Interval-Valued Fuzzy Almost $\alpha$-Continuous Mappings

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Abstract

We introduce the concept of IVF almost $\alpha$-continuity and investigate characterizations for such mappings on the interval-valued fuzzy topological spaces. We study the relationships between IVF almost $\alpha$-continuous mappings and another types of IVF continuous mappings.

Key Words: IVF $\alpha$-continuous, IVF weakly $\alpha$-continuous, IVF almost $\alpha$-continuous, IVF almost open mapping, IVF almost regular

1. Introduction and Preliminaries

Zadeh [9] introduced the concept of fuzzy set and investigated basic properties. Gorzalczyz [2] introduced the concept of interval-valued fuzzy set which is a generalization of fuzzy sets. In [8], Mondal and Samanta introduced the concepts of interval-valued fuzzy topology, continuity and compactness and studied some topological properties. The concept of interval-valued fuzzy topology is a generalization of fuzzy topology in sense of Chang’s fuzzy topology [1]. In [3], Jun et al. introduced the concepts of IVF $\alpha$-open sets and IVF $\alpha$-open mappings and studied some results about them. The concept of IVF strong semi-continuous (or IVF $\alpha$-continuous mapping) was introduced in [4]. The author introduced the concept of IVF weakly $\alpha$-continuous mapping and investigate some properties for them in [6]. In this paper, we introduce the concept of IVF almost $\alpha$-continuous mapping and investigate characterizations for such a mapping. We study the relationships among IVF $\alpha$-continuous mapping, IVF weakly $\alpha$-continuous mapping and IVF almost $\alpha$-continuous mapping.

2. Preliminaries

Let $I$ be the unit interval [0,1] of the real line. A member $A$ of $I^X$ is called a fuzzy set of $X$. For any $A \in I^X$, $A^c$ denotes the complement $1_A - A$. By $0_A$ and $1_A$ we denote constant maps on $X$ with value 0 and 1, respectively.

A Chang’s fuzzy topology $\tau$ [1] is a family $\tau \subseteq I^X$ satisfying the following conditions:

(1) $0_A, 1_A \in \tau$;
(2) for $A, B \in \tau$, if $A, B \in \tau$, then $(A \cap B) \in \tau$;
(3) for every subfamily $\{A_i : i \in J\} \subseteq I^X$, if $A_i \in \tau$, then $\cup_{i \in J} A_i \in \tau$.

Let $D[0,1]$ be the set of all closed subintervals of the interval [0,1]. The elements of $D[0,1]$ are generally denoted by capital letters $M, N, \cdots$ and note that $M= [M^L, M^U]$, where $M^L$ and $M^U$ are the lower and the upper end points respectively. Especially, we denote 0 and 1 as follows: $0= [0,0], 1=[1,1]$. We also note that

(1) $(\forall M, N \in D[0,1]) (M \cap N \Rightarrow M^L \leq N^L, M^U \leq N^U)$,
(2) $(\forall M, N \in D[0,1]) (M \subseteq N \Rightarrow M^L \leq N^L, M^U \leq N^U)$.

For each $M \in D[0,1]$, the complement of $M$, denoted by $M^c$, is defined by $M^c = [1-M^L, 1-M^U]$.

Let $X$ be a nonempty set. A mapping $A : X \rightarrow D[0,1]$ is called an interval-valued fuzzy set (simply, IVF set) in $X$. For each $x \in X$, $A(x)$ is a closed interval whose lower and upper end points are denoted by $[A(x)]^L$ and $[A(x)]^U$, respectively. For any $[a,b] \in D[0,1]$, the IVF set whose value is the interval $[a,b]$ for all $x \in X$ is denoted by $\tilde{a,b}$. In particular, for any $a \in [0,1]$, the IVF set whose value is $a(x) = [a,a]$ for all $x \in X$ is denoted by simply $\tilde{a}$. For a point $p \in X$ and for $[a,b] \subseteq
$D[0,1]$ with $b>0$, the IVF set which takes the value $[a,b]$ at $p$ and $0$ elsewhere in $X$ is called an interval-valued fuzzy point (simply, IVF point) and is denoted by $[a,b]$. In particular, if $b=a$, then it is also denoted by $a$. We denote the set of all IVF sets in $X$ by $IVF(X)$. An IVF point $M$, where $M \in D[0,1]$ is said to belong to an IVF set $A$ in $X$, denoted by $M \subseteq A$, if $|A(x)|^L \geq M^L$ and $|A(x)|^U \geq M^U$. In [8], it has been shown that $A = \cup \{M : M \subseteq A\}$.

For every $A, B \in IVF(X)$, we define

$$A = B \iff (\forall x \in X)(|A(x)|^L = |B(x)|^L),$$
$$A \subseteq B \iff (\forall x \in X)(|A(x)|^L \leq |B(x)|^L),$$
$$A \subseteq B \iff (\forall x \in X)(|A(x)|^U \leq |B(x)|^U).$$

The complement $A^c$ of $A$ is defined by

$$|A(x)|^L = 1 - |A(x)|^U$$
and

$$|A(x)|^U = 1 - |A(x)|^L$$
for all $x \in X$.

For a family of IVF sets $\{A_i : i \in J\}$ where $J$ is an index set, the union $G = \cup_{i \in J} A_i$, and $F = \cap_{i \in J} A_i$, are defined by

$$|G(x)|^L = \sup_{i \in J} |A_i(x)|^L$$
and

$$|F(x)|^U = \inf_{i \in J} |A_i(x)|^U$$
respectively, for all $x \in X$.

Let $f : X \to Y$ be a mapping and let $A$ be an IVF set in $X$. Then the image of $A$ under $f$, denoted by $f(A)$ [8], defined as follows

$$f(A)(y)^L = \begin{cases} 
\sup_{z \in f^{-1}(y)} |A(z)|^L, & \text{if } f^{-1}(y) \neq \emptyset, \\
0, & \text{otherwise},
\end{cases}$$

for all $y \in Y$.

Let $B$ be an IVF set in $Y$. Then the inverse image of $B$ under $f$, denoted by $f^{-1}(B)$ [8], defined as follows

$$|f^{-1}(B(x))|^L = |B(f(x))|^L$$
for all $x \in X$. Then it follows that $f(M) = M_{f^{-1}(y)}$.

Definition 2.1 ([8]). A family $\tau$ of IVF sets in $X$ is called an interval-valued fuzzy topology (simply, IVFT) on $X$ if it satisfies the following properties:

1. $0, 1 \in \tau$.
2. $A, B \in \tau \Rightarrow A \cap B \in \tau$.
3. For $i \in J$, $A_i \in \tau \Rightarrow \cup_{i \in J} A_i \in \tau$.

Every member of $\tau$ is called an IVF open set. An IVF set $A$ is called an IVF closed set if the complement of $A$ is an IVF open set. And the pair $(X, \tau)$ is called an interval-valued fuzzy topological space (simply, IVFTS).

In an IVF topological space $(X, \tau)$, for $A \subseteq IVF(X)$, the IVF closure and the IVF interior of $A$ [8], denoted by $cl(A)$ and $int(A)$, respectively, are defined as

$$cl(A) = \cap \{B \subseteq IVF(X) : B \in \tau \text{ and } A \subseteq B\},$$
$$int(A) = \cup \{B \subseteq IVF(X) : B \in \tau \text{ and } B \subseteq A\}.$$

Theorem 2.2 ([8]). Let $(X, \tau)$ be an IVF topological space and $A, B \subseteq IVF(X)$. Then

1. $A$ is an IVF closed set iff $A = cl(A)$.
2. $cl(A) \cup cl(B) = cl(A \cup B)$.
3. $cl(cl(A)) = cl(A)$.
4. $int(A) = 1 - cl(1 - A)$ and $cl(A) = 1 - int(1 - A)$.

Let $A$ be an IVF set in an IVFTS $(X, \tau)$. Then $A$ is said to be IVF $\alpha$-continuous [3] (resp., IVF semiopen [3], IVF preopen [3], IVF regular open [5] and IVF $\beta$-open [5]) if $A \subseteq int(cl(cl(A)))$ (resp., $A \subseteq cl(int(A))$, $A \subseteq cl(cl(A))$, $A \subseteq cl(cl(cl(A)))$).

Let $(X, \tau_1)$ and $(Y, \tau_2)$ be two IVFTS. Then $f : X \to Y$ is said to be IVF continuous [8] (resp., IVF $\alpha$-continuous or IVF strongly semi-continuous [4]) if for every IVF open set $B$ in $Y$, $f^{-1}(B)$ is IVF open (resp., IVF $\alpha$-open) in $X$. And $f$ is said to be IVF weakly $\alpha$-continuous [6] if for every IVF point $M$ and each IVF open set $V$ containing $f(M)$, there exists an IVF $\alpha$-open set $U$ containing $M$ such that $f(U) \subseteq cl(V)$.

3. IVF Almost $\alpha$-continuous Mappings

Definition 3.1. Let $f : X \to Y$ be a mapping IVFTS’s $(X, \tau_1)$ and $(Y, \tau_2)$. Then $f$ is said to be IVF almost $\alpha$-continuous if for each IVF point $M$ and each IVF open set $V$ containing $f(M)$, there exists an IVF $\alpha$-open set $U$ containing $M$ such that $f(U) \subseteq int(cl(V))$.

Obviously the following implications are obtained but the converses are not true in general:

IVF continuous $\Rightarrow$ IVF $\alpha$-continuous $\Rightarrow$ IVF almost $\alpha$-continuous $\Rightarrow$ IVF weakly $\alpha$-continuous.

Example 3.2. Let $X = \mathbb{I}$ and let $A, B, C, D$ and $E$ be IVF sets defined as follows

$$A(x) = \frac{2}{9}, \ B(x) = \frac{1}{3}, \ C(x) = \frac{2}{3}, \ D(x) = \frac{7}{9}, \ E(x) = \frac{3}{4}.$$
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E(x) = \left(\frac{8}{9}\right).

(1) Consider IVF topologies \( \tau_1 \) and \( \tau_2 \) on \( X \) as follows
\( \tau_1 = \{0, A, B, 1\} \) and \( \tau_2 = \{0, B, E, 1\} \).

Then the identity mapping \( f : (X, \tau_1) \to (X, \tau_2) \) is an IVF almost \( \alpha \)-continuous mapping but it is not IVF \( \alpha \)-continuous.

(2) Consider IVF topologies \( \tau_3 \) and \( \tau_4 \) on \( X \) as follows
\( \tau_3 = \{0, A, C, 1\} \) and \( \tau_4 = \{0, B, E, 1\} \).

Then the identity mapping \( f : (X, \tau_3) \to (X, \tau_4) \) is an IVF weakly \( \alpha \)-continuous mapping but it is not IVF almost \( \alpha \)-continuous.

Lemma 3.3 (Theorem 3.6 in [6]). Let \( (X, \tau) \) be an IVFTS and \( A \) an IVF set in \( X \). Then
(1) \( A \cap \text{int}(\text{cl}(A)) \) is IVF \( \alpha \)-open.
(2) \( A \cup \text{cl}(\text{int}(A)) \) is IVF \( \alpha \)-closed.

Theorem 3.4. Let \( f : X \to Y \) be a mapping between IVFTS's \( (X, \tau_1) \) and \( (Y, \tau_2) \). Then the following statements are equivalent:
(1) \( f \) is IVF almost \( \alpha \)-continuous.
(2) \( f^{-1}(B) \subseteq \text{int}(\text{cl}(f^{-1}(\text{int}(cl(B))))) \) for each IVF open set \( B \) of \( Y \).
(3) \( \text{cl}(\text{int}(f^{-1}(\text{int}(F)))) \subseteq f^{-1}(\text{F}) \) for each IVF closed set \( F \) in \( Y \).
(4) \( \text{cl}(\text{int}(f^{-1}(\text{int}(cl(B)))))) \subseteq f^{-1}(\text{cl}(B)) \) for each \( B \in \text{IVF}(Y) \).
(5) \( f^{-1}(\text{int}(B)) \subseteq \text{int}(\text{cl}(f^{-1}(\text{int}(cl(B)))))) \) for each \( B \in \text{IVF}(Y) \).
(6) \( \text{cl}(\text{int}(f^{-1}(\text{cl}(V)))) \subseteq f^{-1}(\text{cl}(V)) \) for an IVF regular open set \( V \) in \( Y \).
(7) \( f^{-1}(V) \) is IVF \( \alpha \)-closed for an IVF regular closed set \( F \) in \( Y \).
(1) \( f^{-1}(V) \) is IVF \( \alpha \)-open for an IVF regular open set \( V \) in \( Y \).

Proof. (1) \( \Rightarrow \) (2) Let \( B \) be an IVF open set in \( Y \). Then for each \( M_\alpha \in f^{-1}(B) \), there exists a IVF \( \alpha \)-open set \( U \) of \( M_\alpha \), such that \( f(U) \subseteq \text{int}(cl(B)) \). Since \( U \) is IVF \( \alpha \)-open, \( M_\alpha \subseteq \text{int}(\text{cl}(U)) \subseteq \text{int}(\text{cl}(f^{-1}(\text{int}(cl(B))))) \). Hence the statement (2) is obtained.

(2) \( \Rightarrow \) (1) For an IVF point \( M_\alpha \) in \( X \) and \( V \) an IVF open set containing \( f(M_\alpha) \), by (2), \( M_\alpha \subseteq f^{-1}(V) \subseteq \text{int}(\text{cl}(f^{-1}(\text{cl}(V)))) \). Put \( U = f^{-1}(\text{cl}(V)) \cap \text{int}(\text{cl}(f^{-1}(\text{cl}(V)))) \); then by Lemma 3.3, \( U \) is an IVF \( \alpha \)-open set containing \( M_\alpha \) such that \( M_\alpha \subseteq U \subseteq f^{-1}(\text{int}(cl(V))) \). This implies \( f(U) \subseteq \text{int}(cl(V)) \), and so hence \( f \) is IVF almost continuous.

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(2) \( \Rightarrow \) (3) Let \( F \) be any IVF closed set of \( Y \). Then since \( 1-F \) is IVF open, from (2) and Theorem 2.2, it follows
\[
\text{int}(\text{cl}(f^{-1}(\text{int}(cl(1-F))))) = \text{int}(\text{cl}(f^{-1}(1-\text{cl}(F))))
= \text{int}(\text{cl}(1-f^{-1}(\text{cl}(F))))
= \text{cl}(\text{cl}(f^{-1}(\text{cl}(F))))).
\]
It implies \( \text{cl}(\text{cl}(f^{-1}(\text{cl}(F)))) \subseteq f^{-1}(F) \).

(3) \( \Rightarrow \) (4) Obvious.

(4) \( \Rightarrow \) (5) For \( B \in \text{IVF}(Y) \), from hypothesis and Theorem 2.2, it follows
\[
\text{cl}(\text{cl}(f^{-1}(\text{cl}(B)))) = \text{cl}(\text{cl}(f^{-1}(\text{cl}(B))))
= \text{cl}(\text{cl}(\text{cl}(f^{-1}(\text{cl}(B))))).
\]
Hence \( \text{f}^{-1}(\text{cl}(B)) \subseteq \text{cl}(\text{cl}(f^{-1}(\text{cl}(\text{cl}(B))))). \)

(5) \( \Rightarrow \) (6) Let \( V \) be any IVF regular open set of \( Y \). Then since \( 1-V \) is IVF regular closed, it follows
\[
\text{cl}(\text{cl}(f^{-1}(\text{cl}(V)))) \subseteq f^{-1}(\text{cl}(V)).
\]

(6) \( \Rightarrow \) (7) Let \( F \) be any IVF regular closed set of \( Y \). Then \( \text{int}(F) \) is IVF regular open and by (6) and \( \text{cl}(\text{cl}(F)) \subseteq f^{-1}(\text{cl}(V)) \), and so \( F \) is IVF \( \alpha \)-closed.

(7) \( \Rightarrow \) (8) Obvious.

(8) \( \Rightarrow \) (1) Let \( V \) be an IVF open set containing \( f(M_\alpha) \). Since \( \text{int}(\text{cl}(V)) \) is IVF regular open, by (8) and \( \text{V} \subseteq \text{cl}(\text{cl}(V)) \), \( f^{-1}(\text{cl}(V)) \) is an IVF \( \alpha \)-open set containing \( M_\alpha \). Set \( U = \text{f}^{-1}(\text{int}(cl(V))) \). Then \( U \) is an IVF \( \alpha \)-open set satisfying \( f(U) \subseteq \text{int}(cl(V)) \). Thus \( f \) is an IVF almost \( \alpha \)-continuous mapping.

Theorem 3.5. Let \( f : X \to Y \) be a mapping between IVFTS's \( (X, \tau_1) \) and \( (Y, \tau_2) \). Then the following are equivalent:
(1) \( f \) is IVF almost \( \alpha \)-continuous.
(2) \( \text{cl}(\text{cl}(f^{-1}(G))) \subseteq f^{-1}(\text{cl}(G)) \) for each IVF \( \beta \)-open set \( G \) in \( Y \).
(3) \( \text{cl}(\text{cl}(f^{-1}(G))) \subseteq f^{-1}(\text{cl}(G)) \) for each IVF semi-open set \( G \) in \( Y \).
Proof. (1) ⇒ (2) Let \( G \) be an IVF \( \beta \)-open set. Then \( G \subseteq c_l(\text{int}(c_l(G)))\) and \( c_l(G) \) is an IVF regular closed set. Hence from the IVF almost \( \alpha \)-continuity, it follows \( c_l(\text{int}(f^{-1}(c_l(G)))) \subseteq c_l(\text{int}(f^{-1}(c_l(G)))) \subseteq f^{-1}(c_l(G)) \).

(2) ⇒ (3) It is obvious since every IVF semiopen set is IVF \( \beta \)-open.

(3) ⇒ (1) Let \( F \) be an IVF regular closed set. Then \( F \) is IVF semiopen, and so from (3), we have
\[
c_l(\text{int}(f^{-1}(F))) \subseteq f^{-1}(c_l(F)) \subseteq f^{-1}(F).
\]
This implies \( f^{-1}(F) \) is IVF \( \alpha \)-closed. Hence, from Theorem 3.4 (7), \( f \) is an IVF almost \( \alpha \)-continuous mapping.

**Theorem 3.6.** Let \( f : X \to Y \) be a mapping between IVFTS's \((X, \tau_1)\) and \((Y, \tau_2)\). Then \( f \) is IVF almost \( \alpha \)-continuous if and only if \( c_l(\text{int}(f^{-1}(c_l(G)))) \subseteq f^{-1}(c_l(G)) \) for each IVF preopen set \( G \) in \( Y \).

Proof. Suppose \( f \) is IVF almost \( \alpha \)-continuous. Let \( G \) be an IVF preopen set in \( Y \). Then we have \( c_l(G) = \text{cl}(\text{int}(G)) \). Set \( U = \text{int}(\text{cl}(G)) \); then by Theorem 3.4 (4), \( c_l(\text{int}(f^{-1}(\text{cl}(G)))) \subseteq f^{-1}(\text{cl}(G)). \)

From \( \text{cl}(U) \subseteq c_l(G) \) and \( c_l(G) \supseteq \text{cl}(\text{int}(G)) \), it follows \( c_l(\text{int}(f^{-1}(c_l(G)))) \subseteq f^{-1}(c_l(G)). \)

For the converse, let \( A \) be an IVF regular closed set in \( Y \). Then \( \text{int}(A) \) is IVF preopen. From hypothesis and \( \text{cl}(\text{int}(A)) = A \), it follows \( c_l(\text{int}(f^{-1}(A))) \subseteq f^{-1}(A) \). This implies \( f^{-1}(A) \) is IVF \( \alpha \)-closed and hence \( f \) is IVF almost \( \alpha \)-continuous.

**Theorem 3.7.** Let \( f : X \to Y \) be a mapping between IVFTS's \((X, \tau_1)\) and \((Y, \tau_2)\). Then \( f \) is IVF almost \( \alpha \)-continuous if and only if \( f^{-1}(G) \subseteq \text{int}(\text{cl}(f^{-1}(\text{int}(c_l(G)))) \) for each IVF preopen set \( G \) in \( Y \).

Proof. Suppose \( f \) is IVF almost \( \alpha \)-continuous and let \( G \) be an IVF preopen set in \( Y \). Then \( \text{int}(c_l(G)) \) is IVF regular open. From Theorem 3.4, we have
\[
f^{-1}(G) \subseteq f^{-1}(\text{int}(c_l(G))) \subseteq \text{int}(\text{cl}(f^{-1}(\text{int}(c_l(G))))).
\]
For the converse, let \( U \) be IVF regular open. Then \( U \) is obviously IVF preopen. By hypothesis, \( f^{-1}(U) \subseteq \text{int}(\text{cl}(f^{-1}(\text{int}(\text{cl}(U))) \subseteq \text{int}(\text{cl}(f^{-1}(U)))). \) So \( f^{-1}(U) \) is IVF \( \alpha \)-open, and hence \( f \) is IVF almost \( \alpha \)-continuous.

**Definition 3.8 ([7]).** An IVFTS \((X, \tau)\) is said to be IVF semi-regular if for each IVF open set \( U \) of \( X \) and each IVF point \( M \subseteq U \) there exists an IVF regular open set \( V \) of \( X \) such that \( M \subseteq V \subseteq U \).

**Theorem 3.9.** Let \( f : X \to Y \) be a mapping on IVFTS's \((X, \tau_1)\) and \((Y, \tau_2)\). If \( f \) is IVF almost \( \alpha \)-continuous and \( Y \) is IVF semi-regular, then \( f \) is IVF \( \alpha \)-continuous.

Proof. Let \( M \subseteq \text{int}(X) \) be an IVF point in \( X \) and \( U \) be an IVF open set in \( Y \) containing \( f(M) \). By the IVF semi-regularity of \( Y \), there exists an IVF regular open \( V \) of \( Y \) such that \( f(M) \subseteq V \subseteq U \). Since \( f \) is IVF almost \( \alpha \)-continuous, \( f^{-1}(V) \) is an IVF \( \alpha \)-open set containing \( M \). Set \( G = f^{-1}(V) \); then \( G \) is an IVF \( \alpha \)-open set containing \( M \), so \( f(M) \subseteq f(G) \subseteq U \). Hence \( f \) is IVF \( \alpha \)-continuous.

**Definition 3.10.** Let \( f : X \to Y \) be a mapping on IVFTS's \((X, \tau_1)\) and \((Y, \tau_2)\). Then \( f \) is said to be IVF almost \( \alpha^* \)-open if \( f(U) \subseteq \text{int}(\text{cl}(f(U))) \) for each IVF \( \alpha^* \)-open set \( U \) in \( X \).

**Definition 3.11.** Let \( f : X \to Y \) be a mapping on IVFTS's \((X, \tau_1)\) and \((Y, \tau_2)\). If \( f \) is IVF weakly \( \alpha \)-continuous and IVF almost \( \alpha^* \)-open, then \( f \) is IVF almost \( \alpha \)-continuous.

Proof. Let \( M \subseteq \text{int}(X) \) be an IVF point in \( X \) and \( U \) be an IVF open set in \( Y \) containing \( f(M) \). By the IVF weakly \( \alpha \)-continuity, there exists an IVF \( \alpha \)-open set \( V \) in \( Y \) such that \( f(V) \subseteq \text{cl}(U) \). Since \( f \) is IVF almost \( \alpha \)-open, \( f(V) \subseteq \text{int}(\text{cl}(f(V))) \subseteq \text{int}(U) \). This implies \( f \) is IVF almost \( \alpha \)-continuous.

**References**


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