Discrimination of Out-of-Control Condition Using AIC in \((\bar{x}, s)\) Control Chart

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ABSTRACT
The \(\bar{x}\) control chart for the process mean and either the \(R\) or \(s\) control chart for the process dispersion have been used together to monitor the manufacturing processes. However, it has been pointed out that this procedure is flawed by a fault that makes it difficult to capture the behavior of process condition visually by considering the relationship between the shift in the process mean and the change in the process dispersion because the respective characteristics are monitored by an individual control chart in parallel. Then, the \((\bar{x}, s)\) control chart has been proposed to enable the process managers to monitor the changes in the process mean, process dispersion, or both. On the one hand, identifying which process parameters are responsible for out-of-control condition of process is one of the important issues in the process management. It is especially important in the \((\bar{x}, s)\) control chart where some parameters are monitored at a single plane. The previous literature has proposed the multiple decision method based on the statistical hypothesis tests to identify the parameters responsible for out-of-control condition. In this paper, we propose how to identify parameters responsible for out-of-control condition using the information criterion. Then, the effectiveness of proposed method is shown through some numerical experiments.

Keywords: \((\bar{x}, s)\) Control Chart, Kullback-Leibler Information, Information Criterion, Multiple Decision Method, Statistical Process Control

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1. INTRODUCTION
Control charts play an important role in statistical process control (Montgomery, 2005). Control charts are used to distinguish chance and assignable causes in the variability of quality characteristics. The process state is called in-control when the process only has variability by the chance cause, while the process state is called out-of-control when the process has variability by assignable cause. When a control chart signals that there is an assignable cause, process managers must initiate a search for the assignable cause of the process disturbance. At this moment, a clue to identifying the assignable cause may be required at first. In this case, the following information on the out-of-control condition will be useful: 1) change in process mean and no change in process dispersion, 2) no change in process mean and change in process dispersion, 3) both changes in process mean and dispersion. Consequently, identification of the out-of-control condition can simplify the search for the assignable causes, and then appropriate actions needed to improve the quality can be implemented sooner.
The process mean and dispersion are usually monitored on investigating the variability of quality characteristics. Therefore, the \( \bar{x} \) control chart for process mean and either the \( R \) or \( s \) control chart for process dispersion are used together for the purpose of monitoring the process condition. In general, the stability of process dispersion is checked on the \( R \) or \( s \) control chart and then the level of process mean is investigated on the \( \bar{x} \) control chart. However, it has been pointed out that this procedure is flawed by a fault which makes it difficult to capture the behavior of process condition visually by taking into consideration the relationship between the shift in the process mean and the change in the process dispersion, because respective characteristics are monitored by an individual control chart in parallel.

The \( (\bar{x}, s) \) control chart (Kanagawa et al., 1997) is a new type of control chart in which the state of process is represented by a succession of points \( (\bar{x}, s) \) on a rectangular coordinate graph. This chart enables the process managers to monitor the change in the process mean, process dispersion, or both. Various expansions of control chart have been published by Kobayashi et al. (2003), Takemoto et al. (2003), Takemoto and Arizono (2005, 2009). As mentioned already, identifying the out-of-control condition is one of the important issues in the process management. It is especially important in the \( (\bar{x}, s) \) control chart where some parameters are monitored at a single plane. In other words, how to identify which process parameters are responsible for out-of-control signals is essential to the process management. On the one hand, the difference between in-control and present states is evaluated by means of Kullback-Leibler (K-L) information (Kullback, 1959) in the \( (\bar{x}, s) \) control chart. Hence, the multiple decision method based on hypothesis tests has been proposed for the purpose of identifying the out-of-control condition by means of the decomposition of K-L information.

The information criterion (Akaike, 1974) is well known in the expansion of information theory. The information criterion is a criterion for model selection. In this study, we apply the information criterion, especially Akaike information criterion (AIC), to identifying the out-of-control condition. The difference between the multiple decision method and the AIC-based method is shown. Then, through some numerical examples, the effectiveness of the proposed method is compared with the previous method.

### 2. OUTLINE OF \( (\bar{x}, s) \) CONTROL CHART

In this section, we explain the outline of \( (\bar{x}, s) \) control chart presented by Kanagawa et al. (1997). The design procedure is shown in subsection 2.1, then the method of distinguishing the changes in process is illustrated in subsection 2.2.

#### 2.1 Design of \( (\bar{x}, s) \) Control Chart

Define a normal distribution \( N(\mu, \sigma^2) \) by the probability density function \( g(x) \), and a normal distribution \( N(\mu_0, \sigma_0^2) \) by the probability density function \( f(x) \), respectively. In this case, the K-L information \( I(g : f) \) given by

\[
I(g : f) = \int g(x) \log \frac{g(x)}{f(x)} dx = \frac{1}{2} \log \frac{\sigma^2}{\sigma_0^2} - 1 + \frac{\sigma^2}{\sigma_0^2} + \frac{(\mu - \mu_0)^2}{\sigma_0^2}
\]

(1)

is used as a measure of the distance from the distribution \( N(\mu_0, \sigma_0^2) \) to the distribution \( N(\mu, \sigma^2) \). Denote the observation of quality characteristics of items by \( x_i, i = 1, 2, \cdots, n \), where \( n \) is the size of samples. Then, assume that \( x_i \) follows the normal distribution \( N(\mu, \sigma^2) \), where \( \mu \) and \( \sigma^2 \) are unknown. When the distribution of ideal quality characteristics is defined by \( N(\mu_0, \sigma_0^2) \), the estimator \( \hat{I}(g : f) \) of \( I(g : f) \) based on the observation \( \{x_1, x_2, \cdots, x_n\} \) is

\[
n \hat{I}(g : f) = \frac{n}{2} \left( \log \frac{\sigma_0^2}{\sigma^2} - 1 + \frac{s^2}{\sigma_0^2} + \frac{(\bar{x} - \mu_0)^2}{\sigma_0^2} \right) = \lambda,
\]

(2)

where

\[
\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i,
\]

(3)

\[
s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2.
\]

(4)

Both estimators \( \bar{x} \) and \( s^2 \) are the maximum likelihood estimators (MLEs). The reason why the MLEs are used in the estimation of K-L information is shown in Arizono and Ohta (1987). In the \( (\bar{x}, s) \) control chart, the statistic in (2) is adopted with respect to the discrimination between in-control and out-of-control states of process.

Then, we explain the procedure for designing the control limit. The expectation and variance of statistic \( \lambda \) under the in-control condition are obtained in Kanagawa et al. (1997) as follows:

\[
E[\lambda] = \frac{n}{2} \phi^{(0)} \left( \frac{n-1}{2} \right) + \frac{n}{2} \log \frac{n}{2},
\]

(5)

\[
V[\lambda] = \frac{n^2}{4} \phi^{(0)} \left( \frac{n-1}{2} \right) - \frac{n}{2}.
\]

(6)

The function \( \phi^{(0)}(z) \) is the poly-gamma function defined by

\[
\phi^{(0)}(z) = \frac{d^{z+1}}{dz^{z+1}} \log \Gamma(z),
\]

(7)
where $\Gamma(z)$ is the gamma function (Abramowitz and Stegun, 1964). The control limit with the probability of the first kind of error $\alpha$ is given by

$$\lambda(\alpha) = \frac{V[\lambda]}{2E[\lambda]} \chi^2_\alpha(\alpha), \quad (8)$$

where $\chi^2_\alpha(\alpha)$ denotes the 100$\alpha$ percentile of chi-square distribution with $\phi$ degree of freedom:

$$\phi = \frac{2E[\lambda]}{V[\lambda]}, \quad (9)$$

This procedure for designing the control limit is based on Patnaik transformation (Patnaik, 1949) where the percentiles of non-central chi-square distribution are transformed into those of central chi-square distribution by using the expectation and variance. Therefore, the rule of discrimination between in-control and out-of-control states of process is designed as follows (Kangawa et al., 1997; Watakabe and Arizono, 1999):

$$\begin{cases} \text{if } \lambda \leq \lambda(\alpha), \text{ then in-control state,} \\ \text{otherwise, out-of-control state.} \end{cases} \quad (10)$$

### 2.2 Multiple Decision for Distinguishing Change in Process

From the property of K-L information, the statistic is divided as follows:

$$\lambda_1 = \frac{n-1}{2} \left\{ \log \frac{s^2}{\sigma_0} + 1 + \frac{n(\bar{x} - \mu_0)^2}{\sigma_0^2} \right\}, \quad (11)$$

$$\lambda_2 = \frac{n-1}{2} \left\{ \log \frac{s^2}{\sigma_0^2} + 1 + \frac{s^2}{\sigma_0^2} \right\} \quad (12)$$

The statistic $\lambda_1$ provides the K-L information on the process mean while the statistic $\lambda_2$ provides the K-L information on the process dispersion. This division is the decomposition of K-L information because of $\lambda = \lambda_1 + \lambda_2$ (Kullback, 1959).

The expectation and variance of statistic $\lambda_i$ are given as

$$E[\lambda_i] = \frac{1}{2} \phi^{(0)} \left( \frac{n-1}{2} \right) + \frac{1}{2} \log n - \frac{n-1}{2n}, \quad (13)$$

$$V[\lambda_i] = \frac{n^2}{4} \phi^{(0)} \left( \frac{n-1}{2} \right) - \frac{n+1}{2n^2} + \frac{1}{2} \quad (14)$$

Using the same technique as the design of control limit in (8), the following rule with respect to the change in process mean is designed:

$$\begin{cases} \text{if } \lambda_1 > \frac{V[\lambda_1]}{2E[\lambda_1]} \chi^2_\alpha(\alpha_1) = \lambda_1(\alpha_1), \\ \text{then changed in process mean,} \end{cases} \quad (15)$$

where $\alpha_1$ is the probabilistic level of criterion.

On one hand, the expectation and variance of statistic $\lambda_2$ are given as

$$E[\lambda_2] = -\frac{n-1}{2} \phi^{(0)} \left( \frac{n-1}{2} \right) + \frac{n-1}{2} \log n - \frac{n-1}{2n}, \quad (16)$$

$$V[\lambda_2] = \frac{n-1}{2} \phi^{(0)} \left( \frac{n-1}{2} \right) - \frac{(n-1)(n+1)}{2n^2} \quad (17)$$

Similarly, the following rule with respect to the change in process dispersion is designed:

$$\begin{cases} \text{if } \lambda_2 > \frac{V[\lambda_2]}{2E[\lambda_2]} \chi^2_\alpha(\alpha_2) = \lambda_2(\alpha_2), \\ \text{then changed in process dispersion,} \end{cases} \quad (18)$$

where $\alpha_2$ is also the probabilistic level of criterion.

The multiple decision method for distinguishing the changes in process is formulated by combining some hypothesis tests for respective parameters. Therefore, the control limit in (8) and the rules in (15) and (18) are applied to the discrimination of out-of-control condition.

Figure 1 shows the lines regarding the control limit in (8) and the multiple decisions in (15) and (18). In Figure 1, the in-control condition is $(\mu_0, \sigma^2_0) = (0.0, 1.0^2)$ and the size of samples is $n = 5$. The other parameters are as follows:

$$\alpha_1 = 1.00(\%), \alpha_2_1 = 1.00(\%), \alpha_2 = 1.00(\%)$$

The respective areas imply the following conditions:

A) in-control condition,
B) change in the process mean and no change in the process dispersion,
C) no change in the process mean and change in the process dispersion,
D) both changes in the process mean and dispersion,
E) slight change(s) in the process mean and/or dispersion.

![Figure 1. The multiple decision in the $(\tau, s)$ control chart.](image-url)
3. PROPOSAL OF IDENTIFYING OUT-OF-CONTROL CONDITION BY AIC

The information criterion (IC) is a tool of selecting the most appropriate model among some statistical models. The definition of IC is given as

\[ IC = -2 \times (\text{Maximum Log-likelihood}) + 2 \times \text{Bias} \]  \hspace{1cm} (19)

Especially, the AIC is a very famous IC. The AIC gives the number of unknown parameters to bias in (19). In this paper, we apply the AIC to the identification of out-of-control condition.

The statistical models of out-of-control condition are given as follows:

\[ f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\}, \]  \hspace{1cm} (20)

\[ f(x; \mu_0, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(x-\mu_0)^2}{2\sigma^2} \right\}, \]  \hspace{1cm} (21)

\[ f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\}, \]  \hspace{1cm} (22)

where \( \mu \) and \( \sigma^2 \) are unknown. The respective models correspond to the out-of-control condition with the change in process mean, change in process dispersion, and the change of both.

By calculating the maximum log-likelihood and adding the number of unknown parameters, the respective AICs are obtained as follows:

\[ \text{AIC}(\mu, \sigma^2) = n \log(2\pi\sigma^2) + \frac{n\bar{x}^2}{\sigma^2} + 2, \]  \hspace{1cm} (23)

\[ \text{AIC}(\mu_0, \sigma^2) = n \log(2\pi\sigma^2) + \frac{n(\bar{x} - \mu_0)^2}{\sigma^2} + n + 2, \]  \hspace{1cm} (24)

\[ \text{AIC}(\mu, \sigma^2) = n \log(2\pi\sigma^2) + n + 4. \]  \hspace{1cm} (25)

The statistical model with the smallest AIC among (23)–(25) is an appropriate model in the present situation. Therefore, the following rule with respect to the identification of out-of-control condition is designed:

\[
\begin{align*}
\text{if } & \text{AIC}(\mu, \sigma^2) < \text{AIC}(\mu_0, \sigma^2), \text{AIC}(\mu_0, \sigma^2), \\
& \text{then process mean is out-of-control,}
\end{align*}
\]

\[
\begin{align*}
\text{if } & \text{AIC}(\mu_0, \sigma^2) < \text{AIC}(\mu, \sigma^2), \text{AIC}(\mu, \sigma^2), \\
& \text{then process dispersion is out-of-control,}
\end{align*}
\]

\[
\begin{align*}
\text{if } & \text{AIC}(\mu, \sigma^2) < \text{AIC}(\mu_0, \sigma^2), \text{AIC}(\mu_0, \sigma^2), \\
& \text{then process mean and dispersion are out-of-control.}
\end{align*}
\]

Then, we can define the borders between out-of-control conditions by using (23)–(25). From the relationship of (23) and (24), (23) and (25), and (24) and (25), the following three borders are derived:

\[ \log \left\{ \frac{s^2 + (\bar{x} - \mu_0)^2}{\sigma^2_0} \right\} + 1 - \frac{s^2}{\sigma^2_n} = 0, \]  \hspace{1cm} (26)

\[ n \log \left( \frac{s^2}{\sigma^2_n} \right) + n - n\bar{x}^2 + 2 = 0. \]  \hspace{1cm} (28)

Figure 2 shows the respective borders in (26)–(28). Further, Figure 3 shows the \((\bar{x}, s)\) control chart with control limit and decision lines for out-of-control conditions. In Figures 2 and 3, the in-control condition is \((\mu_0, \sigma^2_0) = (0.0, 1.0^2)\), the size of samples is \(n = 5\) and the probability of the first kind of error is \(\alpha = 1.00\%\). Then, note that the proposed borders between out-of-control conditions do not require the probabilistic levels \(\alpha_1, \alpha_2\) with respect to the criterion of discrimination rule in Kana-gawa et al. (1997).

4. NUMERICAL SIMULATIONS

In this section, some numerical examples are shown. The case study in subsection 4.1 illustrates a series of procedures for the proposed method including the operation of \((\bar{x}, s)\) control chart. Then, the numerical comparison with the traditional multiple decision method is shown in subsection 4.2.
4.1 Case Study

A series of proposed procedures are explained using the sample data. The sample data set is quoted from Samuel et al. (1998). The quality characteristic is a piston ring outer diameter in a production process for forged piston rings. The detail of data is provided in Table 1. The in-control condition is assumed as \( \mu = 100.0 \) and \( \sigma = 24.0 \). Then, the size of samples is \( n = 4 \) and \( \alpha = 1.00\% \).

Figure 4 is the result where the sample data is plotted on the \((x, s)\) control chart.

Table 1. The outer diameter for forged piston rings

<table>
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<th>No.</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
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</table>

Figure 4. The behavior of samples in \((x, s)\) control chart.

4.2 Numerical Comparison

The comparison of the proposed and traditional methods is conducted using computer simulations. The condition of simulations is as follows:

- in-control condition: \((\mu, \sigma^2) = (0.0, 1.0^2)\),
- size of samples: \( n = 10 \),
- probabilistic level of discrimination: \( \alpha = 1.00\% \) and \( \alpha_i = 1.00\% \), \( i = 1, 2 \).

The steps of simulations are as follows:

(i) The quality characteristics of \( n \) samples are generated as random variables in a specified out-of-control condition.
(ii) Calculate \( x \) and \( s^2 \) using (3) and (4), respectively.
(iii) Calculate the statistic in (2) and compare its value with control limit in (8). When the process state is judged in-control, return step (i).
(iv) For the purpose of identifying the parameters responsible for out-of-control condition, the traditional and proposed methods are applied.
(v) The result of discrimination is noted.

These steps are iterated 1,000,000 times. Then, the rate of appropriate distinctions is shown in Table 2. The upper and lower parts in the respective cells indicate the results in the traditional and proposed methods, where \((\mu, \sigma^2) \) expresses the parameters in the specified out-of-control condition. For example, 45.79% is the rate of appropriate distinctions when the observations are plotted in the area C in Figure 1 in case of \((\mu, \sigma^2) = (0.0, 21.25^2)\). Also, 53.43% is the rate of appropriate distinctions when the observations are plotted in the area C in Figure 3 in case of \((\mu, \sigma^2) = (0.0, 1.25^2)\). That is, these values show the ratios of the appropriate out-of-control condition when definitely distinguished.

In the performance in Table 2, the following properties are broadly shown.

- The traditional method is totally effective in the change of process mean only.
- The proposed method is effective in the change of process dispersion, and the change of both.

These results are due to the difference of the shape of area with respect to the change in the process mean, in other words, egg shaped and V-shaped.

5. CONCLUSION

In this paper, we have considered the discrimination of out-of-control condition in the \((x, s)\) control chart which enables the process managers to monitor the changes in the process mean, process dispersion, or both. The traditional method based on the multiple decisions has been indicated, and then the proposed method based on the AIC has been explained. Through some numeri-
Table 2. The result of performance in the traditional and proposed methods

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