \). Since \( a, b^{-1} \in H \) is intuitionistic vague closed under \( \circ \), by \((IVG\circ, e) \in H \).

\((\Leftarrow)\): Suppose the necessary condition holds. Since \( H \neq \emptyset \), \( \exists u \in H \). Then \( \circ(u, u^{-1}, e_G) = (1, 0) \). Thus, by the hypothesis, \( e_G \in H \). Let \( a \in H \). Then clearly \( \circ(e_G, a^{-1}, a^{-1}) = (1, 0) \). Thus, by the hypothesis, \( a^{-1} \in H \).

For any \( a, b \in H \) and each \( c \in G \), suppose \( a, b, c \in G \). Then \( \circ(a, b, c) = (1, 0) \). Thus, by the hypothesis, \( c \in H \). Thus \( H \) is intuitionistic vague closed under \( \circ \). Since \((G, \circ)\) is an intuitionistic vague group, it can easily be be seen that \((H, \circ|_{H \times H \times H})\) satisfies the condition \((IVG\circ, e) \in H \).

\( H \) is an intuitionistic vague subgroup of \( G \).

**Theorem 4.4.** Let \((G, \circ)\) be an intuitionistic vague group w.r.t. \( IEG \times G \in IE(G \times G) \) and \( IEG \in IE(G) \), and let \( H \) be a nonempty crisp subset of \( G \). Then \( H \) is an intuitionistic vague subgroup of \( G \) if and only if it satisfies the following conditions:

(i) \( H \) is intuitionistic vague closed under \( \circ \)

(ii) For each \( a \in H \), \( a^{-1} \in H \).

**Proof** The proof can be obtained in a similar manner to that of the classical case in [8]. Thus it is omitted.

The following is the immediate result of Theorem 4.4.

**corollary 4.4-1.** Let \((G, \circ)\) be an intuitionistic vague group w.r.t. \( IEG \times G \in IE(G \times G) \) and \( IEG \in IE(G) \), and let \( \{H_\alpha\}_{\alpha \in \Gamma} \) be a nonempty family of intuitionistic vague subgroup of \( G \) such that \( \bigcap_{\alpha \in \Gamma} H_\alpha \neq \emptyset \). Then \( \bigcap_{\alpha \in \Gamma} H_\alpha \) is an intuitionistic vague subgroup of \( G \).

The following is the immediate result of corollary 4.4-1.

**corollary 4.4-2.** Let \((G, \circ)\) be an intuitionistic vague group w.r.t. \( IEG \times G \in IE(G \times G) \) and
Let \( I_E G \in \text{IE}(G) \), and let \( H \) be a nonempty crisp subset of \( G \). Let \( \{ H _{\alpha} \}_{\alpha \in \Gamma} \) be the family of all intuitionistic vague subgroups of \( G \) containing \( H \). Then \( \bigcup _{\alpha \in \Gamma} H _{\alpha} \) is an intuitionistic vague subgroup of \( G \). In this case, \( \bigcup _{\alpha \in \Gamma} H _{\alpha} \) is called the "intuitionistic vague subgroup of \( G \) generated by \( H \), and it is denoted by \( < H > \).

**Definition 4.5.** Let \((G, \phi)\) and \((G', \phi')\) be two intuitionistic vague groups. Then a mapping (in the classical sense) \( f : G \to G' \) is called an intuitionistic vague homomorphism if it satisfies the following conditions:

\[
\mu _{\varphi}(a, b, c) \leq \mu _{\varphi'}(f(a), f(b), f(c))
\]
and

\[
\nu _{\varphi}(a, b, c) \geq \nu _{\varphi'}(f(a), f(b), f(c)), \ \forall a, b, c \in G.
\]

**Remark 4.5.** (a) If \( f : (G, \phi) \to (G', \phi') \) is a vague homomorphism, then \( f : (G, [\phi]) \to (G', [\phi']) \) is an intuitionistic vague homomorphism.

(b) If \( f : (G, \phi) \to (G', \phi') \) is an intuitionistic vague homomorphism, then \( f : (G, [\phi]) \to (G', [\phi']) \) is an intuitionistic vague homomorphism. Moreover, \( f : (G, \mu _{\alpha}) \to (G', \mu _{\alpha'}) \) [resp. \( f : (G, \nu _{\alpha}) \to (G', \nu _{\alpha'}) \)] is a vague homomorphism.

**Proposition 4.6.** Let \((G, \phi)\) and \((G', \phi')\) be two intuitionistic vague groups w.r.t. \( I_E G \in \text{IE}(G \times G) \) and \( I_E G_{\times G'} \in \text{IE}(G' \times G') \), and let \( f : G \to G' \) be an intuitionistic vague homomorphism. Then:

(a) if \( \phi _{G} \) and \( \phi _{G'} \) are identities of \((G, \phi)\) and \((G', \phi')\), respectively, then \( f(\phi _{G}) = \phi _{G'} \).

(b) For each \( a \in G \), \( f^{-1}(a) = f(a)^{-1} \).

**proof.** (a) Let \( \phi _{G} \) and \( \phi _{G'} \) be identities of \((G, \phi)\) and \((G', \phi')\), respectively, and let \( a \in G \). Then \( \phi(a, \phi _{G}, a) = (1, 0) \). Since \( f : G \to G' \) is an intuitionistic vague homomorphism, \( \phi'(f(a), f(\phi _{G}), f(a)) = (1, 0) \).

On the other hand,

\[
\phi'(f(a), e_{G'}, f(a)) = (1, 0).
\]

Thus, by Proposition 3.8,

\[
\mu _{\phi'}(f(a), f(\phi _{G}), f(a)) \wedge \mu _{\phi'}(f(a), e_{G'}, f(a)) = 1 \leq \mu _{I_E G_{G'}}(f(\phi _{G}), e_{G'})
\]
and

\[
\nu _{\phi'}(f(a), f(\phi _{G}), f(a)) \vee \nu _{\phi'}(f(a), e_{G'}, f(a)) = 0 \geq \nu _{I_E G_{G'}}(f(\phi _{G}), e_{G'}).
\]
So \( I_E G_{G'}(e_{G'}, e_{G'}) = (1, 0) \). Hence \( f(\phi _{G}) = e_{G'} \).

(b) Let \( a \in G \). Then clearly \( \phi(a, a^{-1}, e_{G}) = (1, 0) \). Since \( f : G \to G' \) is an intuitionistic vague homomorphism, by (a),

\[
\phi'(f(a), f(a^{-1}), f(e_{G})) = \phi'(f(a), f(a^{-1}), e_{G'}) = (1, 0).
\]

Thus, by Proposition 3.8,

\[
\mu _{\phi'}(f(a), f(a^{-1}), e_{G'}) \wedge \mu _{\phi'}(f(a), f^{-1}(a), e_{G'}) = 1 \leq \mu _{I_E G_{G'}}(f(a^{-1}), f^{-1}(a))
\]
and

\[
\nu _{\phi'}(f(a), f(a^{-1}), e_{G'}) \vee \nu _{\phi'}(f(a), f^{-1}(a), e_{G'}) = 0 \geq \nu _{I_E G_{G'}}(f(a^{-1}), f^{-1}(a)).
\]

So \( I_E G_{(f(a^{-1}), f^{-1}(a))} = (1, 0) \). Hence \( f(a^{-1}) = f^{-1}(a) \). □

**Definition 4.7.** Let \((G, \phi)\) and \((G', \phi')\) be two intuitionistic vague groups w.r.t. \( I_E G \in \text{IE}(G \times G) \), \( I_E G \in \text{IE}(G) \) and \( I_E G_{\times G'} \in \text{IE}(G' \times G') \), \( I_E G_{G'} \in \text{IE}(G') \), respectively, and let \( f : G \to G' \) be an intuitionistic vague homomorphism. Then crisp set \( \{ a \in X : f(a) = e_{G'} \} \) is called an "intuitionistic vague kernel of \( f \), and it is denoted by \( \text{Ker}_{IV} f \).

**Definition 4.8.** Let \( I_E X \in \text{IE}(X) \) and let \( I_E Y \in \text{IE}(Y) \). Then a mapping \( g : X \to Y \) is said to be intuitionistic vague injective w.r.t. \( I_E X \) and \( I_E Y \) if

\[
\mu _{I_E X}(g(a), g(b)) \leq \mu _{I_E Y}(a, b)
\]
and

\[
\nu _{I_E X}(g(a), g(b)) \geq \nu _{I_E Y}(a, b), \ \forall a, b \in X.
\]

It is clear that an intuitionistic vague injective mapping is injective in the classical sense.

**Definition 4.8'[6].** A mapping \( g : X \to Y \) is said to be vague injective w.r.t. \( I_E X \in \text{IE}(X) \) and \( I_E Y \in \text{IE}(Y) \) if

\[
\phi(g(a), g(b)) \leq \phi(X, a, b), \ \forall a, b \in X.
\]

**Remark 4.8.** (a) If \( g : X \to Y \) is vague injective w.r.t. \( I_E X \in \text{IE}(X) \) and \( I_E Y \in \text{IE}(Y) \), then \( g : X \to Y \) is intuitionistic vague injective w.r.t. \( (I_E X, I_E X) \in \text{IE}(X) \) and \( (I_E Y, I_E Y) \in \text{IE}(Y) \).

(b) If \( g : X \to Y \) is intuitionistic vague injective w.r.t. \( I_E X \in \text{IE}(X) \) and \( I_E Y \in \text{IE}(Y) \) [resp. \( X \in \text{IE}(X) \) and \( Y \in \text{IE}(Y) \)], then \( g : X \to Y \) is intuitionistic vague injective w.r.t. \( (I_E X, I_E Y) \in \text{IE}(X \times Y) \).

**Proposition 4.9.** Let \((G, \phi)\) and \((G', \phi')\) be two intuitionistic vague groups w.r.t. \( I_E G_{\times G} \in \text{IE}(G \times G) \), \( I_E G \in \text{IE}(G) \) and \( I_E G_{\times G'} \in \text{IE}(G' \times G') \), \( I_E G'_{G} \in \text{IE}(G') \), respectively, and let \( f : G \to G' \) be an intuitionistic vague homomorphism. Let \( e_{G} \) be the identity of \((G, \phi)\).

(a) \( f \) is injective if and only if \( \text{Ker}_{IV} f = \{ e_{G} \} \).
(b) If \( \circ \) is intuitionistic transitive if first order, and \( f \) is intuitionistic vague injective and surjective, then the mapping \( f^{-1} : G' \rightarrow G \) is an intuitionistic vague homomorphism.

**Proof.** (a) The proof is the analogue of the classical case in [6].

(b) Suppose \( \circ \) is intuitionistic transitive of first order, and \( f \) is an intuitionistic vague injective and surjective. Let \( u, v, w \in G' \). Since \( f \) is surjective and \( \circ \) is strong, \( \exists a, b, c \in G \) such that \( a = f^{-1}(u) \), \( b = f^{-1}(v) \) and \( \circ(a, b, c) = (1, 0) \). Since \( f \) is an intuitionistic vague homomorphism,

\[
\circ'(f(a), f^{-1}(a), c_G) = \circ'(f(a), f^{-1}(b), c_G) = (1, 0)
\]

Since \( f \) is an intuitionistic vague homomorphism, by Proposition 4.8(b) and the hypothesis,

\[
\circ'(f(a), f^{-1}(b), f(c)) = \circ'(f(a), f^{-1}(b), f(c)) = (1, 0).
\]

Thus

\[
\mu_{G'}((f(a), f^{-1}(b), c_G)) \wedge \nu_{G'}((f(a), f^{-1}(b), f(c)) = 1 \leq \mu_{IE_{G'}}((f(c), e_G))
\]

and

\[
\nu_{G'}((f(a), f^{-1}(b), c_G)) \vee \mu_{G'}((f(a), f^{-1}(b), f(c)) = 0 \geq \nu_{IE_{G'}}((f(c), e_G)).
\]

So \( IE_{G'}((f(c), e_G)) = (1, 0) \), i.e., \( f(c) = e_G \). Hence \( c \in Ker_{TV} f \). Therefore, by Theorem 4.3, \( Ker_{TV} f \) is an intuitionistic vague subgroup of \( G \).

(b) Suppose \( H \) is an intuitionistic vague subgroup of \( G \). For any \( a, b \in f(H) \) and each \( c \in G' \). Suppose \( \circ'(a, b^{-1}, c) = (1, 0) \). \( \circ \) is an intuitionistic fuzzy mapping, \( \exists \mu, \nu \in H \) and \( w \in G \) such that \( f(u) = a \), \( f(v) = b \) and \( \circ(u, v, w) = (1, 0) \). Since \( H \) is an intuitionistic vague subgroup of \( G \), by Theorem 4.3, \( w \in H \). Then \( f(w) \in f(H) \). Since \( f \) is an intuitionistic vague homomorphism, by Proposition 4.8(b),

\[
\circ'(f(u), f(v^{-1}), f(w)) = \circ'(f(u), f(v^{-1}), f(w)) = \circ'(a, b^{-1}, f(w)) = (1, 0).
\]

Thus

\[
\mu_{G'}(a, b^{-1}, f(w)) \wedge \nu_{G'}(a, b^{-1}, c) = 1 \leq \mu_{IE_{G'}}(f(w), e_G)
\]

and

\[
\nu_{G'}(a, b^{-1}, f(w)) \vee \mu_{G'}(a, b^{-1}, c) = 0 \geq \nu_{IE_{G'}}(f(w), c).\]

So \( IE_{G'}(f(w), c) = (1, 0) \), i.e., \( c = f(w) \). Since \( w \in H, c \in f(H) \). Hence, by Theorem 4.3, \( f(H) \) is an intuitionistic vague subgroup of \( G' \).

(c) It can be proved in a similar manner to the proof of (b). \( \square \)

For a mapping \( f : X \rightarrow Y \), let \( Im f = \{ f(a) : a \in X \} \). Then the following is the immediate result of Proposition 4.14(b).

**Corollary 4.14.** Let \( (G, \circ) \) and \( (G', \circ') \) be two intuitionistic vague groups w.r.t. \( IE_{G \times G} \in IE(G \times G), IE_G \in IE(G) \) and \( IE_{G' \times G'} \in IE(G' \times G') \) and \( IE_{G'} \in IE(G') \), respectively, and let \( f : G \rightarrow G' \) be an intuitionistic vague homomorphism. Then \( Im f \) is an intuitionistic vague subgroup of \( G' \).

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