A Study on the Effective Algorithm by Fourier Transform for Numerical Conformal Mapping

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Abstract—Conformal mapping has been a familiar tool of science and engineering for generations. The methods of numerical mapping are usually classified into those which construct the map from standard domain such as the unit disk onto the ‘problem domain’, and those which construct the map in the reverse direction. We treat numerical conformal mapping from the unit disk onto the Jordan regions as the problem domain in this paper. The traditional standard methods of this type are based on Theodorsen integral equation. Wegmann’s method is well known as a Newton-like efficient one for solving Theodorsen equation. An improved method for convergence by applying low frequency pass filter to the Wegmann’s method was proposed. In this paper we propose an effective algorithm for numerical conformal mapping based on the improved method. This algorithm is able to determine the discrete numbers and initial values automatically in accordance with the given region and the required accuracy. This results come from analyzing the shape of given domain as seen in the Fourier Transform.

Index Terms—Conformal mapping, Theodorsen equation, Wegmann’s merhod, Fourier Transform, Discrete number

I. INTRODUCTION

This paper is to discuss the numerical conformal mapping from the unit disk onto the Jordan region, which can be solved by Theodorsen equation, a nonlinear integral equation for boundary function. A lot of studies have been suggested to solve Theodorsen equation, but Wegmann’s method has been known as the most effectively economical one in capacity for calculation an memory. This iterative method is based on a certain Riemann-Hilbert problem to sharply reduce the space on calculation and memory [1]. Wegmann’s method is well known as a Newton-like efficient on for solving Theodorsen equation. The non-discretized Wegmann’s method is quadratic convergent[2]-[3].

However, the discretized iteration method of Wegmann has not so wide of convergence ]. An improved method for convergence by applying a low frequency pass filter to the Wegmann’s method was proposed [4]-[5].

In this paper, we investigate error analysis and propose as effective algorithm based on the improved method in early study. The numerical conformal mapping can be automatically approximated by this algorithm according to once given the problem domain and the required accuracy. This results from analyzing the function, which may decide the shape of the given domain under the assumption that the degree of the problem depends of the transformation of a given domain, as seen in the Fourier Transform.

Results of some test calculations are reported.

II. WEGMANN’S ITERATIVE METHOD

We begin with defining functional space by some used in this paper.

t \in \mathbb{T} is used as a variable for 2\pi periodic function, where \mathbb{T} is the quotient space \mathbb{R}/2\pi\mathbb{Z} (R is real number and \mathbb{Z} is integer).

We define the below notations as follows:

\text{C}(\mathbb{T}) : 2\pi periodic complex continuous function space

\text{C}_p(\mathbb{T}) : 2\pi periodic real continuous function space

\text{C}^m(\mathbb{T}) (m \geq 1) : 2\pi periodic complex function space of m times differentiable

\text{C}_p^m(\mathbb{T}) (m \geq 1) : 2\pi periodic real continuous function space on m times differentiable

\text{A}(\mathbb{D}) : complex continuous function space being analytic in \mathbb{D} and continuous in \partial \mathbb{D} , where \mathbb{D} is the interior of a unit circle

\text{A}(\mathbb{D})_{\partial} : set of boundary functions \text{f}_t , \text{f}(t) = h(e^{it}) for \text{A}(\mathbb{D})

Let \Phi be a conformal map of the unit disk \mathbb{D} with boundary \gamma onto a given Jordan domain \Delta with boundary \Gamma. We assume \Phi is normalized by \Phi(0)=0 and \Phi'(0) > 0. Then the map \Phi can be extended to the closure \overline{\mathbb{D}} of \mathbb{D} inducing a conformal map \Phi: \gamma \rightarrow \Gamma. Therefore the problem of computing \Phi: \gamma \rightarrow \Gamma becomes to the problem of computing \text{s}(=s(t)) satisfying

\eta(s) \in \text{A}(\Delta)_{\partial} , \quad [\text{Im } \eta(s)]_0 = 0 \quad (1)

where [\text{Im } \eta(s)]_0 is 0 dimensional Fourier coefficient.
where \( f(t) \) is a boundary function, \( \text{Im} \ f(t) \) is \( f(t) \)'s dimensional Fourier coefficient.

We define \( K \) as the conjugate operator in the below definition,

\[
Ku(t) = \frac{a_0}{2} + \sum_{\mu=1}^{\infty} (a_\mu \cos \mu t + b_\mu \sin \mu t)
\]

Here is another theorem concerning boundary function and conjugate operator [1].

**[Theorem 1]**

\( f(t) \in A(\hat{D})|_t \leftrightarrow \text{Im} \ f(t) - \text{Im} \ f_0 = K \text{Re} \ f(t) \) ,

where \( f(t) \) is a boundary function, \( \text{Im} \ f(t) \) is \( f(t) \)'s imaginary part, \( \text{Re} \ f(t) \) is \( f(t) \)'s real part and \( f_0 \) is \( f \)'s 0 dimensional Fourier coefficient.

The normalized condition (1) makes \( \text{Im} \ \eta_0 = 0 \) possible. Using Theorem 1 we derive a boundary function \( \eta(s) \) as follows:

\[
\eta(s) \in A(\hat{D})|_t \leftrightarrow \text{Im} \ \eta(s) = K \text{Re} \ \eta(s) = 0 \ ,
\]

Formula (2) leads to the below equation (3) by which we can get \( s \) :

\[
\Psi(s) := \text{Im} \ \eta(s) - K \text{Re} \ \eta(s) = 0 \ .
\]

We call this formula (3) Theodorsen equation.

Among the various solutions for this equation, we discuss Wegmann’s method known as the most effective. Wegmann solved the original nonlinear equation (3) with a Newton’s method as like

\[
\Psi_{s_k}(t) + \Psi_{s_k} \delta_{s_k}(t) = 0 \quad s_{k+1}(t) = s_k(t) + \delta_k(t)
\]

\( \Psi_{s_k} \): differential of \( \Psi \) with respect to \( s_k \), \( k=1,2,\cdots \), \( \delta_k \): correction in the \( k \)th step.

Wegmann reduced calculation and memory space by rendering it a Riemann-Hilbert problem on the unit circle. The iteration scheme can be performed numerically in the following way.

Let \( n \) be a natural number as \( N=2n \). Choose equidistant points \( t_v = 2\pi v/N, v=0,1,2,\cdots,N-1 \) in the interval \( [0, 2\pi] \).

The conjugate function \( Ku \) defined by [Definition 1] is approximated by \( K_N u \) as follows [2]:

\[
K_N u(t) = \sum_{\mu=1}^{n-1} (a_\mu \sin \mu t - b_\mu \cos \mu t)
\]

If initial value \( s_0 \) is determined we can calculate an approximate value \( s_k \) \( (k=1,2,\cdots) \) with following formulas (5) to (10)

\[
\begin{align*}
\psi_k(t) &:= \text{Im}(\eta(s_k(t)) - K \text{Re}(\eta(s_k(t)))) \quad (5) \\
\alpha_k &:= \frac{\delta_k \cot \psi_k(t)}{2} \quad (8) \\
\beta_k(t) &:= \frac{\text{Re}(\eta(s_k(t)) - \text{Im}(\eta(s_k(t)))}{2} \quad (9) \\
\alpha_k + K \beta_k(t) &:= \psi_k(t) \quad (10)
\end{align*}
\]

where \( \eta'(s_k) \) is the differential of \( \eta \) with respect to \( s_k \).

### III. IMPLEMENTATION OF WEGMANN’S ITERATION

We define low frequency filter \( L_f \) as

\[
L_f(e^{imt}) = \begin{cases} e^{imt} : 0 \leq |m| \leq n - f \\ 0 : n - f < |m| \leq n \end{cases}
\]

where \( f \) is the parameter for excluding several high frequency from the back. By applying \( L_f \) to the approximate value (10) in terms of Wegmann’s iteration as follows:

\[
s_{k+1} = L_f(s_{k+1} - t) + t \quad (11)
\]

We can make a class of convergence more wide by use (11) to (10).

Let \( \Pi_R \) be a subspace of \( C_R(T) \) as like following trigonometric polynomials

\[
\Pi_R = \text{span}\{a_{\mu} \cos \mu t + b_{\mu} \sin \mu t + a_n \cos nt/2\}
\]

Then we can obtain the coefficients of \( s_{k+1} - t \) by FFT(Fast Fourier Transform) because \( s_{k+1} - t \in \Pi_R \).

The filter parameter \( f \) can be determined as follows.

Assume the boundary function of Jordan region (problem domain) is given by \( \eta(t) = (1 + \xi(t))e^{it} \), \( \xi(t) \in C_R^2(t) \).

Let \( \xi(t) \) and \( \xi(t)' \) can be represented Fourier series by

\[
\begin{align*}
\xi(t) &= \sum_{v=0}^{\infty} c_v e^{ivt} \\
\xi(t)' &= \sum_{v=0}^{\infty} d_v e^{ivt} = \sum_{v=0}^{\infty} v c_v e^{ivt}
\end{align*}
\]

Using \( c_v \) and \( d_v \) \( (v=1,2,\cdots) \) in (12) amounts such as \( D_0 \) and \( D_\mu \) are defined as

\[
\begin{align*}
D_0 &= |c_0| + 4 \sum_{v=1}^{\infty} |d_v| \\
D_\mu &= 2 |d_\mu| + 4 \sum_{k=\mu+1}^{\infty} |d_k|, \quad 1 \leq \mu \leq \infty
\end{align*}
\]
If we determine \( f \) satisfying \( Df < 1 \) then the iteration from (5) to (11) is convergent and numerical experiments for this improved method are reported in [5].

Let \( A_r \) be the space of analytic functions in \( G_r \) defined by \( G_r := \{ z : |\text{Im } z| < r \} \) \( (r > 0) \)

The norm on \( A_r \) is defined by \( \| f \| := \sup_{z \in \mathbb{C}} |f(z)| \). By Wegmann [6] and a formula \( \Phi_{k+1}(e^{it}) := -(\alpha_k + kv_k + \beta_k \cos n - i\lambda_k) e^{i|\lambda_k|}(it - w_k + i\lambda_k) \) we can obtain \( C_m \) satisfying

\[
\| \Phi_{k+1}(e^{it}) \| \leq C_m \| s \|
\]

where \( s_{k+1} \) is approximate value and \( s \) is real value. That means we can estimate errors by right hand of (14) even if we don't know a real value \( s \).

**IV. ALGORITHM**

Now we propose the automatic algorithm to obtain approximate value \( s_{k+1} \) for conformal mapping that can determine \( s_{k+1} \) according to the given problem domain and required accuracy. We usually get the approximate value which is much more close to the true value as the discrete number is increased. It is possible to derive automatic algorithm satisfying required accuracy because we can estimate error with (14) whether the discrete number has to be increased from the initial discrete number.

The detailed algorithm proposed in this paper is as follows:

- **Step1.** Obtain Fourier coefficient \( \xi' \) from the formula of (12) and decide the parameter of low frequency pass filter from the formula (13) by using of Fourier coefficient for \( \xi' \).

- **Step2.** Initial discrete number \( N \) is to be the number of the coefficient term required at Step1. Decide the proper initial value and the required accuracy \( \varepsilon \).

- **Step3.** Perform the iterations from (5) to (11) until \( \delta_k < \varepsilon \).

- **Step4.** Estimate the error of the approximate value which is obtained at Step3 by means of the right side of formula (14). If the estimate error is smaller than \( \varepsilon \), the iteration method is terminate. Otherwise go to the next step.

- **Step5.** The discrete number is to be double. That is, \( N \) is to be \( 2N \). Let the approximate value \( s_k \) which is obtained at the Step4 be discrete on \( 2N \) and be initial value and go to the Step3.

**Step1** means the parameter of low frequency filter is determined by that determine the boundary function of the given problem domain.

This proposed algorithm have some properties as follows:

1. The parameter of low frequency filter and the initial discrete number are automatically decided by the boundary function of the given problem domain.
2. The degree of difficulty for a given problem could be estimate according to the size of low frequency filter parameter.
3. We can get much more fast speed in this algorithm by means of approximate value obtained at the previous step as the initial value for the next iteration.
4. It is possible to decide automatically the discrete number with sufficient accuracy required.

**V. EXAMPLES**

We treat eccentric circle, which is a problem domain, as the example of the numerical experiment. We are able to estimate the correct errors because the real value of this example is known.

The eccentric circle has the following function on the boundary:

\[
\eta(s) = \rho(s)e^{i\theta} = \frac{R_0 + \sqrt{1 - R_0^2 \sin^2 \theta}}{R_0 + 1} 
\]

Real value:

\[
s(t) = \arctan \frac{R_s \sin t}{1 - R_s \cos t} + t
\]

This is an example of problem which is getting difficult as configuration parameter \( R \) is larger toward 1 because the transformation is more and more serious.

The meaning of signs which show the result of the experiments is as follows:

<table>
<thead>
<tr>
<th>k: iteration number</th>
<th>R : configuration parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>N : discrete number</td>
<td>s : real value of the iteration</td>
</tr>
<tr>
<td>( | \delta_k |<em>2 = | s_k - s</em>{k+1} |_2 ) : correction</td>
<td></td>
</tr>
</tbody>
</table>

Table 1 shows the results which we had experiments both with the method of low frequency pass filter (L) and with Wegmann’s method (W). From the Table 1 we know the corrections in according to iteration number and the errors when the iteration is convergent.

There we can see that Wegmann’s iteration has resulted in linear convergence with a configuration parameter \( R = 0.1 \) or divergence after a convergence at \( R = 0.5 \). It makes us not to solve the problem.

However, the iteration by low frequency pass filter (11) has demonstrated that quadratic convergence at \( R = 0.1 \) could make its convergence speed still faster, and that at \( R = 0.5 \) the iteration comes to be converges with stability enough for solution.
TABLE 1.
The Comparison of Corrections by Wegmann Method (W) and Low Frequency Pass Filter Method (L)

<table>
<thead>
<tr>
<th>R</th>
<th>N</th>
<th>k</th>
<th>W</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>32</td>
<td>1</td>
<td>0.40</td>
<td>0.40</td>
</tr>
<tr>
<td>0.1</td>
<td></td>
<td>2</td>
<td>0.10E-1</td>
<td>0.10E-1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>0.70E-5</td>
<td>0.67E-5</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>0.48E-6</td>
<td>0.37E-11</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>0.11E-6</td>
<td>0.11E-14</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6</td>
<td>0.26E-7</td>
<td>0.10E-14</td>
</tr>
<tr>
<td></td>
<td></td>
<td>7</td>
<td>0.62E-8</td>
<td>0.95E-15</td>
</tr>
<tr>
<td></td>
<td></td>
<td>8</td>
<td>0.15E-8</td>
<td>0.21E-14</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9</td>
<td>0.34E-9</td>
<td>0.10E-14</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10</td>
<td>0.81E-10</td>
<td>0.11E-14</td>
</tr>
</tbody>
</table>

0.5 64 1 0.28E+1 0.28E+1
    2 0.40
    3 0.13E-1
    4 0.68E-3
    5 0.15E-2
    6 0.32E-2
    7 0.70E-2
    8 0.15E-1
    9 0.33E-1
   10 0.72E-1
   11 0.49E-1
   12 0.65E-10
   13 0.323E-14
   14 0.402E-13

Next, let us fix notations of numerical estimate error E and real error ER in Table 2 as follows:

\[ ER := \max_{\nu=0,1,\ldots, N-1} |s(t_\nu) - s_{k+1}(t_\nu)| \]
\[ E := \max_{\nu=0,1,\ldots, N-1} |\phi_{k+1}(t_\nu) - \phi_{k+1}(t_\nu)| \]

\[ t_\nu = 2\pi\nu/N \]

Table 2 shows results which we had experiments with the proposed algorithm from Step 1 to Step 5. In Table 2, we show the initial discrete number is fixed with N=64 when R=0.6 is given and it is automatically increased into N=128 so that approximate value can be obtained satisfying required accuracy. The number of iteration k has been reduced steeply by use of the approximate value which is obtained from the previous discrete number as initial value in next iteration step.

TABLE 2.
The Result of Iteration by Automatic Algorithm at R=0.6

<table>
<thead>
<tr>
<th>N</th>
<th>k</th>
<th>ER</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>1</td>
<td>0.649E+0</td>
<td>0.185E+0</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.251E-1</td>
<td>0.240E-1</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.365E-3</td>
<td>0.444E-3</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.110E-5</td>
<td>0.492E-5</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.427E-7</td>
<td>0.191E-6</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.944E-8</td>
<td>0.108E-7</td>
</tr>
<tr>
<td>128</td>
<td>1</td>
<td>0.323E-14</td>
<td>0.402E-13</td>
</tr>
</tbody>
</table>

VI. CONCLUSIONS

A conformal mapping from the unit disk onto a Jordan region is determined to solve the Theodorsen equation which is as integral equation for the boundary correspondence function. Wegmann’s method has been well known as the efficient one in many methods for the Theodorsen equation. However, as a result of numerical experiments by Wegmann’s method, it is found that divergence occurs in some problems.

We found that the convergence is not resulted due to the rate of magnification of high frequency factors during the process of the iteration. Based on this fact we proposed as improved method by low frequency pass filter applied to the Wegmann’s method. This improved method makes convergence fast and has high accuracy even in the case of problems for slow convergence or divergence by Wegmann’s method.

In this paper, we proposed an effective algorithm for numerical conformal mapping based on the improved method. The algorithm proposed here is able to determine the discrete numbers and initial value automatically accordance with the given region and required accuracy.

REFERENCES

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