Fuzzy Pairwise \( \beta-(r, s) \)-continuous Mappings

Eun Pyo Lee\(^1\) and Seung On Lee\(^2\)

\(^1\) Department of Mathematics, Seonam University, Namwon 590-711, Korea
\(^2\) Department of Mathematics, Chungbuk National University, Cheongju 361-763, Korea

Abstract

We introduce the concepts of fuzzy pairwise \( \beta-(r, s) \)-continuous mappings and fuzzy pairwise \( \beta-(r, s) \)-open mappings in smooth bitopological spaces and then we investigate some of their characteristic properties.

**Key words**: fuzzy \( \beta-(r, s) \)-open sets, fuzzy \( \beta-(r, s) \)-closures, fuzzy pairwise \( \beta-(r, s) \)-continuous mappings

1. Introduction

After the introduction of fuzzy sets by Zadeh [9] in his classical paper, Chang [1] was the first to introduce the concept of a fuzzy topology on a set \( X \) by axiomatizing a collection \( T \) of fuzzy subsets of \( X \), where he referred to each member of \( T \) as an open set. In his definition of fuzzy topology, fuzziness in the concept of openness of a fuzzy subset was absent. These spaces and its generalizations are later studied by several authors, one of which, developed by Šostak [8], used the idea of degree of openness. This type of generalization of fuzzy topological spaces was later rephrased by Chattopadhyay, Hazra, and Samanta [2], and by Ramadan [7]. Kandil [3] introduced and studied the notion of fuzzy bitopological spaces as a natural generalization of fuzzy topological spaces. Lee [4] introduced the concept of smooth bitopological spaces as a generalization of smooth topological spaces and Kandil’s fuzzy bitopological spaces.

In this paper, we introduce the concepts of fuzzy pairwise \( \beta-(r, s) \)-continuous, fuzzy pairwise \( \beta-(r, s) \)-open and fuzzy pairwise \( \beta-(r, s) \)-closed mappings in smooth bitopological spaces and then we investigate some of their characteristic properties.

2. Preliminaries

Let \( I \) be the closed unit interval \([0, 1]\) of the real line and let \( I_0 \) be the half open interval \((0, 1]\) of the real line. For a set \( X \), \( I^X \) denotes the collection of all mapping from \( X \) to \( I \). A member \( \mu \) of \( I^X \) is called a fuzzy set of \( X \). By \( \hat{0} \) and \( \hat{1} \) we denote constant mappings on \( X \) with value 0 and 1, respectively. For any \( \mu \in I^X \), \( \mu^c \) denotes the complement \( \hat{1} - \mu \). All other notations are the standard notations of fuzzy set theory.

A Chang’s fuzzy topology on \( X \) [1] is a family \( T \) of fuzzy sets in \( X \) which satisfies the following properties:

1. \( \hat{0}, \hat{1} \in T \).
2. If \( \mu_1, \mu_2 \in T \) then \( \mu_1 \land \mu_2 \in T \).
3. If \( \mu_k \in T \) for all \( k \), then \( \bigvee \mu_k \in T \).

The pair \((X, T)\) be called a Chang’s fuzzy topological space. Members of \( T \) are called \( T \)-fuzzy open sets of \( X \) and their complements \( T \)-fuzzy closed sets of \( X \).

A system \((X, T_1, T_2)\) consisting of a set \( X \) with two Chang’s fuzzy topologies \( T_1 \) and \( T_2 \) on \( X \) is called a Kandil’s fuzzy bitopological space.

A smooth topology on \( X \) is a mapping \( T : I^X \to I \) which satisfies the following properties:

1. \( T(\hat{0}) = T(\hat{1}) = 1 \).
2. \( T(\mu_1 \land \mu_2) \geq T(\mu_1) \land T(\mu_2) \).
3. \( T(\bigvee \mu_i) \geq \bigwedge T(\mu_i) \).

The pair \((X, T)\) is called a smooth topological space. For \( r \in I_0 \), we call \( \mu \) a \( T \)-fuzzy \( r \)-open set of \( X \) if \( T(\mu) \geq r \) and \( \mu \) a \( T \)-fuzzy \( r \)-closed set of \( X \) if \( T(\mu^c) \geq r \).

A system \((X, T_1, T_2)\) consisting of a set \( X \) with two smooth topologies \( T_1 \) and \( T_2 \) on \( X \) is called a smooth bitopological space. Throughout this paper the indices \( i, j \) take values in \( \{1, 2\} \) and \( i = j \).

Let \((X, T)\) be a smooth topological space. Then it is easy to see that for each \( r \in I_0 \), an \( r \)-cut

\[ T_r = \{ \mu \in I^X \mid T(\mu) \geq r \} \]

is a Chang’s fuzzy topology on \( X \).
Let \((X, T)\) be a Chang’s fuzzy topological space and \(r \in I_0\). Then the mapping \(T^r : I^X \rightarrow I\) is defined by
\[
T^r(\mu) = \begin{cases} 
1 & \text{if } \mu = \emptyset, 1, \\
r & \text{if } \mu \in T - \{\emptyset, 1\}, \\
0 & \text{otherwise}
\end{cases}
\]
becomes a smooth topology.

Hence, we obtain that if \((X, T_1, T_2)\) is a smooth bitopological space and \(r, s \in I_0\), then \((X, (T_1)_r, (T_2)_s)\) is a Kandil’s fuzzy bitopological space. Also, if \((X, T_1, T_2)\) is a Kandil’s fuzzy bitopological space and \(r, s \in I_0\), then \((X, (T_1)^r, (T_2)^s)\) is a smooth bitopological space.

**Definition 2.1.** [4] Let \((X, T)\) be a smooth topological space. For each \(r \in I_0\) and for each \(\mu \in I^X\), the \(T\)-fuzzy \(r\)-closure is defined by
\[
T\text{-Cl}(\mu, r) = \{\rho \in I^X | \mu \leq \rho, T(\rho) \geq r\}
\]
and the \(T\)-fuzzy \(r\)-interior is defined by
\[
T\text{-Int}(\mu, r) = \{\rho \in I^X | \mu \geq \rho, T(\rho) \geq r\}.
\]

**Lemma 2.2.** [4] Let \(\mu\) be a fuzzy set of a smooth topological space \((X, T)\) and let \(r \in I_0\). Then we have:
1. \(T\text{-Cl}(\mu, r)\) is \(T\text{-Int}(\mu, r)\).
2. \(T\text{-Int}(\mu, r)\) is \(T\text{-Cl}(\mu, r)\).

**Definition 2.3.** [6] Let \(\mu\) be a fuzzy set of a smooth bitopological space \((X, T_1, T_2)\) and \(r, s \in I_0\). Then \(\mu\) is said to be
1. a \((T_1, T_2)\)-fuzzy \(\beta\)-(r, s)-open set if \(\mu \leq T_1\text{-Cl}(T_2\text{-Int}(T_1\text{-Cl}(\mu, s), r), s)\),
2. a \((T_1, T_2)\)-fuzzy \(\beta\)-(r, s)-closed set if \(T_2\text{-Int}(T_1\text{-Cl}(T_1\text{-Cl}(\mu, s), r), s) \leq \mu\).

**Definition 2.4.** [4, 5] Let \(f : (X, T_1, T_2) \rightarrow (Y, U_1, U_2)\) be a mapping from a smooth bitopological space \(X\) to a smooth bitopological space \(Y\) and \(r, s \in I_0\). Then \(f\) is said to be
1. a fuzzy pairwise \((r, s)\)-continuous mapping if the induced mapping \(f : (X, T_1) \rightarrow (Y, U_1)\) is a fuzzy \(r\)-continuous mapping and the induced mapping \(f : (X, T_2) \rightarrow (Y, U_2)\) is a fuzzy \(s\)-continuous mapping,
2. a fuzzy pairwise \((r, s)\)-semicontinuous mapping if \(f^{-1}(\mu)\) is a \((T_1, T_2)\)-fuzzy \((r, s)\)-seminopen set of \(X\) for each \(U_1\)-fuzzy \(r\)-open set \(\mu\) of \(Y\) and \(f^{-1}(\nu)\) is a \((T_2, T_1)\)-fuzzy \((s, r)\)-seminopen set of \(X\) for each \(U_2\)-fuzzy \(s\)-open set \(\nu\) of \(Y\).

3. Fuzzy pairwise \(\beta\)-(r, s)-continuous mappings

**Definition 3.1.** Let \(f : (X, T_1, T_2) \rightarrow (Y, U_1, U_2)\) be a mapping from a smooth bitopological space \(X\) to a smooth bitopological space \(Y\) and \(r, s \in I_0\). Then \(f\) is called
1. a fuzzy pairwise \(\beta\)-(r, s)-continuous mapping if \(f^{-1}(\mu)\) is a \((T_1, T_2)\)-fuzzy \((r, s)\)-preopen set of \(X\) for each \(U_1\)-fuzzy \(r\)-open set \(\mu\) of \(Y\) and \(f^{-1}(\nu)\) is a \((T_2, T_1)\)-fuzzy \((s, r)\)-preopen set of \(X\) for each \(U_2\)-fuzzy \(s\)-open set \(\nu\) of \(Y\).
2. a fuzzy pairwise \(\beta\)-(r, s)-open mapping if \(f\) is a \((U_1, U_2)\)-fuzzy \((r, s)\)-open set of \(Y\) for each \(T_1\)-fuzzy \(r\)-open set \(\rho\) of \(X\) and \(f(\lambda)\) is a \((U_2, U_1)\)-fuzzy \((s, r)\)-open set of \(Y\) for each \(T_2\)-fuzzy \(s\)-open set \(\lambda\) of \(X\).
3. a fuzzy pairwise \(\beta\)-(r, s)-closed mapping if \(f\) is a \((U_1, U_2)\)-fuzzy \((r, s)\)-closed set of \(Y\) for each \(T_1\)-fuzzy \(r\)-closed set \(\rho\) of \(X\) and \(f(\lambda)\) is a \((U_2, U_1)\)-fuzzy \((s, r)\)-closed set of \(Y\) for each \(T_2\)-fuzzy \(s\)-closed set \(\lambda\) of \(X\).

**Remark 3.2.** It is clear that every fuzzy pairwise \((r, s)\)-semicontinuous mapping is a fuzzy pairwise \(\beta\)-(r, s)-continuous mapping and every fuzzy pairwise \((r, s)\)-precontinuous mapping is a fuzzy pairwise \(\beta\)-(r, s)-continuous mapping. However, the following example show that all of the converses need not be true.

**Example 3.3.** Let \(X = \{x, y\}\) and \(\mu_1, \mu_2, \mu_3, \mu_4\) be fuzzy sets of \(X\) defined as
\[
\begin{align*}
\mu_1(x) &= 0.4, & \mu_1(y) &= 0.7; \\
\mu_2(x) &= 0.1, & \mu_2(y) &= 0.2; \\
\mu_3(x) &= 0.8, & \mu_3(y) &= 0.5; \\
\mu_4(x) &= 0.7, & \mu_4(y) &= 0.6.
\end{align*}
\]
Define \(T_1 : I^X \rightarrow I\) and \(T_2 : I^X \rightarrow I\) by
\[
T_1(\mu) = \begin{cases} 
1 & \text{if } \mu = \emptyset, 1, \\
\frac{1}{2} & \text{if } \mu = \mu_1, \\
0 & \text{otherwise;}
\end{cases}
\]
and
\[
T_2(\mu) = \begin{cases} 
1 & \text{if } \mu = \emptyset, 1, \\
\frac{1}{4} & \text{if } \mu = \mu_2, \\
0 & \text{otherwise.}
\end{cases}
\]
Then clearly \((T_1, T_2)\) is a smooth bitopology on \(X\). Define \(U_1 : I^X \rightarrow I\) and \(U_2 : I^X \rightarrow I\) by

\[
U_1(\mu) = \begin{cases} 
1 & \text{if } \mu = \hat{0}, \hat{1}, \\
\frac{1}{2} & \text{if } \mu = \mu_3, \\
0 & \text{otherwise};
\end{cases}
\]

and

\[
U_2(\mu) = \begin{cases} 
1 & \text{if } \mu = \hat{0}, \hat{1}, \\
\frac{1}{2} & \text{if } \mu = \mu_4, \\
0 & \text{otherwise};
\end{cases}
\]

Then clearly \((U_1, U_2)\) is a smooth bitopology on \(X\). Consider the identity mapping \(1_X : (X, T_1, T_2) \rightarrow (X, U_1, U_2)\). Then it is a fuzzy pairwise \(\beta-(\frac{1}{2}, \frac{1}{2})\)-continuous mapping which is not a fuzzy pairwise \(\beta-(\frac{1}{2}, \frac{1}{2})\)-semicontinuous mapping.

Define \(V_1 : I^X \rightarrow I\) and \(V_2 : I^X \rightarrow I\) by

\[
V_1(\mu) = \begin{cases} 
1 & \text{if } \mu = \hat{0}, \hat{1}, \\
\frac{1}{2} & \text{if } \mu = \mu_4, \\
0 & \text{otherwise};
\end{cases}
\]

and

\[
V_2(\mu) = \begin{cases} 
1 & \text{if } \mu = \hat{0}, \hat{1}, \\
\frac{1}{2} & \text{if } \mu = \mu_3, \\
0 & \text{otherwise};
\end{cases}
\]

Then clearly \((V_1, V_2)\) is a smooth bitopology on \(X\). Consider the identity mapping \(1_X : (X, T_1, T_2) \rightarrow (X, V_1, V_2)\). Then it is a fuzzy pairwise \(\beta-(\frac{1}{2}, \frac{1}{2})\)-continuous mapping which is not a fuzzy pairwise \(\beta-(\frac{1}{2}, \frac{1}{2})\)-precontinuous mapping.

**Definition 3.4.** Let \((X, T_1, T_2)\) be a smooth bitopological space and \(r, s \in I_0\). For each \(\mu \in I^X\), the \((T_1, T_2)\)-fuzzy \(\beta-(r, s)\)-closure is defined by

\[
(T_1, T_2)-\beta Cl(\mu, r, s) = \bigwedge \{ \rho \in I^X | \rho \leq \mu, \rho \text{ is } (T_1, T_2)\text{-fuzzy } \beta-(r, s)\text{-closed}\}
\]

and the \((T_1, T_2)\)-fuzzy \(\beta-(r, s)\)-interior is defined by

\[
(T_1, T_2)-\beta Int(\mu, r, s) = \bigvee \{ \rho \in I^X | \rho \geq \mu, \rho \text{ is } (T_1, T_2)\text{-fuzzy } \beta-(r, s)\text{-open}\}
\]

**Lemma 3.5.** For a fuzzy set \(\mu\) of a smooth bitopological space \((X, T_1, T_2)\) and let \(r, s \in I_0\), we have:

1. \((T_1, T_2)\)-Cl(\(\mu, r, s\)) = \((T_1, T_2)\)-Int(\(\mu^c, r, s\)).
2. \((T_1, T_2)\)-Int(\(\mu, r, s\)) = \((T_1, T_2)\)-Cl(\(\mu^c, r, s\)).

**Proof.** (1) Since \((T_1, T_2)\)-Int(\(\mu, r, s\)) is a \((T_1, T_2)\)-fuzzy \(\beta-(r, s)\)-open set and \((T_1, T_2)\)-Cl(\(\mu^c, r, s\)) \(\leq \mu\), we have \((T_1, T_2)\)-Int(\(\mu, r, s\)) \(\subseteq \) \((T_1, T_2)\)-cl(\(\mu^c, r, s\)) \(\subseteq \) \((T_1, T_2)\)-cl(\(\mu, r, s\)). Thus

\[
(T_1, T_2)\text{-Cl}(\mu^c, r, s) \leq (T_1, T_2)\text{-Cl}((T_1, T_2)\text{-cl}(\mu^c, r, s), r, s) = (T_1, T_2)\text{-cl}(\mu, r, s).
\]

Conversely, \((T_1, T_2)\text{-Cl}(\mu^c, r, s) \leq \mu\). Thus

\[
(T_1, T_2)\text{-Cl}(\mu^c, r, s) \leq (T_1, T_2)\text{-cl}(\mu, r, s).
\]

and hence

\[
(T_1, T_2)\text{-cl}(\mu, r, s) \leq (T_1, T_2)\text{-Cl}(\mu^c, r, s).
\]

(2) Similar to (1).

**Theorem 3.6.** Let \(f : (X, T_1, T_2) \rightarrow (Y, U_1, U_2)\) be a mapping and \(r, s \in I_0\). Then the following statements are equivalent:

1. \(f\) is a fuzzy pairwise \(\beta-(r, s)\)-continuous mapping.
2. \(f^{-1}(\mu)\) is a \((T_1, T_2)\)-fuzzy \(\beta-(r, s)\)-closed set of \(X\) for each \(U_1\)-fuzzy \(r\)-closed set \(\mu\) of \(Y\) and \(f^{-1}(\nu)\) is a \((T_2, T_1)\)-fuzzy \(\beta-(s, r)\)-closed set of \(X\) for each \(U_2\)-fuzzy \(s\)-closed set \(\nu\) of \(Y\).
3. For each fuzzy set \(\rho\) of \(X\),

\[
f((T_1, T_2)-\text{Cl}(\rho, r, s)) \leq U_1-\text{Cl}(f(\rho), r)
\]

and

\[
f((T_2, T_1)-\text{Cl}(\rho, s, r)) \leq U_2-\text{Cl}(f(\rho), s).
\]

4. For each fuzzy set \(\mu\) of \(Y\),

\[
(T_1, T_2)-\text{Cl}(f^{-1}(\mu), r, s) \leq f^{-1}(U_1-\text{Cl}(\mu, r))
\]

and

\[
(T_2, T_1)-\text{Cl}(f^{-1}(\mu), s, r) \leq f^{-1}(U_2-\text{Cl}(\mu, s)).
\]

5. For each fuzzy set \(\mu\) of \(Y\),

\[
f^{-1}(U_1-\text{Int}(\mu, r)) \leq (T_1, T_2)-\text{Int}(f^{-1}(\mu), r, s)
\]

and

\[
f^{-1}(U_2-\text{Int}(\mu, s)) \leq (T_2, T_1)-\text{Int}(f^{-1}(\mu), s, r).
\]

**Proof.** (1) \(\Rightarrow\) (2) Let \(\mu\) be any \(U_1\)-fuzzy \(r\)-closed set and \(\nu\) any \(U_2\)-fuzzy \(s\)-closed set of \(Y\). Then \(\mu^c\) is a \(U_1\)-fuzzy \(r\)-open set and \(\nu^c\) is a \(U_2\)-fuzzy \(s\)-open set of \(Y\). Since \(f\) is a fuzzy pairwise \(\beta-(r, s)\)-continuous mapping, \(f^{-1}(\mu^c)\) is a \((T_1, T_2)\)-fuzzy \(\beta-(r, s)\)-open set and \(f^{-1}(\nu^c)\) is a \((T_2, T_1)\)-fuzzy \(\beta-(s, r)\)-open set of \(X\). Thus \(f^{-1}(\mu)\) is a \((T_1, T_2)\)-fuzzy \(\beta-(r, s)\)-closed set and \(f^{-1}(\nu)\) is a \((T_2, T_1)\)-fuzzy \(\beta-(s, r)\)-closed set of \(X\).

(2) \(\Rightarrow\) (3) Let \(\rho\) be any fuzzy set of \(X\). Then \(U_1\)-Cl(\(f(\rho), r\)) is a \(U_1\)-fuzzy \(r\)-closed set and \(U_2\)-Cl(\(f(\rho), s\)) is a \(U_2\)-fuzzy \(s\)-closed set of \(Y\). By (2),
(4) Let $\mu$ be any fuzzy set of $Y$. Then $f^{-1}(\mu)$ is a fuzzy set of $X$. By (3),
\[
{f^{-1}(U_f-Cl(f(\mu), r))} \leq f^{-1}(U_1-Cl(f(\mu), r)) \leq U_1-Cl(f(\mu), r)
\]
and
\[
f((T_1, T_2)-\beta Cl(f^{-1}(\mu), s, r)) \leq U_2-Cl(f^{-1}(\mu), s, r)
\]
Thus
\[
(T_1, T_2)-\beta Cl(f^{-1}(\mu), r, s)
\]
and
\[
(T_2, T_1)-\beta Cl(f^{-1}(\mu), s, r)
\]
By Lemma 3.5,
\[
{f^{-1}(U_f-Int(\mu, r))} = f^{-1}(U_f-Cl(f^{-1}(\mu), r, s)) \leq (T_1, T_2)-\beta Cl(f^{-1}(\mu), r, s)
\]
and
\[
{f^{-1}(U_2-Int(\mu, s))} = f^{-1}(U_2-Cl(f^{-1}(\mu), s, r)) \leq (T_2, T_1)-\beta Int(f^{-1}(\mu), s, r)
\]
(5) $\Rightarrow$ (4) Let $\mu$ be any $U_1$-fuzzy $r$-open set and $\nu$ any $U_2$-fuzzy $s$-open set of $Y$. Then $U_1-Int(\mu, r) = \mu$ and $U_2-Int(\nu, s) = \nu$. By (5),
\[
f^{-1}(\mu) = f^{-1}(U_f-Int(\mu, r)) \leq (T_1, T_2)-\beta Int(f^{-1}(\mu), r, s)
\]
and
\[
f^{-1}(\nu) = f^{-1}(U_f-Int(\nu, s)) \leq (T_2, T_1)-\beta Int(f^{-1}(\nu), s, r)
\]
So $f^{-1}(\mu) = (T_1, T_2)-\beta Int(f^{-1}(\mu), r, s)$ and $f^{-1}(\nu) = (T_2, T_1)-\beta Int(f^{-1}(\nu), s, r)$. Hence $f^{-1}(\mu)$ is a $(T_1, T_2)$-fuzzy $\beta$-continuous mapping of $X$. Thus $f$ is a fuzzy pairwise $\beta$-continuous mapping.

Theorem 3.7. Let $f : (X, T_1, T_2) \to (Y, U_1, U_2)$ be a bijection and $r, s \in I_0$. Then $f$ is a fuzzy pairwise $\beta$-continuous mapping if and only if $U_f-Int(f(\mu), r) \leq f((T_1, T_2)-\beta Int(\mu, r, s))$ and $U_f-Int(f(\nu), s) \leq f((T_2, T_1)-\beta Int(\nu, s, r))$ for each fuzzy set $X$.

Proof. Let $f$ be a fuzzy pairwise $\beta$-continuous mapping and $\mu$ any fuzzy set of $X$. Then $U_f-Int(f(\mu), r)$ is a $U_1$-fuzzy $r$-open set and $U_f-Int(f(\mu), s)$ is a $U_2$-fuzzy $s$-open set of $Y$. Since $f$ is a fuzzy pairwise $\beta$-continuous mapping, we have $f^{-1}(U_f-Int(f(\mu), r))$ is a $(T_1, T_2)$-fuzzy $\beta$-continuous mapping of $X$. Since $f$ is fuzzy pairwise $\beta$-continuous and one-to-one, we have
\[
f^{-1}(U_f-Int(f(\mu), r)) \leq (T_1, T_2)-\beta Int(f^{-1}(\mu), r, s)
\]
and
\[
\begin{align*}
    f^{-1}(U_2{-}\text{Int}(f(\rho), s)) \\
    \leq (T_2, T_1){-}\beta\text{Int}(f^{-1}f(\rho), s, r) \\
    = (T_2, T_1){-}\beta\text{Int}(\rho, s, r).
\end{align*}
\]
Since \( f \) is onto,
\[
\begin{align*}
    U_1{-}\text{Int}(f(\rho), r) \\
    = f f^{-1}(U_1{-}\text{Int}(f(\rho), r)) \\
    \leq f((T_1, T_2){-}\beta\text{Int}(\rho, s, r))
\end{align*}
\]
and
\[
\begin{align*}
    U_2{-}\text{Int}(f(\rho), s) \\
    = f f^{-1}(U_2{-}\text{Int}(f(\rho), s)) \\
    \leq f((T_2, T_1){-}\beta\text{Int}(\rho, s, r)).
\end{align*}
\]
Conversely, let \( U_1 \)-fuzzy \( r \)-open set and \( \nu \) any \( U_2 \)-fuzzy \( s \)-open set of \( Y \). Then \( U_1{-}\text{Int}(\mu, r) = \mu \) and \( U_2{-}\text{Int}(\nu, s) = \nu \). Since \( f \) is onto,
\[
\begin{align*}
    f((T_1, T_2){-}\beta\text{Int}(f^{-1}(\mu), r, s)) \\
    \geq U_1{-}\text{Int}(f^{-1}(\mu), r) \\
    = U_1{-}\text{Int}(\mu, r) \\
    = \mu
\end{align*}
\]
and
\[
\begin{align*}
    f((T_2, T_1){-}\beta\text{Int}(f^{-1}(\nu), s, r)) \\
    \geq U_2{-}\text{Int}(f^{-1}(\nu), s) \\
    = U_2{-}\text{Int}(\nu, s) \\
    = \nu.
\end{align*}
\]
Since \( f \) is one-to-one, we have
\[
\begin{align*}
    f^{-1}(\mu) \leq f^{-1}f((T_1, T_2){-}\beta\text{Int}(f^{-1}(\mu), r, s)) \\
    = (T_1, T_2){-}\beta\text{Int}(f^{-1}(\mu), r, s) \\
    \leq f^{-1}(\mu)
\end{align*}
\]
and
\[
\begin{align*}
    f^{-1}(\nu) \leq f^{-1}f((T_2, T_1){-}\beta\text{Int}(f^{-1}(\nu), s, r)) \\
    = (T_2, T_1){-}\beta\text{Int}(f^{-1}(\nu), s, r) \\
    \leq f^{-1}(\nu).
\end{align*}
\]
So \( f^{-1}(\mu) = (T_1, T_2){-}\beta\text{Int}(f^{-1}(\mu), r, s) \) and \( f^{-1}(\nu) = (T_2, T_1){-}\beta\text{Int}(f^{-1}(\nu), s, r) \). Hence \( f^{-1}(\mu) \) is a \( (T_1, T_2){-}\beta\text{Int}(\mu, r, s) \)-open set and \( f^{-1}(\nu) \) is a \( (T_2, T_1){-}\beta\text{Int}(\nu, s, r) \)-open set of \( X \). Therefore \( f \) is a fuzzy pairwise \( \beta-(r, s) \)-continuous mapping. \( \square \)

**Theorem 3.8.** Let \( f : (X, T_1, T_2) \to (Y, U_1, U_2) \) be a mapping and \( r, s \in I_0 \). Then the following statements are equivalent:

1. \( f \) is a fuzzy pairwise \( \beta-(r, s) \)-open mapping.

2. For each fuzzy set \( \rho \) of \( X \),
\[
    f((T_1{-}\text{Int}(\rho, r))) \leq (U_1, U_2){-}\beta\text{Int}(f(\rho), r, s)
\]
and
\[
    f((T_2{-}\text{Int}(\rho, s))) \leq (U_2, U_1){-}\beta\text{Int}(f(\rho), s, r).
\]

3. For each fuzzy set \( \mu \) of \( Y \),
\[
    T_1{-}\text{Int}(f^{-1}(\mu), r) \leq f^{-1}((U_1, U_2){-}\beta\text{Int}(\mu, r, s))
\]
and
\[
    T_2{-}\text{Int}(f^{-1}(\mu), s) \leq f^{-1}((U_2, U_1){-}\beta\text{Int}(\mu, s, r)).
\]

**Proof.** (1) \( \Rightarrow \) (2) Let \( \rho \) be any fuzzy set of \( X \). Clearly \( T_1{-}\text{Int}(\rho, r) \) is a \( T_1{-}\beta\text{Int}(r, s) \)-open set and \( T_2{-}\text{Int}(\rho, s) \) is a \( T_2{-}\beta\text{Int}(r, s) \)-open set of \( Y \). Since \( f \) is a fuzzy pairwise \( \beta-(r, s) \)-open mapping, \( f(T_1{-}\text{Int}(\rho, r)) \) is a \( (U_1, U_2)-\beta\text{Int}(r, s) \)-open set and \( f(T_2{-}\text{Int}(\rho, s)) \) is a \((U_1, U_2)-\beta\text{Int}(r, s) \)-open set of \( Y \). Thus
\[
\begin{align*}
    f(T_1{-}\text{Int}(\rho, r)) \\
    = (U_1, U_2){-}\beta\text{Int}(f(T_1{-}\text{Int}(\rho, r)), r, s) \\
    \leq (U_1, U_2){-}\beta\text{Int}(f(\rho), r, s)
\end{align*}
\]
and
\[
\begin{align*}
    f(T_2{-}\text{Int}(\rho, s)) \\
    = (U_2, U_1){-}\beta\text{Int}(f(T_2{-}\text{Int}(\rho, s)), s, r) \\
    \leq (U_2, U_1){-}\beta\text{Int}(f(\rho), s, r)
\end{align*}
\]
(2) \( \Rightarrow \) (3) Let \( \mu \) be any fuzzy set of \( Y \). Then \( f^{-1}(\mu) \) is a fuzzy set of \( X \). By (2),
\[
\begin{align*}
    f(T_1{-}\text{Int}(f^{-1}(\mu), r)) \\
    \leq (U_1, U_2){-}\beta\text{Int}(f f^{-1}(\mu), r, s) \\
    \leq (U_1, U_2){-}\beta\text{Int}(\mu, r, s)
\end{align*}
\]
and
\[
\begin{align*}
    f(T_2{-}\text{Int}(f^{-1}(\mu), s)) \\
    \leq (U_2, U_1){-}\beta\text{Int}(f f^{-1}(\mu), s, r) \\
    \leq (U_2, U_1){-}\beta\text{Int}(\mu, s, r).
\end{align*}
\]
Thus we have
\[
\begin{align*}
    T_1{-}\text{Int}(f^{-1}(\mu), r) \\
    \leq f^{-1}f(T_1{-}\text{Int}(f^{-1}(\mu), r)) \\
    \leq f^{-1}((U_1, U_2){-}\beta\text{Int}(\mu, r, s))
\end{align*}
\]
and
\[
\begin{align*}
    T_2{-}\text{Int}(f^{-1}(\mu), s) \\
    \leq f^{-1}f(T_2{-}\text{Int}(f^{-1}(\mu), s)) \\
    \leq f^{-1}((U_2, U_1){-}\beta\text{Int}(\mu, s, r)).
\end{align*}
\]
(3) ⇒ (1) Let \( \rho \) be any \( T_1 \)-fuzzy \( r \)-open set and \( \lambda \) any \( T_2 \)-fuzzy \( s \)-open set of \( X \). Then \( T_1 \)-Int(\( \rho \), \( r \)) = \( \rho \) and \( T_2 \)-Int(\( \lambda \), \( s \)) = \( \lambda \). By (3),

\[
\rho = T_1 \text{-Int}(\rho, r) \\
\leq T_1 \text{-Int}(f^{-1}f(\rho), r) \\
\leq f^{-1}((U_1, U_2) - \beta \text{Int}(f(\rho), r, s))
\]

and

\[
\rho = T_2 \text{-Int}(\lambda, s) \\
\leq T_2 \text{-Int}(f^{-1}f(\lambda), s) \\
\leq f^{-1}((U_2, U_1) - \beta \text{Int}(f(\lambda), s, r))
\]

Hence we have

\[
f(\rho) \leq f f^{-1}((U_1, U_2) - \beta \text{Int}(f(\rho), r, s)) \\
\leq (U_1, U_2) - \beta \text{Int}(f(\rho), r, s) \\
\leq f(\rho)
\]

and

\[
f(\lambda) \leq f f^{-1}((U_2, U_1) - \beta \text{Int}(f(\lambda), s, r)) \\
\leq (U_2, U_1) - \beta \text{Int}(f(\lambda), s, r) \\
\leq f(\lambda)
\]

Thus \( f(\rho) = (U_1, U_2) - \beta \text{Int}(f(\rho), r, s) \) and \( f(\lambda) = (U_2, U_1) - \beta \text{Int}(f(\lambda), s, r) \). Hence \( f(\rho) \) is a \((U_1, U_2)\)-fuzzy \( \beta-(r, s)\)-open set and \( f(\lambda) \) is a \((U_2, U_1)\)-fuzzy \( \beta-(s, r)\)-open set of \( Y \). Therefore \( f \) is a fuzzy pairwise \( \beta-(r, s)\)-open mapping. \(\square\)

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**References**


**Eun Pyo Lee**
Professor of Seonam University
E-mail: eplee55@paran.com

**Seung On Lee**
Professor of Chungbuk National University
E-mail: solee@chungbuk.ac.kr
Corresponding author