Interval-Valued Fuzzy Cosets

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Abstract

First, we prove a number of results about interval-valued fuzzy groups involving the notions of interval-valued fuzzy cosets and interval-valued fuzzy normal subgroups which are analogs of important results from group theory. Also, we introduce analogs of some group-theoretic concepts such as characteristic subgroup, normalizer and abelian groups. Secondly, we prove that if \( A \) is an interval-valued fuzzy subgroup of a group \( G \) such that the index of \( A \) is the smallest prime dividing the order of \( G \), then \( A \) is an interval-valued fuzzy normal subgroup. Finally, we show that there is a one-to-one correspondence the interval-valued fuzzy cosets of an interval-valued fuzzy subgroup \( A \) of a group \( G \) and the cosets of a certain subgroup \( H \) of \( G \).

Key Words: interval-valued fuzzy normal subgroup, interval-valued fuzzy coset, interval-valued fuzzy characteristic fuzzy subgroup, normalizer, abelian group.

1. Introduction

The concept of a fuzzy set was introduced by Zadeh[9], and in 1965, he[10] introduced the notion of interval-valued fuzzy set as a generalization of fuzzy sets. After that time, Mondal and Samanta[8], and choi et al.[3] applied it to topology. Also, several researchers [1,2, 4-7] applied one to algebra.

The present paper is a sequel to [4]. We obtain a number of further analogs of the properties of groups, thereby enriching the theory of interval-valued fuzzy groups and, in particular, corroborating the concept of interval-valued fuzzy normal subgroups and interval-valued fuzzy cosets introduced in [4,5]. Moreover, we obtain an analog of the following standard result from group theory that if \( \theta \) is an automorphism of a group \( G \) which leaves invariant some normal subgroup \( N \), then \( \theta \) induces an automorphism of the quotient group \( G/N \).

Some variations of this result are also considered, for which we obtain analogs for interval-valued fuzzy groups. Also we show that there is a natural one-to-one correspondence between the interval-valued fuzzy cosets of an interval-valued fuzzy subgroup \( A \) of a group \( G \) and the cosets of a subgroup \( G_A \) of \( G \) defined by \( G_A = \{ g \in G : A(g) = A(e) \} \), where \( e \) denotes, as usual, the identity element of the group \( G \). Our analysis illustrates that the subgroup \( G_A \) defined above plays a significant role in investigating the structure of the corresponding interval-valued fuzzy subgroup.

2. Preliminaries

In this section, we list some basic concepts and well-
known results which are needed in the later sections.

Let $D(I)$ be the set of all closed subintervals of the unit interval $I = [0, 1]$. The elements of $D(I)$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M_L, M_U]$, where $M_L$ and $M_U$ are the lower and the upper end points respectively. Especially, we denote $0 = [0, 0], 1 = [1, 1]$, and $\alpha = [a, a]$ for every $a \in (0, 1)$. We also note that

(i) $(\forall M, \alpha \in D(I)) (M = N \iff M_L = N_L, M_U = N_U)$,

(ii) $(\forall M, \alpha \in D(I)) (M \subseteq N \iff M_L \subseteq N_L, M_U \subseteq N_U)$.

For every $M \in D(I)$, the complement of $M$, denoted by $M^c$, is defined by $M^c = 1 - M = [1 - M_U, 1 - M_L]$ (See [8]).

Definition 2.1 [8, 10]. A mapping $A : X \to D(I)$ is called an interval-valued fuzzy set (in short, IVS) in $X$, denoted by $A = [A_L, A_U]$, if $A_L, A_U \in I^X$ such that $A_L \leq A_U$, i.e., $A_L(x) \leq A_U(x)$ for each $x \in X$, where $A_L(x)$[resp. $A_U(x)$] is called the lower resp. upper end point of $x$ to $A$. For any $[a, b] \in D(I)$, the interval-valued fuzzy set $A$ in $X$ defined by $A(x) = [A_L(x), A_U(x)] = [a, b]$ for each $x \in X$ is denoted by $[a, b]_I$ and if $a = b$, then the IVS $[a, b]$ is denoted by simply $a$. In particular, $\tilde{0}$ and $\tilde{1}$ denote the interval-valued fuzzy empty set and the interval-valued fuzzy whole set in $X$, respectively.

We will denote the set of all IVSs in $X$ as $D(I)^X$. It is clear that set $A = [A_L, A_U] \in D(I)^X$ for each $A \in I^X$.

Definition 2.2 [8]. Let $A, B \in D(I)^X$ and let $\{A_\alpha\}_{\alpha \in \Gamma} \subset D(I)^X$. Then

(i) $A \subseteq B$ iff $A_L \leq B_L$ and $A_U \leq B_U$.

(ii) $A = B$ iff $A \subseteq B$ and $B \subseteq A$.


(iv) $\bigcup_{\alpha \in \Gamma} A_\alpha = [\bigvee_{\alpha \in \Gamma} A^\alpha_L, \bigvee_{\alpha \in \Gamma} A^\alpha_U]$.

(v) $A \cap B = [A_L \wedge B_L, A_U \wedge B_U]$.

Result 2.A [8, Theorem 1]. Let $A, B, C \in D(I)^X$ and let $\{A_\alpha\}_{\alpha \in \Gamma} \subset D(I)^X$. Then

(a) $\tilde{0} \subseteq A \subseteq \tilde{1}$.

(b) $A \cup B = B \cup A$, $A \cap B = B \cap A$.

(c) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$.

(d) $A, B \subset A \cup B, A \cap B \subset A, B$.

(e) $A \cap (\bigcup_{\alpha \in \Gamma} A_\alpha) = \bigcup_{\alpha \in \Gamma} (A \cap A_\alpha)$.

(f) $A \cup (\bigcap_{\alpha \in \Gamma} A_\alpha) = \bigcap_{\alpha \in \Gamma} (A \cup A_\alpha)$.

(g) $A_0 = \tilde{1}$. $A_0 = \tilde{0}$.

(h) $(A^c)^c = A$.

Definition 2.3 [8]. Let $f : X \to Y$ be a mapping, let $A = [A_L, A_U] \in D(I)^X$ and let $B = [B_L, B_U] \in D(I)^Y$. Then

(a) the image of $A$ under $f$, denoted by $f(A)$, is an IVS in $Y$ defined as follows: For each $y \in Y$,

$$f(A)(y) = \begin{cases} \bigvee_{y' = f(x)} A^L(x), & \text{if } f^{-1}(y) \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$f(A^c)(y) = \begin{cases} \bigwedge_{y' = f(x)} A^U(x), & \text{if } f^{-1}(y) \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

(b) the preimage of $B$ under $f$, denoted by $f^{-1}(B)$, is an IVS in $Y$ defined as follows: For each $y \in Y$,

$$f^{-1}(B^L)(y) = (B^L \circ f)(x) = B^L(f(x))$$

and

$$f^{-1}(B^c)(y) = (B^U \circ f)(x) = B^U(f(x)).$$

It can be easily seen that $f(A) = [f(A^L), f(A^U)]$ and $f^{-1}(B) = [f^{-1}(B^L), f^{-1}(B^U)]$.

Result 2.B [8, Theorem 2]. Let $f : X \to Y$ be a mapping and $g : Y \to Z$ be a mapping. Then

(a) $f^{-1}(B^c) = (f^{-1}(B))^c, \forall B \in D(I)^Y$. 

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(b) \([f(A)]^c \subset f(A^c), \forall A \in D(I)^Y\).
(c) \(B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2), \) where \(B_1, B_2 \in D(I)^Y\).
(d) \(A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2), \) where \(A_1, A_2 \in D(I)^X\).
(e) \(f(f^{-1}(B)) \subset B, \forall B \in D(I)^Y.\)
(f) \(A \subset f(f^{-1}(A)), \forall A \in D(I)^Y.\)
(g) \((g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C)), \forall C \in D(I)^Z.\)
(h) \(f^{-1}(\bigcup_{\alpha \in \Gamma} B_\alpha) = \bigcup_{\alpha \in \Gamma} f^{-1}B_\alpha, \) where \(\{B_\alpha\}_{\alpha \in \Gamma} \in D(I)^Y.\)
(h) \(f^{-1}(\bigcap_{\alpha \in \Gamma} B_\alpha) = \bigcap_{\alpha \in \Gamma} f^{-1}B_\alpha, \) where \(\{B_\alpha\}_{\alpha \in \Gamma} \in D(I)^Y.\)

3. Interval-valued fuzzy subgroups

**Definition 3.1** [1, 6]. Let \(G\) be a group with the identity \(e\) and let \(A \in D(I)^G.\) Then \(A\) is called an interval-valued fuzzy subgroup (in short, IVG) of \(G\) if
(i) \(A^L(xy) \geq A^L(x) \wedge A^L(y)\) and \(A^U(xy) \geq A^U(x) \wedge A^U(y)\) for any \(x, y \in G.\)
(ii) \(A^L(x^{-1}) \geq A^L(x)\) and \(A^U(x^{-1}) \geq A^U(x)\) for each \(x \in G.\)

We will denote the set of all IVGs of \(G\) as IVG(G).

**Definition 3.2** [6]. Let \(G\) be a group with the identity \(e\) and let \(A \in IVG(G).\) Then \(A\) is called an interval-valued fuzzy subgroup (in short, IVG) of \(G\) if \(A(xy) = A(yx)\) for any \(x, y \in G.\)

We will denote the set of all IVNGs of \(G\) as IVNG(G).

**Definition 3.3.** Let \(A\) be an IVG of a group \(G\) and let \(\theta : G \to G\) be a mapping. We define a mapping \(\theta^\theta = [(\theta^\theta)^L, (\theta^\theta)^U] : G \to D(I)\) as follows: For each \(g \in G,\)
\[
\theta^\theta(g) = A(\theta(g)).
\]
For a group \(G,\) a subgroup \(K\) is called a characteristic subgroup if \(\theta(K) = K\) for every automorphism \(\theta\) of \(G.\) We now define an analog.

**Definition 3.4.** Let \(A\) be an IVG of a group \(G.\) Then \(A\) is called an interval-valued fuzzy characteristic subgroup of \(G\) if \(\theta^\theta = A\) for every automorphism \(\theta\) of \(G.\)

**Proposition 3.5.** Let \(G\) be a group, let \(A \in D(I)^G\) and let \(\theta : G \to G\) be a mapping.
(a) If \(A \in IVG(G)\) and \(\theta\) is a homomorphism, then \(\theta^\theta \in IVG(G)\).
(b) If \(A\) is an interval-valued fuzzy subgroup of \(G,\) then \(A \in IVNG(G)\).

**Proof.** (a) Let \(x, y \in G.\) Then
\[
A^\theta(xy) = A(\theta(xy)) = A(\theta(x)) \cdot A(\theta(y)).\]\[Since \(\theta\) is a homomorphism]
\[
\text{Since } A \in IVG(G),
\]
\[
A^L(\theta(x)) \cdot A^L(\theta(y)) \geq A^L(\theta(xy)) \wedge A^L(\theta(xy)) = (\theta^\theta)^L(x) \wedge (\theta^\theta)^L(y).
\]
Similarly, we have that
\[
A^U(\theta(x)) \cdot A^U(\theta(y)) \geq A^U(\theta(xy)) \wedge A^U(\theta(xy)) = (\theta^\theta)^U(x) \wedge (\theta^\theta)^U(y).
\]
Thus
\[
(\theta^\theta)^L(xy) \geq (\theta^\theta)^L(x) \wedge (\theta^\theta)^L(y)
\]
and
\[
(\theta^\theta)^U(xy) \geq (\theta^\theta)^U(x) \wedge (\theta^\theta)^U(y).
\]
On the other hand,
\[
A^\theta(x^{-1}) = A(\theta(x^{-1})) = A(\theta(x))^{-1} \quad \text{[Since } \theta \text{ is a homomorphism]}
\]
\[
A^\theta(x) = A(\theta(x)) \quad \text{[By Result 3.1]}
\]
Hence $A^\theta \in \text{IVG}(G)$.

(b) Let $\theta : G \to G$ be the automorphism of $G$ defined by $\theta(g) = x^{-1}gx$ for each $g \in G$. Then clearly it is standard result that $\theta$ is an automorphism of $G$, called the \textit{inner automorphism} induced by $x$. Let $x, y \in G$. Since $A$ is interval-valued fuzzy characteristic, $A^\theta = A$. Thus

$$A(xy) = A^\theta(xy) = A(\theta(xy))$$
$$= A(x^{-1}(xy)x) \quad \text{[By the definition of $\theta$]}
= A(yx).$$

Hence $A \in \text{IVNG}(G)$. This completes the proof. \hfill $\square$

\textbf{Remark 3.6.} Proposition 3.5(b) is an analog of the result that a characteristic subgroup of a group is normal.

Now we obtain analogs of the concepts of conjugacy, normalizer regarding a group, and their properties.

\textbf{Definition 3.7.} Let $G$ be a group and let $A_1, A_2 \in \text{IVG}(G)$. Then we say that $A_1$ is \textit{conjugate} to $A_2$ if there exists an $x \in G$ such that $A_1(y) = A_2(x^{-1}yx)$ for each $g \in G$.

It is easy to show that the relation of conjugacy is an equivalence relation on IVG(G). Hence IVG(G) is a union of pairwise disjoint classes of interval-valued fuzzy subgroups each consisting of interval-valued fuzzy subgroups which are equivalent to one another. Now we shall obtain an expression giving the number of distinct conjugates of an interval-valued fuzzy subgroups.

\textbf{Notation.} Let $G$ be a group, let $A \in \text{IVG}(G)$ and let $g \in G$. We define a mapping $A^g = [(A^g)^L, (A^g)^U] : G \to D(I)$ as follows: for each $u \in G$, $A^g(u) = A(g^{-1}ug)$, i.e., $(A^g)^L(u) = A^L(g^{-1}ug)$ and $(A^g)^U(u) = A^U(g^{-1}ug)$.

From Proposition 3.5(a), it is clear that $A^g \in \text{IVG}(G)$.

\textbf{Definition 3.8.} Let $A$ be an IVG of a group $G$. Then the set $N(A) = \{g \in G : A^g = A\}$ is called the \textit{normalizer} of $A$.

\textbf{Proposition 3.9.} Let $A$ be an IVG of a group $G$. Then

(a) $N(A)$ is a subgroup of $G$.

(b) $A \in \text{IVNG}(G)$ id and only if $N(A) = G$.

(c) If $G$ is a finite group, then the number of distinct conjugates of $A$ is equal to the index of $N(A)$ in $G$.

\textbf{Proof.} (a) Let $g, h \in N(A)$ and let $u \in G$. Then $A^g(u) = A((gh)^{-1}u(gh)) = A(h^{-1}(g^{-1}ug)h) = A^b(g^{-1}ug) = (A^b)^g(u)$. Thus $A^g = (A^g)^h = A^h = A$. So $gh \in N(A)$. Let $x \in N(A)$ and let $y = x^{-1}$. Let $u \in G$. Then

$$A^u(u) = A(y^{-1}uy) = A(xux^{-1}) = A((x^{-1}u^{-1}x)^{-1})$$
$$= A(x^{-1}u^{-1}x) \quad \text{[By Result 3.1]}
= A^u(u^{-1}) \quad \text{[By the definition of $A^u$]}
= A(u^{-1}) \quad \text{[Since $A^u = A$]}
= A(u). \quad \text{[By Result 3.1]}
$$

Thus $A^g = A$. So $y = x^{-1} \in N(A)$. Hence $N(A)$ is a subgroup of $G$.

(b)$\Rightarrow$: Suppose $A \in \text{IVNG}(G)$ and let $g \in G$. Let $u \in G$. Then

$$A^g(u) = A(g^{-1}ug) = A((g^{-1}u)g)$$
$$= A(gg^{-1}u) \quad \text{[Since $A \in \text{IVNG}(G)$]}
= A(u).$$

Thus $A^g = A$. So $g \in N(A)$, i.e., $G \subset N(A)$. Hence $N(A) = G$.

(c)$\Rightarrow$: Suppose $N(A) = G$ and let $x, y \in G$. Then

$$A(xy) = A(xyxx^{-1}) = A(xyx)x^{-1}$$
$$= A^x(yx) \quad \text{[By the definition of $A^x$]}
= A(xy). \quad \text{[By the hypothesis]}
$$

Hence $A \in \text{IVNG}(G)$.

(c) Consider the decomposition of $G$ as a union of cosets of $N(A)$,

$$G = x_1N(A) \cup x_2N(A) \cup \cdots \cup x_kN(A), \quad (3.1)$$

where $k$ is the number of distinct cosets, i.e., the index of $N(A)$ in $G$. Let $x \in N(A)$ and choose $i$ such that $1 \leq i \leq k$. Then

$$x \in x_iN(A) \quad \text{and hence} \quad x \in N(A).$$

Thus $N(A) = G$. Hence $A \in \text{IVNG}(G)$.
k. Let \( g \in G \). Then
\[
A^{x_i}(g) = A((x_i x)^{-1} g(x_i x)) = A(x_i^{-1}(x_i^{-1} g x_i)) = A^x(g)
\]
Thus \( A^{x_i} = A^x \) for each \( x \in N(A) \) and \( 1 \leq i \leq k \). So any two elements of \( G \) which lie in the same coset \( x, N(A) \) give rise to the same conjugate \( A^{x_i} \) of \( A \). We now show that two distinct cosets give two distinct conjugates of \( A \). Assume that \( A^{x_i} = A^{x_j} \), where \( i \neq j \) and \( 1 \leq i \leq k \), \( 1 \leq j \leq k \). Let \( g \in G \). Then
\[
A^{x_i}(g) = A^{x_j}(g), \text{ i.e., } A(x_i^{-1} g x_i) = A(x_j^{-1} g x_j). \tag{3.2}
\]
Let \( h \in G \) such that \( g = x h x_j^{-1} \). Then, by (3.2),
\[
A(x_i^{-1} x h x_j^{-1} x_i) = A(x_j^{-1} x h x_j^{-1} x_i)\]
\(
\Rightarrow A((x_i^{-1} x h x_j^{-1} x_i) = A(h), \text{ i.e., } A((x_j^{-1} x h x_j^{-1} x_i)) = A(h)\]
\(
\Rightarrow A^{x_i^{-1} x h x_j^{-1} x_i} = A(h), \text{ i.e., } A^{x_j^{-1} x_i} = A
\)
Thus \( x_i^{-1} x_i \in N(A) \). So \( x_i N(A) = x_j N(A) \). Since (3.1) represent a partition of \( G \) into pairwise disjoint cosets and \( i \neq j \), this is not possible. Hence the number of distinct conjugates of \( A \) is equal to the index of \( N(A) \) in \( G \). This completes the proof.

**Remark 3.10.** Proposition 3.9(b) illustrates the motivation behind the term "normalizer" and it shows the analogy with the fact that a subgroup \( H \) of a group \( G \) is normal in \( G \) if and only if the normalizer of \( H \) in \( G \) is equal to \( G \) itself. And Proposition 3.9(c) is an analog of a basic result in group theory.

**Definition 3.11 [4].** Let \( A \) be an IVG of a group \( G \) and let \( x \in G \). We define two mappings \( A_x = [A x, A x^2] : G \rightarrow D(I) \) and \( x A = [A x, x A^2] : G \rightarrow D(I) \) as follows, respectively: For each \( g \in G \), \( A x(g) = A(g x^{-1}) \) and \( x A(g) = A(x^{-1} g) \). Then \( A x \) [resp. \( x A \)] is called the interval-valued fuzzy right [resp. left] coset of \( G \) determined by \( x \) and \( A \).

**Lemma 3.12.** Let \( A \) be an IVG of a group \( G \) and let \( K = \{ x \in G : A x = A e \} \), where \( e \) denotes the identity element of \( G \). Then \( K \) is a subgroup of \( G \). Furthermore, \( G_A = K \).

**Proof.** Let \( k \in K \) and let \( g \in G \). Then \( A k(g) = A e(g) \). Thus \( A k^{-1} = A(g) \). In particular, \( A(k^{-1}) = A(e) \), i.e., \( A(k^{-1}) = A(e) \). Thus \( k^{-1} \in G_A \). By Result 3.B, \( G_A \) is a subgroup of \( G \). Thus \( k \in G_A \). So \( K \subseteq G_A \). Now let \( h \in G_A \). Then
\[
A(h) = A(e). \tag{3.3}
\]
Let \( g \in G \). Then \( A h(g) = A(gh^{-1}) \) and \( A e(g) = A(g) \). Thus
\[
A^L(gh^{-1}) \geq A^L(g) \wedge A^L(h^{-1})
\]
\[
= A^L(g) \wedge A^L(h) \quad \text{[By Result 3.A]}
\]
\[
= A^L(g) \wedge A^L(e) \quad \text{[By (3.3)]}
\]
\[
= A^L(g) \quad \text{[By Result 3.A]}
\]
Similarly, we have that \( A^U(gh^{-1}) \geq A^U(g) \). Also,
\[
A^L(g) = A^L(gh^{-1} h) \geq A^L(gh^{-1}) \wedge A^L(h)
\]
\[
= A^L(gh^{-1}) \wedge A^L(e) \quad \text{[By (3.3)]}
\]
\[
= A^L(gh^{-1}) \quad \text{[By Result 3.A]}
\]
Similarly, we have that \( A^U(g) \geq A^U(gh^{-1}) \). So \( A(gh^{-1}) = A(g) \), i.e., \( A h = A e \), i.e., \( h \in K \). Hence \( G_A \subseteq K \). Therefore \( G_A = K \). This completes the proof.

**Corollary 3.12 [6, Proposition 5.4].** Let \( G \) be a group. If \( A \in \text{IVNG}(G) \), then \( G_A < G \).

**Proof.** Let \( g \in G \) and let \( x \in G_A \). Then
\[
A(g^{-1} x g) = A(g g^{-1} x) \quad \text{[Since } A \in \text{IVNG}(G)]
\]
\[
= A(x) = A(e) \quad \text{[Since } x \in G_A]\]
Thus \( g^{-1} x g \in G_A \). Hence \( G_A < G \).
For a group $G$, the commutator $[x, y]$ of two elements $x, y$ in $G$ is defined as $[x, y] = x^{-1}y^{-1}xy$. If $xy = yx$, then obviously $[x, y] = e$. Thus $G$ is abelian if $[x, y] = e$ for all $x, y \in G$. This motivates the following definition.

**Remark 3.13.** A special case of Lemma 3.12 is implicit in Theorem 2.12 in [4], where it was tacitly assumed that $A$ is interval-valued fuzzy normal. But, as we see now, it is not necessarily to assume that $A$ is interval-valued fuzzy normal, and this fact straightens the proof of the interval-valued fuzzy Lagrange’s theorem [4, Theorem 4.12].

**Definition 3.14.** Let $A$ be an IVG of a group $G$. Then $A$ is said to be interval-valued fuzzy abelian if $A([x, y]) = A(e)$ for any $x, y \in G$.

**Result 3.D** [4, Theorem 2.12]. Let $A \in$ IVG$(G)$. Then $A \in$ IVNG$(G)$ if and only if $A^L([x, y]) \geq A^L(x)$ and $A^U([x, y]) \geq A^U(x)$ for any $x, y \in G$.

Analogous to some well-known properties of abelian group, we prove.

**Theorem 3.15.** (a) An interval-valued fuzzy abelian subgroup of a group is interval-valued fuzzy normal.

(b) Given an interval-valued fuzzy abelian subgroup of $G$, there is a normal subgroup $N$ of $G$ such that $G/N$ is abelian.

**Proof.** (a) Let $A$ be an interval-valued fuzzy abelian subgroup of $G$. Let $x, y \in G$. Then, by Result 3.A, $A^L([x, y]) = A^L(e) \geq A^L(x)$ and $A^U([x, y]) = A^U(e) \geq A^U(x)$. Hence, by Result 3.D, $A \in$ IVNG$(G)$.

(b) Let $A$ be an interval-valued fuzzy abelian subgroup of $G$. Then, by (a), $A \in$ IVNG$(G)$. Thus, by Corollary 3.12, $G_A \leq G$. Also, it is easy to see that $G' \leq G_A$, where $G'$ denotes the commutator subgroup of $G$ (i.e., the subgroup generated by all elements $[x, y], x, y \in G$). Hence $G/A$ is abelian.

The following is the immediate result of Definition 3.2 and Result 3.C.

**Proposition 3.16.** If $\{A_\alpha\}_{\alpha \in \Gamma}$ is a family of IVNGs of a group $G$, then $\bigcap_{\alpha \in \Gamma} A_\alpha \in$ IVNG$(G)$. Furthermore, if $A, B \in$ IVNG$(G)$, then $A \cap B \in$ IVNG$(G)$.

It is a standard result in group theory that if $G$ is a group, $H \leq G$, $K \leq G$ and $H \vartriangleleft G$, then $H \cap K \vartriangleleft K$ is normal in $K$. Now we derive an analog for interval-valued fuzzy subgroups.

**Proposition 3.17.** Let $G$ be a group and let $A \in$ IVG$(G)$, $B \in$ IVNG$(G)$. Then $A \cap B$ is an interval-valued fuzzy normal subgroup of the group $G_A$.

**Proof.** It is clear that $G_A$ is a subgroup of $G$ by Result 3.B. By Proposition 3.16, $A \cap B \in$ IVNG$(G)$. Thus $A \cap B \in$ IVNG$(G_A)$. Let $x, y \in G_A$. Since $G_A$ is a subgroup of $G$, $xy \in G_A$ and $yx \in G_A$. Thus $A(xy) = A(yx) = A(e)$. Since $B \in$ IVNG$(G)$, $B(xy) = B(yx)$. So

$$(A \cap B)(xy) = [A^L(xy) \cap B^L(xy), A^U(xy) \cap B^U(xy)]$$

$$= [A^L(yx) \cap B^L(yx), A^U(yx) \cap B^U(yx)]$$

$$= (A \cap B)(yx).$$

Hence $A \cap B \in$ IVNG$(G_A)$.

### 4. Interval-valued fuzzy cosets

**Result 4.A** [4, Theorem 2.9]. Let $A$ be an IVG of a group $G$. Then the followings are equivalent:

(a) $A^L(xy^{-1}) \geq A^L(y)$ and $A^U(xy^{-1}) \geq A^U(y)$ for any $x, y \in G$.

(b) $A(xy^{-1}) = A(y)$ for any $x, y \in G$.

(c) $A \in$ IVNG$(G)$.

(d) $xA = Ax$ for each $x \in G$.

(e) $xAx^{-1} = A$ for each $x \in G$.

**Remark 4.1.** We shall restrict ourselves in the subsequent discussion, without any loss of generality, with
interval-valued fuzzy right cosets only (corresponding results for interval-valued fuzzy left cosets could be obtained without any difficulty). Consequently from now on we call an interval-valued fuzzy right coset an interval-valued fuzzy coset and denote it as $Ax$ for each $x \in G$.

**Definition 4.2** [4]. Let $A$ be an IVG of a finite group $G$. Then the cardinality $|G/A|$ of $G/A$ is called an index of $A$, where $G/A$ denotes the set of all interval-valued fuzzy cosets of $A$.

**Result 4.B** [4, Proposition 3.7]. Let $A$ be an IVNG of a group $G$. We define an operation $*$ on $G/A$ as follows: For any $x, y \in G$, $Ax * Ay = Axy$. Then $(G/A, *)$ is a group. In this case, $G/A$ is called the interval-valued fuzzy quotient group by $A$.

**Result 4.C** [4, Theorem 4.12]. Let $A$ be an IVG of a finite group $G$. Then the index of $A$ divides the order of $G$.

It is a well-known result in group theory that subgroup $A$ of index $2$ is a normal subgroup. We now obtain an analog of a generalization of this result.

**Proposition 4.3.** Let $A$ be an IVG of a finite group $G$ such that the index of $A$ is $p$, where $p$ is the smallest prime dividing the order of $G$. Then $A \in IVNG(G)$.

**Proof.** By Result 3.B, $G_A$ is a subgroup of $G$. Since $A$ is an IVG of $G$ such that the index of $A$ is $p$, by Lemma 3.12 and Result 4.C, $G_A$ has index $p$ in $G$, i.e., $G_A$ has $p$ distinct (right) cosets, say, $\{G_Ax_i : 1 \leq i \leq p\}$. Now consider the permutation representation of $G$ on the cosets of $G_A$ given by the map $\pi : x \mapsto \pi_{x^{-1}}$, where $\pi_{x^{-1}} : G_Ax_i \mapsto G_Ax_i x^{-1}$, $1 \leq i \leq p$. Since the index of $G_A$ in $G$ is $p$, $\pi$ is an isomorphism of $G$ into the symmetric group $S_p$. Furthermore, $\text{Ker}\pi = \text{Core}(G_A)$, where $\text{Core}(G_A)$ denotes the intersection of all the conjugates $g^{-1}G_Ag$, $g \in G$. By the fundamental theorem of homomorphisms of groups and using Lagrange’s theorem, the order of $G/\text{Core}(G_A)$ divides $p!$ which is the order of $S_p$. Furthermore,

$$G/\text{Core}(G_A) \cong (G/G_A)(G_A/\text{Core}(G_A))$$

and the order of $G/G_A$ is $p$. Thus it follows that the order of $G_A/\text{Core}(G_A)$ divides $(p - 1)!$. Since the order of $G_A$ divides the order of $G$, $G_A = \text{Core}(G_A)$; otherwise we get a contradiction to the fact that $p$ is the smallest prime dividing the order of $G$. Since $\text{Core}(G_A)$ is a normal subgroup of $G$, $G_A$ is a normal subgroup of $G$.

Now consider the quotient group $G/H$. Since the order of $G/G_A$ is $p$, $G/G_A$ is abelian. Let $x, y \in G$. Then $G_Ax(G_Ay) = (G_Ay)(G_Ax)$. Thus $G_Axy = G_Ayx$. So there exists an $h \in G_A$ such that $xy = hyx$. Then

$$A^L(xy) = A^L(hyx) \geq A^L(h) \land A^L(yx) = A^L(e) \land A^L(yx) = A^L(yx).$$

Similarly, we have that $A^U(xy) \geq A^U(yx)$. Also, we have that $A^L(xy) \geq A^L(xy) \land A^L(yx) \geq A^U(xy)$. So $A(xy) = A(yx)$ for any $x, y \in G$. Hence $A \in IVNG(G)$.

This completes the proof. □

The following is the immediate result of Proposition 4.3.

**Corollary 4.3.** Let $A$ be an IVG of a group $G$ such that the index of $A$ is $2$, then $A \in IVNG(G)$.

It is well-known in group theory that $\theta$ is a homomorphism of a group $G$ into itself whose kernel is $N$, then $\theta$ induces a homomorphism from $G/N$ into itself. Now we derive an analog of the following result.

**Proposition 4.4.** Let $A$ be an IVNG of a group $G$ and let $\theta$ be an homomorphism of $G$ into itself such that $\theta(G_A) = G_A$. Then $\theta$ induces a homomorphism $\tilde{\theta}$ of the interval-valued fuzzy cosets of $A$ defined as follows: $\tilde{\theta}(Ax) = A\theta(x)$ for each $x \in G$.

**Proof.** Suppose $x, y \in G$ such that $Ax = Ay$. Then $Ax(x) = Ay(x)$ and $Ax(y) = Ay(y)$. Thus $A(e) = A(xy^{-1}) = A(yx^{-1})$. So $xy^{-1}, yx^{-1} \in G_A$. Since
\[ \theta(G_A) = G_A, \theta(xy^{-1}), \theta(yx^{-1}) \in G_A. \]

Then

\[ A(\theta(xy^{-1})) = A(\theta(yx^{-1})) = A(e). \quad (4.1) \]

Let \( y \in G. \) Then

\[ (A\theta(x))^L(y) = A^L(g\theta(x)^{-1}) = A^L(g\theta(x^{-1})) \text{ [Since } \theta \text{ is a homomorphism]} \]

\[ \geq A^L(g\theta(y)(\theta(x^{-1})) \text{ [Since } \theta \text{ is a homomorphism]} \]

\[ = A^L(g\theta(y^{-1})) \wedge A^L(\theta(y)(x^{-1})) \]

[Since \( A \in \text{IVG}(G) \)]

\[ = A^L(g\theta(y^{-1})) \wedge A^L(\theta(y)(x^{-1})) \]

Similarly, we have that \( (A\theta(x))L(y) \geq (A\theta(y))L(g). \]

Also, we have that \( (A\theta(y))L(y) \geq (A\theta(x))L(g). \) Thus \( A\theta(x) = A\theta(y). \) So \( \bar{\theta} \) is well-defined. Now let \( x, y \in G. \) Then

\[ \bar{\theta}(Ax \ast Ay) = \bar{\theta}(Ax \ast ) \text{ [By Result 4.B]} \]

\[ = A\theta(xy) \text{ [By the definition of } \bar{\theta} \text{]} \]

\[ = A\theta(x) \theta(y) \text{ [Since } \theta \text{ is a homomorphism]} \]

\[ = A\theta(Ax) \ast A\theta(Ay) \text{ [By Result 4.B]} \]

\[ = \bar{\theta}(Ax) \ast \bar{\theta}(Ay). \text{ [By the definition of } \bar{\theta} \text{]} \]

Hence \( \bar{\theta} \) is a homomorphism. This completes the proof. \( \square \)

Corollary 4.4-1. In the same hypothesis as in Proposition 4.4, if \( \theta \) is an automorphism and \( G \) is finite, then \( \bar{\theta} \) is an automorphism.

Proof. Since \( G \) has finite order, it is easy to see that \( \theta \) has finite order. Suppose that \( \theta \) has order \( k. \) Then \( \theta^k = \text{id}_G, \)

where \( \text{id}_G \) denotes the identity mapping. Let \( x, y \in G \) such that \( \bar{\theta}(Ax) = \bar{\theta}(Ay). \) Then, by the definition of \( \bar{\theta}, \)

\[ A\theta(x) = A\theta(y). \]

Since \( \theta^k = \text{id}_G, \theta^k(x) = x \) and \( \theta^k(y) = y. \) Thus \( Ax = A\theta^k(x) = A\theta^k(y) = Ay. \)

So \( \bar{\theta} \) is injective. Hence \( \bar{\theta} \) is an automorphism. \( \square \)

Corollary 4.4-2. In the same hypothesis as in Proposition 4.4, if \( \bar{\theta} \) is an automorphism and \( G_A = \{e\}, \) then \( \theta \) is an automorphism.

Proof. Let \( x, y \in G \) such that \( \theta(x) = \theta(y). \) Then

\[ A\theta(x) = A\theta(y), \text{ i.e., } \bar{\theta}(Ax) = \bar{\theta}(Ay). \]

Since \( \bar{\theta} \) is injective, \( Ax = Ay. \) Then \( Ax(y) = Ay(y). \) Thus \( A(yx^{-1}) = A(e). \) So \( yx^{-1} \in G_A. \) Since \( G_A = \{e\}, yx^{-1} = e. \) Thus \( x = y. \) So \( \theta \) is injective. Hence \( \theta \) is an automorphism. \( \square \)

The motivation of the following result stems from the standard theorem in group theory that if \( \theta \) is an automorphism of \( G \) and \( N \) is a normal subgroup of \( G \) such that \( N^\theta \subset N, \) then \( \theta \) induces an automorphism of the quotient group \( G/N \) into itself.

Remark 4.5. In Proposition 4.4, we have assumed \( A \) to be interval-valued fuzzy normal instead of assuming only that \( A \) is an interval-valued fuzzy subgroup. This has been done to ensure that the law of composition of interval-valued fuzzy cosets is well-defined, and this fact is used in the proof of Proposition 4.4 to show that \( \bar{\theta} \) is a homomorphism (refer to Result 4.B). However, it is clear from the proof that to show \( \bar{\theta} \) is well-defined it is not necessary to assume \( A \) to be interval-valued fuzzy normal.

Proposition 4.6. Let \( A \) be an IVNG of a group \( G \) and let \( \theta \) be an automorphism of \( G \) such that \( A^\theta = A \) (recall the definition of \( A^\theta \) given by Definition 3.3). Then \( \theta \) induces an automorphism \( \bar{\theta} \) of \( G/A \) defined as follows: for each \( x \in G, \theta(Ax) = A\theta(x). \)

Proof. Let \( x, y \in G \) such that \( Ax = Ay. \) We show that

\[ \bar{\theta}(Ax) = \bar{\theta}(Ay), \text{ i.e., } A\theta(x)(y) = A\theta(y)(y) \text{ for each } g \in G. \]

Let \( g \in G. \) Since \( \theta \) is an automorphism of \( G, \) there exists a \( g^* \in G \) such that \( \theta(g^*) = g. \)

Since \( Ax = Ay, \)

\( Ax(g^*) = Ay(g^*), \) i.e., \( A(g^*)x^{-1} = A(g^*)y^{-1}. \) Since \( A^\theta = A, \)

\( A^\theta(g^*)x^{-1} = A^\theta(g^*)y^{-1}. \) By Definition 3.3,

\( A^\theta(g^*)y^{-1}) = A^\theta(g^*)y^{-1}). \) Since \( \theta \) is an automorphism of \( G, \)

\( A(\theta(g^*)x^{-1}) = A(\theta(g^*)y^{-1}). \) Thus \( A(g^*)x^{-1}) = A(g^*)y^{-1}). \text{ i.e., } A\theta(x)(y) = A\theta(y)(y). \)

So \( \bar{\theta}(Ax) = \bar{\theta}(Ay). \) Hence \( \bar{\theta} \) is well-defined. The proof
of the fact that $\theta$ is a homomorphism is analogous to the corresponding part of the proof of Proposition 4.4, and thus we omit the details. Now suppose $Ax \in Ker\theta$ for each $x \in G$. Then $\theta(Ax) = A\theta(x) = Ae$. Let $g \in G$. Then $A\theta(x)(\theta(g)) = Ae\theta(g)$, i.e., $A(\theta(g)\theta(x^{-1})) = A\theta(g)$. Thus $A\theta(gx^{-1}) = A\theta(g)$, i.e., $A\theta(gx^{-1}) = A\theta(g)$. Since $A\theta = A, A(gx^{-1}) = A(g)$. Then $Ax(g) = Ae(g)$. Thus $Ax = Ae$, i.e., $Ker\bar{\theta} = \{Ae\}$. So $\bar{\theta}$ is injective. Hence $\bar{\theta}$ is an automorphism of $G/A$. This completes the proof. 

Theorem 4.7. Let $A$ be an IVG of a finite group $G$ and let $x, y \in G$. Then $G_{Ax} = G_{Ay}$ if and only if $Ax = Ay$.

Proof. By Result 3.B and Lemma 3.12, $G_A$ is a subgroup of $G$ and $G_A = \{x \in G : Ax = Ae\}$.

$(\Rightarrow)$: Suppose $G_{Ax} = G_{Ay}$ for any $x, y \in G$. Then $xy^{-1} \in G_A$. Thus $Axy^{-1} = Ae$. Let $g \in G$. Then $Axy^{-1}(g) = Ae(g)$, i.e., $A(gyx^{-1}) = A(g)$. Replacing $g$ by $gy^{-1}$, which is also an arbitrary element of $G$, we get that $A(gx^{-1}) = A(gy^{-1})$ for each $y \in G$. Thus $Ax(g) = Ay(g)$ for each $y \in G$. So $Ax = Ay$.

$(\Leftarrow)$: Suppose $Ax = Ay$ for any $x, y \in G$ and let $g \in G$. Then $Ax(g) = Ay(g)$, i.e., $A(gx^{-1}) = A(gy^{-1})$. In particular, $A(gy^{-1}) = A(yx^{-1}) = A(e)$. Thus $yx^{-1} \in G_A$. So $G_{Ax} = G_{Ay}$. This completes the proof.

Remark 4.8. Proposition 4.6 shows that there is a one-to-one correspondence between the (right) cosets of $G_A$ in $G$ and the interval-valued fuzzy cosets of $A$, given by the mapping $x \leftrightarrow Ax$ for each $x \in G$. Hence we see that the subgroup $G_A$ plays a key role in the analysis of interval-valued fuzzy cosets.

References


