Abstract— This paper proposes a novel linearization method for Takagi–sugeno (TS) fuzzy model. A T-S fuzzy controller consists of linear controllers based on local linear models and the local linear controllers cannot be designed independently because of overall stability conditions which are usually conservative. To use linear control theories easily for T-S fuzzy system, the linearization of T-S fuzzy model is required. However, The linearization of T-S fuzzy model is difficult to be achieved by using existing linearization methods because fuzzy rules and membership functions are included in T-S fuzzy models. So, a new linearization method is proposed for the T-S fuzzy system based on the idea of T-S fuzzy state transformation. For the T-S fuzzy system linearized with uncertainties, a robust optimal controller with the robustness of sliding model control(SMC) is designed.

Index Terms— T-S fuzzy control, linearization, robust optimal, sliding mode control.

I. INTRODUCTION

T-S fuzzy model is based on using a set of fuzzy rules to describe a global nonlinear system in terms of a set of local linear models which are smoothly connected by fuzzy membership functions. A T-S fuzzy controller is also described by using a set of local linear controllers. A global controller is constructed from the local controllers in such a way that global stability with various performance indexes of the closed-loop fuzzy control system is guaranteed. The major techniques that have been used include quadratic stabilization, linear matrix inequalities (LMIs), Lyapunov stability theory, bilinear matrix inequalities, and so on [1]-[4]. Definitely, T-S fuzzy models provide a basis for development of systematic approaches to stability analysis and controller design of fuzzy control systems in view of powerful conventional control theory and techniques.

However, the requirement of stability condition is conservative for the designing of a T-S fuzzy controller. So, a great deal of attention is focused on the stability analysis of T-S fuzzy system. The requirement of stability conditions and their conservatism makes it difficult to use the linear conventional control techniques for the T-S fuzzy model. This difficulty is more severe in the case of existence of uncertainties and time delay in the T-S fuzzy system [6]-[8]. These difficulties are not shown in control of linear systems. Once the T-S fuzzy models have been linearized, then the conventional linear controllers can be used without such difficulties.

Therefore, the purpose of this paper is to linearize T-S fuzzy model and makes direct use of linear control theories possible. To the best of our knowledge, there is no linearization technique applicable for T-S fuzzy model. The conventional linearization methods can not be used for T-S fuzzy model because fuzzy rules and membership functions are included in the T-S fuzzy model. So, this paper proposes new linearization technique for T-S fuzzy model by using T-S fuzzy state transformation. A T-S fuzzy model is transformed into linearizable form and then fuzzy feedback changes it into a linear controllable canonical form.

The linearizable condition of the proposed linearization technique is just controllability of linear local models which is easy to be checked and guaranteed. In the conventional linearization techniques, checking the size of the class of linearizable systems is considered as an open problem [9]-[11]. Therefore, The result of this paper can be considered as a new approximate linearization which has easier linearizable conditions for nonlinear systems. In this paper, the T-S fuzzy system with uncertainties is also considered. The SMC and optimal control are applied for the linearized T-S fuzzy system.

The important property of the sliding mode control is that the dynamics of overall system are determined by the sliding surface. The SMC input push the state onto the sliding surface and states can have the desirable dynamics of the surface in spite of uncertainties. Therefore, the system can be robust to parameter uncertainties and disturbances [12]-[15]. In this paper, a robust optimal controller is designed for T-S fuzzy model by using the proposed linearization and SMC.

The rest of this paper is organized as follows. In Section 2, T-S fuzzy system and controllers are presented and problem is formulated. In Section 3, the proposed linearization method is presented. In Section 4, robust optimal controller is designed for uncertain T-S model based on the proposed linearization and SMC. In section 5, numerical examples are given to illustrate the
achievement of the proposed linearization. In Section 6, a conclusion is drawn.

II. PROBLEM FORMULATION

The T-S fuzzy model been proposed by Takagi and Sugeno [1] to represent the local linear dynamic relations of nonlinear systems. The local linear model is described by fuzzy If-Then rules and will be employed to deal with the control design problem of nonlinear systems. The ith rule of the fuzzy model is of the following form:

**Plant Rule i:**

If \( w_i(t) \) is \( F_{i1} \) and \( \cdots \) and \( w_g(t) \) is \( F_{ig} \)

Then \( \dot{x}(t) = A_i x(t) + B_i u(t) \) \hspace{1cm} (1)

for \( i = 1, 2, \cdots, L \)

where \( F_{ig} \) is the fuzzy set, \( A_i \in \mathbb{R}^{m \times m}, \) \( B_i \in \mathbb{R}^{m \times n}, \) \( L \) is the number of If-Then rules, and \( w_i(t), w_2(t), \cdots, w_g(t) \) are premise variables.

By using a standard fuzzy inference method, that is, using a singleton fuzzifier, product fuzzy inference, and center-average defuzzifier, the T-S fuzzy model in (1) can be rewritten as

\[
\dot{x}(t) = \frac{\sum_{i=1}^{L} \mu_i(w(t)) \{A_i x(t) + B_i u(t)\}}{\sum_{i=1}^{L} \mu_i(w(t))}
\]

\[
= \sum_{i=1}^{L} \mu_i(w(t)) \{A_i x(t) + B_i u(t)\}
\]

where \( w(t) = [w_i(t), w_2(t), \cdots, w_g(t)] \), \( F_{y}(w(t)) \) is the grade of membership of \( w_i(t) \) in \( F_y \) and

\[
\mu_i(w(t)) = \prod_{j=1}^{g} F_{y}(w_j(t))
\]

In this paper, it is assumed that \( \mu_i(w(t)) \geq 0 \) for \( i = 1, 2, \cdots, L \) and \( \sum_{i=1}^{L} \mu_i(w(t)) > 0 \) for all \( t \). Therefore, we get \( h_i(w(t)) \geq 0 \) for \( i = 1, 2, \cdots, L \) and

\[
\sum_{i=1}^{L} h_i(w(t)) = 1.
\]

Classical T-S controller has the following form

\[
u(t) = \sum_{j=1}^{L} h_j(w(t)) u_j(t)
\]

Overall T-S fuzzy system is written as follows.

\[
\dot{x}(t) = \sum_{i=1}^{L} h_i(w(t)) \{A_i x(t) + B_i u(t)\}
\]

(5)

Each local linear controller must be designed based on the stability conditions of the above system. Typical stability conditions are found in [1]-[3]. The conditions are more conservative in the case of the system with uncertainties and time delays[7][8]. This makes it difficult to use linear control theories for T-S fuzzy system.

The best way of using linear control theories freely in the control of the T-S fuzzy system is linearization of them. Therefore, the problem of this paper is formulated as linearization of T-S fuzzy system into the following controllable form.

\[
\dot{z}(t) = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1 \\
\end{bmatrix}
+ \begin{bmatrix}
u(t) \\
\end{bmatrix}
\]

(6)

III. PROPOSED T-S FUZZY FEEDBACK LINEARIZATION

In this section, a new linearization technique is proposed for T-S fuzzy model under the following assumption:

**Assumption 1.** All of the linear models are controllable. i.e., the following condition is satisfied:

\[
\text{rank}\begin{bmatrix}
B_{i1} & A_{i1} B_{i} & \ldots & A_{i1}^{n-1} B_{i}
\end{bmatrix} = n
\]

(7)

This is a very common condition for most control systems.

Under the above assumption, the following i-th rules of the fuzzy model are proposed to deal with the coordinate change for linearization.

**Plant rule i:**

If \( w_i(t) \) is \( F_{i1} \) and \( \cdots \) and \( w_g(t) \) is \( F_{ig} \)

Then \( \dot{x}(t) = A_i x(t) + B_i u(t) \)

\[
z(t) = T_i x(t)
\]

(8)

and \( \dot{z}(t) = A_{ci} z(t) + B_{ci} u(t) \)

(9)

for \( i = 1, 2, \cdots, L \)

where \( A_{ci} = T_i A_i T_i^{-1}, B_{ci} = T_i B_i \).

After coordinate change, the overall fuzzy system is inferred from (9) as follows:

\[
\dot{z}(t) = \sum_{i=1}^{L} h_i(w(t)) \{A_{ci} z(t) + B_{ci} u(t)\}
\]

(10)
By using product fuzzy inference and center-average defuzzifier, the overall z(t) can be rewritten as

\[ z(t) = \sum_{i=1}^{L} h_i(w(t))T_i x(t) \quad (11) \]

which is considered as the summation of the states x(t) through transformation \( T_i \) with the proportion of \( h_i \).

The above system can be a linearizable form by determining \( T_i \) as follows.

Under the assumption 1, the following \( T_i \) is obtained as follows:

\[
T_i = \begin{bmatrix} B_{c_i} & A_{c_i} B_{c_i} & \cdots & A_{c_i} B_{c_i}^{n-1} \end{bmatrix} \begin{bmatrix} B_i & A_i B_i & \cdots & A_i B_i^{n-1} \end{bmatrix}^{-1}
\]

where the parameters in \( A_{c_i} \) are obtained from the following characteristic equation of the i-th local linear model:

\[
|sI - A_{c_i}| = s^n - a_{i_0}s^{n-1} - \cdots - a_{i_2}s - a_{i_1} = 0 \quad (13)
\]

Then the above state transformation \( T_i \) changes the i-th linear system into the following controllable canonical form:

\[
\dot{z}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ a_{i_1} & a_{i_2} & a_{i_3} & \cdots & a_{i_m} \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \quad (14)
\]

The overall fuzzy system with local linear models (14) is now linearizable T-S fuzzy system.

\[
\dot{z}(t) = \sum_{i=1}^{L} h_i(w(t))\{A_{c_i}z(t) + B_{c_i}u(t)\}
\]

\[
= \sum_{i=1}^{L} h_i z_2 + \sum_{i=1}^{L} h_i z_3 + \cdots + \sum_{i=1}^{L} h_i a_i z(t)
\]

\[
= \sum_{i=1}^{L} h_i(2) + \sum_{i=1}^{L} h_i(3) + \cdots + \sum_{i=1}^{L} h_i a_i z(t)
\]

where \( a_i = [a_{i_1} \ a_{i_2} \ \cdots \ a_{i_m}] \) and remind that \( \sum_{i=1}^{L} h_i(w(t)) = 1 \) from (3).

The following theorem presents a controller which linearizes T-S fuzzy system.

**Theorem 1.** The following nonlinear feedback input linearizes the T-S fuzzy system into the controllable canonical form (6):

\[
u(t) = -\sum_{i=1}^{L} h_i(w(t))a_i z(t) + v(t) \quad (16)
\]

where the overall state \( z(t) \) is inferred in (1). The proof is obvious from (15).

The Fig. 1 shows the overall description for the proposed linearization method.

**IV. ROBUST OPTIMAL CONTROL WITH SMC AND LINEARIZATION**

In this section, the robustness of SMC is added to the linearized T-S fuzzy system. Consider T-S fuzzy system with uncertainties.

\[
\dot{x}(t) = \sum_{i=1}^{L} h_i(w(t))\{A_i x(t) + B_i u(t)\} + h(t) \quad (17)
\]

where \( h(t) \) is lumped uncertainties including parameter uncertainties and disturbances and bounded by

\[
\|w(t)\| \leq w_{max} \quad (18)
\]

satisfying the following matching condition.

\[
w(t) = B_i w_i(t) \quad \text{for} \quad i = 1, \cdots, L \quad (19)
\]

In order to design robust optimal controller with SMC, the sliding surface which has the dynamics of optimal controlled system must be used. Such a sliding surface is designed as follows.

First, the virtual state is defined based on the following controllable canonical form of the nominal system:

\[
\begin{bmatrix} \dot{z}_{c_1}(t) \\ \dot{z}_{c_2}(t) \\ \vdots \\ \dot{z}_{c_{n-1}}(t) \\ \dot{z}_c(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} z_{c_1}(t) \\ z_{c_2}(t) \\ \vdots \\ z_{c_{n-1}}(t) \\ z_c(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} v(t) \quad (20)
\]
In this paper, the control objective of $v_o(t)$ is to minimize the following cost function.

$$ J = \int_0^1 x^T(t)Qx(t) + r^2v_o(t)^2 dt $$ (21)

The optimal input $v_o(t)$ is determined as the following form:

$$ v_o(t) = -\frac{1}{r}B^T S z_o(t) = -K z_o(t) $$ (22)

where $K = [k_1 \ldots k_n]$ and $S$ is the solution of the following Riccati equation.

$$ -SA - A^T S - Q + \frac{1}{r}SB^T BS = 0 $$ (23)

The virtual state is defined as follows:

$$ z_v(t) = -k_1z_1(t) - k_2z_2(t) - \cdots - k_n z_n(t) $$ (24)

Sliding surface is defined as

$$ S(z, z_v) = z_v(t) + Kz(t) $$

$$ = z_v(t) + k_1z_1(t) + k_2z_2(t) + \cdots + k_n z_n(t) $$ (25)

If the state of the system (17) is on the sliding surface (25), the state has the dynamic of the nominal system (20) [14]. To guarantee the state on the sliding surface, the following hitting condition must be satisfied:

$$ S(x)S(x) < 0 $$ (26)

The above condition is satisfied by the discontinuous control input presented in Theorem 2.

**Theorem 2.** The SMC system with the following control input has the dynamics of the nominal optimal control system.

$$ v(t) = (KB)^{-1}(k_n z_v(t) - KBw(t))_{\max} \cdot \text{sgn}(s) $$ (27)

**Proof** See the proof of theorem 2 in [14].

As mentioned in theorem 2 and sliding mode control theory, the state $x(t)$ follows the trajectory of the nominal system controlled by sliding mode control input $v(t)$ which is designed to put the states of the system onto the sliding surface.

The following initial virtual state makes the initial value of $s(z, z_v)$ equal to zero without reaching phase.

$$ z_v(t_0) = -k_1z_1(t_0) - k_2z_2(t_0) - \cdots - k_n z_n(t_0) $$ (28)

### V. NUMERICAL EXAMPLE AND SIMULATION RESULTS

To show the robustness of the proposed robust optimal controller and its performance, consider an example of a dc motor controlling an inverted pendulum via a gear train [4].

$$ \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ 9.8\sin x_1(t) + x_3(t) \\ -10x_2(t) - 10x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} h(t) $$ (29)

where $w(t)$ is a pulse train with the size of 0.5.

The above system can be described as follows:

$$ \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 9.8\sin x_1(t) \\ x_1(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} h(t) $$ (30)

The local linear models in the T-S fuzzy model are as follows:

$$ A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 9.8 & 1 & 0 \\ -10 & -10 & 10 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -10 & 10 \end{bmatrix}, B_1 = B_2 = 0 $$ (31)

The angle of the pendulum is $x_1(t)$, $x_3(t)$ is the current of the motor. The $h_1(t)$ and $h_2(t)$ are fuzzy sets defined as

$$ h_1(x_1(t)) = \begin{cases} \frac{\sin(x_1(t))}{x_1}, & x_1(t) \neq 0 \\ 1, & x_1(t) = 0 \end{cases} $$

$$ h_2(x_1(t)) = 1 - h_1(x_1(t)) $$ (32)

The membership functions are shown in Fig. 2.

![Membership function](image)

**Fig. 2. Membership function**

This fuzzy model exactly represents the dynamics of the nonlinear mechanical system under $-\pi \leq x_1(t) \leq \pi$. 
The state transformation matrices is obtained as

\[
T_1 = \begin{bmatrix}
0.1 & 0 & 0 \\
0 & 0.1 & 0 \\
0.98 & 0 & 0.1
\end{bmatrix}
\quad \text{and} \quad
T_2 = \begin{bmatrix}
0.1 & 0 & 0 \\
0 & 0.1 & 0 \\
0 & 0 & 0.1
\end{bmatrix}
\] (33)

The controllable canonical forms are

\[
A_{c1} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-0.2 & -10 & 1
\end{bmatrix}, \quad A_{c2} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-10 & -10 & 1
\end{bmatrix}
\] (34)

From the above, the following vectors are identified:

\[
a_1 = [98 \ -0.2 \ -10] \quad \text{and} \quad a_2 = [0 \ -10 \ -10]
\] (35)

The following optimal controller for the linearized system is used.

\[
v_o = [-3.1623 \ 4.6054 \ 3.1954] z
\] (36)

The virtual state is defined as follows.

\[
\dot{x}_v = -3.1623z_2 - 4.6054z_3 - 3.1954z_v
\] (37)

Sliding surface is given by

\[
s(z, z_v) = z_v + z_1 + 2z_2 + 3z_3
\] (38)

By differentiating (38), the following is obtained.

\[
\dot{s}(z, z_v) = -k_3z_v + k_3v + k_3h
\] (39)

The following novel sliding mode control input is obtained from the hitting condition.

\[
v = \frac{1}{k_3} [k_3z_v - k_3h_{\max} \cdot \text{sgn}(s)]
\] (40)

Therefore the following initial virtual state can be obtained.

\[
z_v(t_0) = -z_1(t_0) - 2z_2(t_0) - 3z_3(t_0)
\] (41)

As perfect linearization of the nonlinear system results in the Brunovsky canonical form, to verify the validity of the linearization, the responses of the system linearized by the proposed method is compared to the case of the Brunovsky canonical system.

The simulation results are shown in figures from Fig. 3 to Fig. 7.
For the T-S fuzzy system with uncertainty, the results of SMC optimal control based on the proposed linearization and the optimal control results for the uncertain T-S fuzzy system are compared to the response of the optimal control for the ideal Brunovsky canonical system with optimal controller. The proposed control case is almost same with the case of the optimal control with ideal Brunovsky canonical system. This means that the T-S fuzzy model has been linearized with sufficient exactness and the proposed robust optimal controller is robust and gives optimal performance. However, the optimal control reveals its weakness for uncertainties.

VI. CONCLUSIONS

Novel linearization of T-S fuzzy model is proposed by using the T-S fuzzy state transformation and fuzzy feedback and consequently, linear control theories can be used easily for T-S fuzzy systems. For the linearized system, robust optimal controller with SMC is designed and gives good performance in the presence of uncertainties. The linearization of T-S fuzzy model can be considered as a approximate linearization of a nonlinear system approximated by T-S fuzzy system.

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