Normal fuzzy probability for generalized triangular fuzzy sets

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Abstract

A fuzzy set \( A \) defined on a probability space \( (\Omega, \mathcal{F}, P) \) is called a fuzzy event. Zadeh defines the probability of the fuzzy event \( A \) using the probability \( P \). We define the generalized triangular fuzzy set and apply the extended algebraic operations to these fuzzy sets. A generalized triangular fuzzy set is symmetric and may not have value 1. For two generalized triangular fuzzy sets \( A \) and \( B \), \( A(+B) \) and \( A(-B) \) become generalized trapezoidal fuzzy sets, but \( A(\cdot)B \) and \( A(/)B \) need not to be a generalized triangular fuzzy set or a generalized trapezoidal fuzzy set. We define the normal fuzzy probability on \( \mathbb{R} \) using the normal distribution. And we calculate the normal fuzzy probability for generalized triangular fuzzy sets.

Key Words: fuzzy event, generalized triangular fuzzy set, normal fuzzy probability

1. Introduction

We define the generalized triangular fuzzy set and calculate four operations of two generalized triangular fuzzy sets([4]). Four operations are based on the Zadeh’s extension principle([6], [7], [8]). And Zadeh defines the probability of fuzzy event as follows.

Let \( (\Omega, \mathcal{F}, P) \) be a probability space, where \( \Omega \) denotes the sample space, \( \mathcal{F} \) the \( \sigma \)-algebra on \( \Omega \), and \( P \) a probability measure. A fuzzy set \( A \) on \( \Omega \) is called a fuzzy event. Let \( \mu_\lambda(\cdot) \) be the membership function of the fuzzy event \( A \).

Then the probability of the fuzzy event \( A \) is defined by Zadeh([9]) as

\[
\bar{P}(A) = \int_{\Omega} \mu_\lambda(\omega) dP(\omega), \quad \mu_\lambda(\cdot): \Omega \to [0, 1]
\]

We defined the normal fuzzy probability using the normal distribution and calculated the normal fuzzy probability for quadratic fuzzy number([2]). And then we had the explicit formula for the normal fuzzy probability for trigonometric fuzzy number. Furthermore we calculated the normal fuzzy probability for trigonometric fuzzy numbers driven by the above four operations([3]).

In this paper, we calculate the normal fuzzy probability for generalized triangular fuzzy sets.

2. Preliminaries

Let \( (\Omega, \mathcal{F}, P) \) be a probability space, and \( X \) be a random variable defined on it. Let \( g \) be a real-valued Borel-measurable function on \( \mathbb{R} \). Then \( g(X) \) is also a random variable.

Definition 2.1. We say that the mathematical...
expectation of \(g(X)\) exists if

\[
E[g(X)] = \int g(X) dP = \int g(X) dP_X(t)
\]

is finite.

We note that a random variable \(X\) defined on \((\Omega, \mathcal{F}, P)\) induces a measure \(P_X\) on a Borel set \(B \subseteq \mathcal{B}\) defined by the relation \(P_X(B) = P(X^{-1}(B))\). Then \(P_X\) becomes a probability measure on \(\mathcal{B}\) and is called the probability distribution of \(X\). It is known that if \(E[g(X)]\) exists, then \(g\) is also integrable over \(\Omega\) with respect to \(P_X\). Moreover, the relation

\[
\int_{\Omega} g(X) dP = \int_{\mathbb{R}} g(t) dP_X(t)
\]  

holds. We note that the integral on the right-hand side of (2.1) is the Lebesgue--Stieltjes integral of \(g\) with respect to \(P_X\). In particular, if \(g\) is continuous on \(\mathbb{R}\) and \(E[g(X)]\) exists, we can rewrite (2.1) as follows

\[
\int_{\Omega} g(X) dP = \int_{\mathbb{R}} g(t) dP_X = \int_{-\infty}^{\infty} g(x) dF(x),
\]

(2.2)

where \(F\) is the distribution function corresponding to \(P_X\) and the last integral is a Riemann--Stieltjes integral.

Let \(F\) be absolutely continuous on \(\mathbb{R}\) with probability density function \(f(x) = F'(x)\). Then \(E[g(X)]\) exists if and only if the integral

\[
\int_{-\infty}^{\infty} |g(x)|f(x) dx
\]

is finite and in that case we have

\[
E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx
\]

Example 2.2. Let the random variable \(X\) (denoted \(X \sim N(m, \sigma^2)\)) have the normal distribution with the probability density function

\[
f(x) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}, \ x \in \mathbb{R}
\]

where \(\sigma^2 > 0\) and \(m \in \mathbb{R}\). Then \(E[|X|] < \infty\) for every \(\gamma > 0\), and we have

\[
E[X] = m \quad \text{and} \quad E[(X-m)^2] = \sigma^2
\]

The induced measure \(P_X\) is called the normal distribution.

A fuzzy set \(A\) on \(\Omega\) is called a fuzzy event. Let \(\mu_\cdot(\cdot)\) be the membership function of the fuzzy event \(A\). Then the probability of the fuzzy event \(A\) is defined by Zadeh(9) as

\[
\widetilde{P}(A) = \int_\Omega \mu_\cdot(\omega) dP(\omega), \ \mu_\cdot(\cdot) : \Omega \rightarrow [0,1]
\]

Definition 2.3. The normal fuzzy probability \(\widetilde{P}(A)\) of a fuzzy set \(A\) on \(\mathbb{R}\) is defined by

\[
\widetilde{P}(A) = \int_\mathbb{R} \mu_\cdot(x) dP_X,
\]

(2.3)

where \(P_X\) is the normal distribution.

Definition 2.4. A triangular fuzzy number is a fuzzy number \(A\) having membership function

\[
\mu_\cdot(x) = \begin{cases} 
0, & x < a_1, \ a_1 \leq x, \\
-x-a_1, & a_1 \leq x < a_2, \\
-a_2, & a_2 \leq x < a_3
\end{cases}
\]

The above triangular fuzzy number is denoted by \(A = (a_1, a_2, a_3)\).

Definition 2.5. The addition, subtraction, multiplication, and division of two fuzzy sets are defined as

1. Addition \(A(+B)\):
\[
\mu_{A(+B)}(x) = \sup_{y \in \mathbb{R}} \min \{\mu_A(x+y), \mu_B(y)\}
\]

2. Subtraction \(A(-B)\):
\[
\mu_{A(-B)}(x) = \sup_{y \in \mathbb{R}} \min \{\mu_A(x-y), \mu_B(y)\}
\]

3. Multiplication \(A(\cdot)B\):
\[
\mu_{A(\cdot)B}(x) = \sup_{y \in \mathbb{R}} \min \{\mu_A(x) \cdot \mu_B(y)\}
\]

4. Division \(A(/)B\):
\[
\mu_{A(/)B}(x) = \sup_{y \in \mathbb{R}} \min \{\mu_A(x) / \mu_B(y)\}
\]

Example 2.6. (2.1) For two triangular fuzzy number \(A = (1, 2, 4)\) and \(B = (2, 4, 5)\), we have

1. Addition : \(A(+)B = (3, 6, 9)\).
2. Subtraction : \(A(-)B = (-4, -2, 2)\).
3. Multiplication :
\[
\mu_{A(\cdot)B}(x) = \begin{cases} 
0, & x < 2, \ 20 \leq x, \\
\frac{-2 + \sqrt{2x}}{2}, & 2 \leq x < 8, \\
\frac{7 - \sqrt{9+2x}}{2}, & 8 \leq x < 20.
\end{cases}
\]

Note that \(A(\cdot)B\) is not a triangular fuzzy number.
4. Division :
\[ \mu_{A/B}(x) = \begin{cases} 0, & x \leq \frac{1}{5}, \ 2 \leq x, \\ \frac{5x-1}{x+1}, & 1 \leq x < \frac{1}{2}, \\ -\frac{x+2}{x+1}, & \frac{1}{2} \leq x < 2. \end{cases} \]

Note that \( A/B \) is not a triangular fuzzy number.

3. Generalized triangular fuzzy sets

In this section, we define the generalized triangular fuzzy set and apply the extended algebraic operations to these fuzzy sets. A generalized triangular fuzzy set is symmetric and may not have value 1.

**Definition 3.1.** A generalized triangular fuzzy set is a symmetric fuzzy set \( A \) having membership function

\[
\mu_A(x) = \begin{cases} 0, & x < a_1, a_2 \leq x, \\ 2c(x-a_1), & a_1 \leq x < \frac{1}{2}(a_1+a_2) \\ -2c(x-a_2), & a_1+a_2 \geq x < a_2 \end{cases}
\]

(3.1)

where \( a_1, a_2 \in \mathbb{R} \) and \( 0 < c < 1 \).

The above generalized triangular fuzzy set is denoted by \( A = (a_1, c, a_2) \).

**Theorem 3.2.** ([4]) For two generalized triangular fuzzy sets \( A = (a_1, c, a_2) \) and \( B = (b_1, c, b_2) \), if \( c_1 \leq c_2 \) and \( \mu_B(x) \geq c_1 \) in \( [k_1, k_2] \), we have the followings.

1. \( A(+)B = (a_1+b_1, \frac{1}{2}(a_1+a_2)+k_1, c, \frac{1}{2}(a_1+a_2)+k_2) \), i.e., \( A(+)B \) is a generalized trapezoidal fuzzy set.
2. \( A(-)B = (a_1-b_2, \frac{1}{2}(a_1-a_2)-k_2, c, \frac{1}{2}(a_1-a_2)-k_1) \), i.e., \( A(-)B \) is a generalized trapezoidal fuzzy set.
3. \( A(\cdot)B \) is a fuzzy set on \( (a_1, a_2, a_3) \), but need not to be a generalized triangular fuzzy set or a generalized trapezoidal fuzzy set.
4. \( A(\div)B \) is a fuzzy set on \( (a_1, b_2, b_1) \), but need not to be a generalized triangular fuzzy set or a generalized trapezoidal fuzzy set.

**Example 3.3.** ([4]) Let \( A = (2, \frac{1}{2}, 8) \) and \( B = ((1, \frac{4}{9}, 5)) \) be generalized triangular fuzzy sets, i.e.,

\[
\mu_A(x) = \begin{cases} 0, & x < 2, 8 \leq x, \\ \frac{1}{6}(x-2), & 2 \leq x < 5 \\ \frac{1}{6}(x-8), & 5 \leq x \leq 8 \end{cases}
\]

and

\[
\mu_B(x) = \begin{cases} 0, & x < 1, 5 \leq x, \\ \frac{2}{5}(x-1), & 1 \leq x < 3 \\ \frac{2}{5}(x-5), & 3 \leq x < 5 \end{cases}
\]

Let \( A_\alpha \) and \( B_\alpha \) be the \( \alpha \) - cuts of \( A \) and \( B \), respectively. Let \( A_\alpha = [a_1^\alpha, a_2^\alpha] \) and \( B_\alpha = [b_1^\alpha, b_2^\alpha] \). Since \( \alpha = \frac{1}{6}(a_1^\alpha-2) \) and \( \alpha = -\frac{1}{6}(a_2^\alpha-8) \), we have \( A_\alpha = [a_1^\alpha, a_2^\alpha] = [6a + 2, -6a + 8] \). Since \( \alpha = \frac{2}{5}(b_1^\alpha-1) \) and \( \alpha = -\frac{2}{5}(b_2^\alpha-5) \), we have \( B_\alpha = [b_1^\alpha, b_2^\alpha] = [\frac{5}{2}a + 1, -\frac{5}{2}a + 5] \). Then we have the followings.

1. Addition:

\[
\mu_{A_\alpha + B_\alpha}(x) = \begin{cases} 0, & x < 3, 13 \leq x, \\ \frac{2}{17}(x-3), & 3 \leq x < \frac{29}{4}, \\ \frac{29}{4}, & \frac{29}{4} \leq x < \frac{35}{4}, \\ \frac{29}{4}, & \frac{35}{4} \leq x < 13, \end{cases}
\]

i.e., \( A(+)B = (3, \frac{29}{4}, \frac{35}{4}, 13) \).

2. Subtraction:

\[
\mu_{A_\alpha - B_\alpha}(x) = \begin{cases} 0, & x < -3, 7 \leq x, \\ \frac{2}{17}(x+3), & -3 \leq x < \frac{5}{4}, \\ \frac{5}{4}, & \frac{5}{4} \leq x < \frac{11}{4}, \\ \frac{11}{4}, & \frac{11}{4} \leq x < 7. \end{cases}
\]

i.e., \( A(-)B = (-3, \frac{5}{4}, \frac{11}{4}, 7) \).

3. Multiplication:

\[
\mu_{A_\alpha \cdot B_\alpha}(x) =
\]
Thus \( A(\cdot)B \) is a fuzzy set on \((2, 40)\), but not a generalized triangular fuzzy set or a generalized trapezoidal fuzzy set.

4. Division:

\[
\mu_{A(\cdot)/B}(x) = \begin{cases} 
0, & x < \frac{2}{5}, \ 8 \leq x, \\
10 - 4, & \frac{2}{5} \leq x < \frac{4}{3}, \\
1, & \frac{4}{3} \leq x < \frac{20}{9}, \\
-2(8 - x), & \frac{20}{9} \leq x < 8.
\end{cases}
\]

Thus \( A(/)B \) is a fuzzy set on \((\frac{2}{5}, 8)\), but not a generalized triangular fuzzy set or a generalized trapezoidal fuzzy set.

4. Normal fuzzy probability

In this section, we derive the explicit formula for the normal fuzzy probability for generalized triangular fuzzy sets and give some examples.

**Theorem 41.** Let \( X \sim N(m, \sigma^2) \) and \( A = ((a_1, c, a_2)) \) be generalized triangular fuzzy set. Then the normal fuzzy probability of a generalized triangular fuzzy set \( A \) is

\[
\tilde{P}(A) = \frac{-2mc}{\sqrt{2\pi} (a_2 - a_1)} \left( 2N\left( \frac{a_1 + a_2 - 2m}{2\sigma} \right) - N\left( \frac{a_1 - m}{\sigma} \right) - N\left( \frac{a_2 - m}{\sigma} \right) \right)
\]

where \( N(a) \) is the standard normal distribution, that is,

\[
N(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} \exp\left(-\frac{t^2}{2}\right) dt.
\]

**Proof.** Since

\[
\mu_A(x) = \begin{cases} 
0, & x < a_1, a_2 \leq x, \\
\frac{2c(x - a_1)}{a_2 - a_1}, & a_1 \leq x < \frac{a_1 + a_2}{2}, \\
\frac{-2c(x - a_2)}{a_2 - a_1}, & \frac{a_1 + a_2}{2} \leq x < a_2.
\end{cases}
\]

we have

\[
\tilde{P}(A) = \int_{-\infty}^{\frac{a_1 + a_2}{2}} g_1(x) f(x) dx + \int_{\frac{a_1 + a_2}{2}}^{a_2} g_2(x) f(x) dx,
\]

where \( g_1(x) = \frac{2c(x - a_1)}{a_2 - a_1}, \ g_2(x) = \frac{-2c(x - a_2)}{a_2 - a_1} \) and

\[
f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x - m)^2}{2\sigma^2}\right).
\]

Putting \( \frac{x - m}{\sigma} = t \), then

\[
\tilde{P}(A) = \int_{a_1}^{a_2} \frac{2c}{\sqrt{2\pi}\sigma} \left( \frac{a_1 + a_2 - 2m}{2\sigma} - \frac{a_1 - m}{\sigma} \right) e^{-\frac{t^2}{2\sigma^2}} dt
\]

\[
- \int_{a_1}^{a_2} \frac{2c}{\sqrt{2\pi}\sigma} \left( \frac{a_1 - m}{\sigma} \right) e^{-\frac{t^2}{2\sigma^2}} dt
\]

\[
+ \int_{a_1}^{a_2} \frac{2c}{\sqrt{2\pi}\sigma} \left( \frac{a_2 - m}{\sigma} \right) e^{-\frac{t^2}{2\sigma^2}} dt
\]

\[
= \frac{2mc}{\sqrt{2\pi} (a_2 - a_1)} \left( N\left( \frac{a_1 + a_2 - 2m}{2\sigma} \right) - N\left( \frac{a_1 - m}{\sigma} \right) - N\left( \frac{a_2 - m}{\sigma} \right) \right)
\]

\[
- \frac{2mc}{\sqrt{2\pi} (a_2 - a_1)} \left( N\left( \frac{a_1 - m}{\sigma} \right) - N\left( \frac{a_1 + a_2 - 2m}{2\sigma} \right) \right)
\]

\[
- \frac{2mc}{\sqrt{2\pi} (a_2 - a_1)} \left( N\left( \frac{a_2 - m}{\sigma} \right) - N\left( \frac{a_1 + a_2 - 2m}{2\sigma} \right) \right)
\]

\[
+ \frac{2mc}{\sqrt{2\pi} (a_2 - a_1)} \left( N\left( \frac{a_1 - m}{\sigma} \right) - N\left( \frac{a_1 + a_2 - 2m}{2\sigma} \right) \right)
\]

\[
- \frac{2mc}{\sqrt{2\pi} (a_2 - a_1)} \left( N\left( \frac{a_1 + a_2 - 2m}{2\sigma} \right) - N\left( \frac{a_1 - m}{\sigma} \right) \right)
\]

\[
+ \frac{2mc}{\sqrt{2\pi} (a_2 - a_1)} \left( N\left( \frac{a_1 + a_2 - 2m}{2\sigma} \right) - N\left( \frac{a_1 - m}{\sigma} \right) \right)
\]

\[
= \frac{-2mc}{\sqrt{2\pi} (a_2 - a_1)} \left( N\left( \frac{a_1 + a_2 - 2m}{2\sigma} \right) - N\left( \frac{a_1 - m}{\sigma} \right) \right)
\]

\[
- \frac{2mc}{\sqrt{2\pi} (a_2 - a_1)} \left( N\left( \frac{a_1 - m}{\sigma} \right) - N\left( \frac{a_1 + a_2 - 2m}{2\sigma} \right) \right)
\]

\[
+ \frac{2mc}{\sqrt{2\pi} (a_2 - a_1)} \left( N\left( \frac{a_2 - m}{\sigma} \right) - N\left( \frac{a_1 + a_2 - 2m}{2\sigma} \right) \right)
\]

\[
- \frac{2mc}{\sqrt{2\pi} (a_2 - a_1)} \left( N\left( \frac{a_1 + a_2 - 2m}{2\sigma} \right) - N\left( \frac{a_1 - m}{\sigma} \right) \right)
\]

\[
+ \frac{2mc}{\sqrt{2\pi} (a_2 - a_1)} \left( N\left( \frac{a_1 + a_2 - 2m}{2\sigma} \right) - N\left( \frac{a_1 - m}{\sigma} \right) \right)
\]
= \frac{2mc}{\sqrt{2\pi (a_x-a_y)}} (2N(\frac{a_1 + a_2 - 2m}{2\sigma}) - N(\frac{a_1-m}{\sigma}))
- \frac{2c(a_1+\sigma)}{\sqrt{2\pi (a_x-a_y)}} (N(\frac{a_1 + a_2 - 2m}{2\sigma}) - N(\frac{a_1-m}{\sigma}))
+ \frac{2c(a_2+\sigma)}{\sqrt{2\pi (a_x-a_y)}} (N(\frac{a_2-m}{\sigma}) - N(\frac{a_1+a_2-2m}{2\sigma})).

Thus the proof is complete.

**Example 4.2.** Let \( A= ((2, \frac{1}{2}, 8)) \) be a generalized triangular fuzzy number. Then the normal fuzzy probability of \( A \) with respect to \( X \sim N(3, 2^2) \) is 0.1777. In fact, putting \( \frac{x-3}{2} = t \), we have

\[ \overline{P}(A) = \int_{-2}^{2} \frac{x-2}{6} \frac{1}{2\sqrt{\pi}} e^{-\frac{(x-3)^2}{8}} dx \]

\[ + \int_{-2}^{2} \frac{8-x}{6} \frac{1}{2\sqrt{\pi}} e^{-\frac{(x-3)^2}{8}} dx \]

\[ = \frac{1}{6\sqrt{2\pi}} \int_{-1/2}^{1} (2t+1) e^{-\frac{t^2}{2}} dt \]

\[ - \frac{1}{6\sqrt{2\pi}} \int_{1/2}^{5/2} (2t-5) e^{-\frac{t^2}{2}} dt \]

\[ = \frac{1}{3\sqrt{2\pi}} \int_{-1/2}^{1} te^{-\frac{t^2}{2}} dt \]

\[ + \frac{1}{6\sqrt{2\pi}} \int_{-1/2}^{1/2} e^{-\frac{t^2}{2}} dt \]

\[ - \frac{1}{3\sqrt{2\pi}} \int_{1/2}^{5/2} te^{-\frac{t^2}{2}} dt \]

\[ + \frac{5}{6\sqrt{2\pi}} \int_{1/2}^{5/2} e^{-\frac{t^2}{2}} dt \]

\[ = \frac{1}{3\sqrt{2\pi}} \left( e^{-\frac{1}{8}} - 2e^{-\frac{1}{2}} + e^{-\frac{25}{8}} \right) \]

\[ + \frac{1}{6\sqrt{2\pi}} \left( N(1) - N(-1/2) \right) \]

\[ + \frac{5}{6\sqrt{2\pi}} \left( N(5/2) - N(1) \right) \]

\[ = 0.1777. \]

2. Let \( B= ((1, \frac{3}{4}, 5)) \) be a generalized triangular fuzzy number. Then the normal fuzzy probability of \( B \) with respect to \( X \sim N(3, 3^2) \) is 0.1924. In fact, putting \( \frac{x-3}{3} = t \), we have

\[ \overline{P}(B) = \int_{1}^{3} 3(x-1) \frac{1}{3\sqrt{2\pi}} e^{-\frac{(x-3)^2}{18}} dx \]

\[ + \int_{1}^{3} 3(5-x) \frac{1}{3\sqrt{2\pi}} e^{-\frac{(x-3)^2}{18}} dx \]

\[ = \frac{3}{8\sqrt{2\pi}} \int_{-2/3}^{0} (3t-2) e^{-\frac{t^2}{2}} dt \]

\[ - \frac{3}{8\sqrt{2\pi}} \int_{0}^{2/3} (3t-2) e^{-\frac{t^2}{2}} dt \]

\[ = \frac{9}{8\sqrt{2\pi}} \int_{-2/3}^{0} te^{-\frac{t^2}{2}} dt \]

\[ + \frac{3}{4\sqrt{2\pi}} \int_{-2/3}^{0} e^{-\frac{t^2}{2}} dt \]

\[ - \frac{9}{8\sqrt{2\pi}} \int_{0}^{2/3} te^{-\frac{t^2}{2}} dt \]

\[ + \frac{3}{4\sqrt{2\pi}} \int_{0}^{2/3} e^{-\frac{t^2}{2}} dt \]

\[ = \frac{9}{4\sqrt{2\pi}} \left( \frac{e^{-2}}{3} - 1 \right) \]

\[ + \frac{3}{4\sqrt{2\pi}} \left( N(2/3) - N(-2/3) \right) \]

\[ = 0.1924. \]

References


[8] L. A. Zadeh, "The concept of a linguistic variable