ON A CERTAIN CLASS OF $p$-VALENT UNIFORMLY CONVEX FUNCTIONS USING DIFFERENTIAL OPERATOR

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Abstract. In this paper, using differential operator, we have introduce new class of $p$-valent uniformly convex functions in the unit disc $U = \{z : |z| < 1\}$ and obtain the coefficient bounds, extreme bounds and radius of starlikeness for the functions belonging to this generalized class. Furthermore, partial sums $f_k(z)$ of functions $f(z)$ in the class $S^*(\lambda, \alpha, \beta)$ are considered. The various results obtained in this paper are sharp.

1. Introduction

Let $S$ denote the class of functions of the form

\[
f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad a_n \geq 0, \quad (p \in \mathbb{N})
\]

which are analytic and $p$-valent in the unit disc $U = \{z : |z| < 1\}$. Also $S^*$ denote the subclass of $S$ consisting of functions of the form

\[
f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n, \quad a_n \geq 0, \quad (p \in \mathbb{N}).
\]

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G. Murugusundaramoorthy([1]), Goodman([3], [4]) and Ronning([5], [6]) have studied the following subclasses

i) A function \( f(z) \in S \) is said to be in the class \( S_p(\alpha, \beta) \) of uniformly \( \beta \)-starlike function if it satisfies the condition

\[
\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} - \alpha \right\} > \beta \left| \frac{zf'(z)}{f(z)} - 1 \right| \quad (z \in U),
\]

where \(-1 < \alpha \leq 1, \beta \geq 0\) and \( p \in \mathbb{N} \).

ii) A function \( f(z) \in S \) is said to be in the class \( UCV(\alpha, \beta) \) of uniformly \( \beta \)-convex function if it satisfies the condition

\[
\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} - \alpha \right\} > \beta \left| \frac{zf'(z)}{f'(z)} \right| \quad (z \in U),
\]

where \(-1 < \alpha \leq 1, \beta \geq 0\) and \( p \in \mathbb{N} \). It follows from (1.3) and (1.4) that \( f(z) \in UCV(\alpha, \beta) \) is equivalent to

\[
zf'(z) \in S_p(\alpha, \beta).
\]

For the function \( f(z) \in S \) is given by (1.1) and \( g(z) \in S \) is given by,

\[
g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n.
\]

We define the Hadamard product (convolution) of \( f(z) \) and \( g(z) \) given by,

\[
(f \ast g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n \quad (z \in U, \ p \in \mathbb{N}).
\]

For the function \( f(z) \in S \), we define the following,

\[
I^0 f(z) = f(z), \quad I^1 f(z) = zf'(z) + \frac{1+p}{z^p}
\]

and for \( k = 2, 3, \ldots \)

\[
I^k f(z) = z(I^{k-1} f(z))' + \frac{1+p}{z^p} = z^p + \sum_{n=p+1}^{\infty} n(k) a_n z^n \quad (p \in \mathbb{N}),
\]
where $I^k$ is called as differential operator. Ghanim and Darus([2]) have studied this operator extensively.

Let $S^*(\alpha, \beta)$ be the subclass of $S$ consisting of the functions of the form (1.1) and satisfying the condition

\begin{equation}
\left| \frac{z(I^k f(z))'}{I^k f(z)} - p - \frac{\beta z(I^k f(z))'}{I^k f(z)} - \alpha p \right| < \mu \quad (z \in U, \ p \in N),
\end{equation}

where $-1 \leq \alpha < \beta \leq 1$ and $0 < \mu \leq 1$. Also let $S^{**}(\alpha, \beta) = S^*(\alpha, \beta) \cap S^*$.

The main object of this paper is to study the coefficient estimates, extreme points and radius of starlikeness for the functions belonging to the generalized class $S^{**}(\alpha, \beta)$. Furthermore, partial sums $f_k(z)$ of functions $f(z)$ in the class $S^*(\alpha, \beta)$ are considered and sharp lower bounds for the ratios of real part of $f(z)$ to $f_k(z)$ and $f'(z)$ to $f_k'(z)$ are determined.

In this paper, all the investigated results are motivated by Ronning([5], [6]), K. G. Subramanian ([10], [11]).

### 2. Basic properties

In this we obtain a necessary and sufficient condition for functions $f(z)$ in the classes $S^*(\alpha, \beta)$ and $S^{**}(\alpha, \beta)$.

**Theorem 2.1.** A function $f(z)$ of the form (1.1) is in $S^*(\alpha, \beta)$ if

\begin{equation}
\sum_{n=p+1}^{\infty} \left[ [(n-p) + \mu(n\beta - \alpha p)]n(k)|a_n| \leq \mu p(\beta - \alpha),
\end{equation}

where $-1 \leq \alpha < \beta \leq 1$, $0 < \mu \leq 1$ and $p \in N$.

**proof.** Since $f(z) \in S^*(\alpha, \beta)$, it is sufficient to show that

\begin{equation}
\left| \frac{z(I^k f(z))'}{I^k f(z)} - p - \frac{\beta z(I^k f(z))'}{I^k f(z)} - \alpha p \right| < \mu.
\end{equation}
We have

\[
\left| \frac{(I^k f(z))'}{I^k f(z)} - p \right| = \frac{p z^p + \sum_{n=p+1}^{\infty} n(k)a_n z^n - p}{z^p + \sum_{n=p+1}^{\infty} n(k)a_n z^n} - p \beta \left( \frac{p z^p + \sum_{n=p+1}^{\infty} n(k)a_n z^n}{z^p + \sum_{n=p+1}^{\infty} n(k)a_n z^n} \right) - \alpha p
\]

\[
\leq \frac{p z^p + \sum_{n=p+1}^{\infty} n(k)a_n z^n - p z^p - p \sum_{n=p+1}^{\infty} n(k)a_n z^n}{\beta \left( p z^p + \sum_{n=p+1}^{\infty} n(k)a_n z^n \right) - \alpha p [z^p + \sum_{n=p+1}^{\infty} n(k)a_n z^n]} \leq \mu.
\]

Allowing value of \( z \) tends to 1 minus on the real axis, we get

\[
\sum_{n=p+1}^{\infty} [(n - p) + \mu(n - \alpha)] n(k) |a_n| \leq \mu p(\beta - \alpha).
\]

\[\square\]

**Theorem 2.2.** A necessary and sufficient condition for \( f(z) \) of the form (1.2) to be in the \( S^{**}(\alpha, \beta) \)

\[
(2.2) \sum_{n=p+1}^{\infty} [(n - p) + \mu(n - \alpha)] n(k) |a_n| \leq \mu p(\beta - \alpha),
\]

where \(-1 \leq \alpha < \beta \leq 1, 0 < \mu \leq 1 \) and \( p \in N \).

**Proof.** In view of theorem (2.1), we need only to prove that necessity. If \( f(z) \in S^{**}(\alpha, \beta) \) and \( z \) is real then

\[
\left| \frac{(I^k f(z))'}{I^k f(z)} - p \right| = \frac{p z^p + \sum_{n=p+1}^{\infty} n(k)a_n z^n - p}{z^p + \sum_{n=p+1}^{\infty} n(k)a_n z^n} - p \beta \left( \frac{p z^p + \sum_{n=p+1}^{\infty} n(k)a_n z^n}{z^p + \sum_{n=p+1}^{\infty} n(k)a_n z^n} \right) - \alpha p
\]

We have

\[
\left| \frac{(I^k f(z))'}{I^k f(z)} - p \right| = \frac{p z^p + \sum_{n=p+1}^{\infty} n(k)a_n z^n - p}{z^p + \sum_{n=p+1}^{\infty} n(k)a_n z^n} - p \beta \left( \frac{p z^p + \sum_{n=p+1}^{\infty} n(k)a_n z^n}{z^p + \sum_{n=p+1}^{\infty} n(k)a_n z^n} \right) - \alpha p
\]

\[
\leq \frac{p z^p + \sum_{n=p+1}^{\infty} n(k)a_n z^n - p z^p - p \sum_{n=p+1}^{\infty} n(k)a_n z^n}{\beta \left( p z^p + \sum_{n=p+1}^{\infty} n(k)a_n z^n \right) - \alpha p [z^p + \sum_{n=p+1}^{\infty} n(k)a_n z^n]} \leq \mu.
\]
The above expression is bounded by \( \mu \) then we obtain the inequality
\[
\sum_{n=p+1}^{\infty} [(n-p) + \mu(n\beta - \alpha p)]n(k)|a_n| \leq \mu p(\beta - \alpha),
\]
where \(-1 \leq \alpha < \beta \leq 1, 0 < \mu \leq 1 \) and \( p \in \mathbb{N} \).

In the following theorem we show that the class \( S^{**}(\alpha, \beta) \) is closed under convex linear combination.

**Theorem 2.3.** Let \( f(z) \) defined by (1.2) and \( g(z) = z^p - \sum_{n=p+1}^{\infty} b_n z^n \) be in the class \( S^{**}(\alpha, \beta) \). Then the function
\[
h(z) = (1 - \xi) f(z) + \xi g(z) = z^p - \sum_{n=p+1}^{\infty} \eta_n z^n
\]
is also in the class \( S^{**}(\alpha, \beta) \), where \( \eta_n = (1 - \xi)a_n + \xi b_n, \ 0 \leq \xi < 1 \).

**Proof.** Since the function \( f(z) \) and \( g(z) \) belongs to \( S^{**}(\alpha, \beta) \), we have
\[
\sum_{n=p+1}^{\infty} [(n-p) + \mu(n\beta - \alpha p)]n(k)|a_n| \leq \mu p(\beta - \alpha) \tag{2.3}
\]
and
\[
\sum_{n=p+1}^{\infty} [(n-p) + \mu(n\beta - \alpha p)]n(k)|b_n| \leq \mu p(\beta - \alpha) \tag{2.4}
\]
Clearly,
\[
h(z) = (1 - \xi) f(z) + \xi g(z)
\]
\[
= (1 - \xi) \left( z^p - \sum_{n=p+1}^{\infty} a_n z^n \right) + \xi \left( z^p - \sum_{n=p+1}^{\infty} b_n z^n \right)
\]
\[
= z^p - \sum_{n=p+1}^{\infty} [(1 - \xi)a_n + \xi b_n] z^n
\]
\[
= z^p - \sum_{n=p+1}^{\infty} c_n z^n,
\]
where \( c_n = (1 - \xi)a_n + \xi b_n \). Using (2.3) and (2.4),

\[
\sum_{n=p+1}^{\infty} [(n - p) + \mu(n\beta - \alpha p)]n(k)c_n
\]

\[
= \sum_{n=p+1}^{\infty} [(n - p) + \mu(n\beta - \alpha p)]n(k)[(1 - \xi)a_n + \xi b_n]
\]

\[
= (1 - \xi) \sum_{n=p+1}^{\infty} [(n - p) + \mu(n\beta - \alpha p)]n(k)a_n
\]

\[
+ \xi \sum_{n=p+1}^{\infty} [(n - p) + \mu(n\beta - \alpha p)]n(k)b_n
\]

\[
= (1 - \xi)\mu p(\beta - \alpha) + \xi\mu p(\beta - \xi)
\]

\[
\leq \mu p(\beta - \alpha).
\]

Thus we have \( \sum_{n=p+1}^{\infty} [(n - p) + \mu(n\beta - \alpha p)]n(k)c_n \leq \mu p(\beta - \alpha) \). Hence 

\( h(z) \in S^{**}(\alpha, \beta). \) \hfill \Box

**Theorem 2.4 (Extreme Points).** Let \( f_1(z) = z^p \) and for \( n = 2, 3, 4, \ldots \)

\[
f_n(z) = z^p - \frac{\mu p(\beta - \alpha)}{[(n - p) + \mu(n\beta - \alpha p)]n(k)}z^n
\]

Then \( f(z) \in S^{**}(\alpha, \beta) \) if and only if \( f(z) \) can be expressed in the form

\[
f(z) = \sum_{n=1}^{\infty} \xi_n f_n(z), \text{ where } \xi_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \xi_n = 1.
\]

**Proof.** Suppose that \( f(z) = \sum_{n=1}^{\infty} \xi_n f_n(z) \), then

\[
f(z) = z^p - \sum_{n=p+1}^{\infty} \frac{\mu p(\beta - \alpha)}{[(n - p) + \mu(n\beta - \alpha p)]n(k)}\xi_n z^n = z^p - \sum_{n=p+1}^{\infty} c_n z^n
\]
where \( c_n = \frac{\mu p(\beta - \alpha)}{[n - p + \mu(n \beta - \alpha p)] n(k)} \xi_n \). Thus

\[
\sum_{n=p+1}^{\infty} [(n - p) + \mu(n \beta - \alpha p)n(k)]c_n
\]

\[
= \sum_{n=p+1}^{\infty} [(n - p) + \mu(n \beta - \alpha p)]^{n(k) \mu p(\beta - \alpha) \xi_n} \frac{n(k) \mu p(\beta - \alpha) \xi_n}{[(n - p) + \mu(n \beta - \alpha p)] n(k)}
\]

\[
\leq \mu p(\beta - \alpha) \sum_{n=p+1}^{\infty} \xi_n \leq \mu p(\beta - \alpha),
\]

since \( 0 \leq \sum_{n=p+1}^{\infty} \xi_n \leq 1 \). Hence \( f(z) \in S^{**}(\alpha, \beta) \).

Conversely, suppose that \( f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n \in S^{**}(\alpha, \beta) \). Therefore we have, for \( n = 2, 3, 4, \ldots \)

\[
a_n \leq \frac{\mu p(\beta - \alpha)}{[(n - p) + \mu(n \beta - \alpha p)] n(k)}.
\]

Setting \( \xi_n = \frac{[(n - p) + \mu(n \beta - \alpha p)] n(k)}{\mu p(\beta - \alpha)} a_n \) for \( n = 2, 3, 4, \ldots \) and \( \xi_1 = 1 - \sum_{n=p+1}^{\infty} \xi_n \), we find that \( \xi_n \geq 0 \) for \( n = 1, 2, 3, \ldots \)

\[
\sum_{n=p+1}^{\infty} \xi_n = \sum_{n=p+1}^{\infty} \frac{[(n - p) + \mu(n \beta - \alpha p)] n(k)}{\mu p(\beta - \alpha)} a_n \leq 1,
\]

since \( f(z) \in S^{**}(\alpha, \beta) \). And so \( \xi_1 = 1 - \sum_{n=1}^{\infty} \xi_n \geq 0 \). Thus \( \xi_n \geq 0 \) for
\[ n = 1, 2, 3, \ldots \] and \( \sum_{n=1}^{\infty} \xi_n = 1. \) Now

\[
f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n
= z^p - \sum_{n=p+1}^{\infty} \frac{\mu p(\beta - \alpha)}{[(n - p) + \mu(n\beta - \alpha p)]n(k)} \xi_n z^n
= \sum_{n=1}^{\infty} \xi_n f_n(z).
\]

Hence we complete the proof of theorem. The proof of the Theorem 2.4 follows on lines similar to the proof of theorem on extreme points given in Silverman([8]).

**Theorem 2.5 (Closure Theorem).** Let the function \( f_j(z) \) defined by

\[
f_j(z) = z^p - \sum_{n=p+1}^{\infty} a_{n,j} z^n, \quad a_{n,j} \geq 0, \quad j = 1, 2, 3 \cdots m
\]

be in the class \( S^{**}(\alpha, \beta) \). Then the function

\[
h(z) = z^p - \frac{1}{m} \sum_{n=p+1}^{\infty} \left( \sum_{j=1}^{m} a_{n,j} \right) z^n
\]

is in the class \( S^{**}(\alpha, \beta) \), where \( \alpha = \min_{1 \leq j \leq m} \{ \alpha_j \} , \quad -1 \leq \alpha_j < 1. \)

**Proof.** Since \( f_j(z) \in S^{**}(\alpha, \beta) \) for \( j = 1, 2, 3 \cdots m \), by Theorem 2.2
we have
\[
\sum_{n=p+1}^{\infty} \left[ (n - p) + \mu(n\beta - \alpha p) \right] n(k) \left( \frac{1}{m} \sum_{j=1}^{m} a_{n,j} \right) = \frac{1}{m} \sum_{j=1}^{m} \left( \sum_{n=p+1}^{\infty} \left[ (n - p) + \mu(n\beta - \alpha p) \right] n(k) a_{n,j} \right) \leq \frac{1}{m} \sum_{j=1}^{m} \mu p(\beta - \alpha j) \leq \mu p(\beta - \alpha).
\]

Hence \( h(z) \in S^{**}(\alpha, \beta) \). \qed

Next we prove the theorem for the radius of starlikeness and convexity.

**THEOREM 2.6.**

Let \( f(z) \in S^{**}(\alpha, \beta) \). Then

i) \( f(z) \) is starlike of order \( \delta(0 \leq \delta < 1) \) in the disc

\[
|z| \leq r = \inf_{n \geq p+1} \left\{ \frac{\left[ (n - p) + \mu(n\beta - \alpha p) \right] (2 - p - \delta)n(k)}{\mu p(\beta - \alpha)(2 - n - \delta)} \right\}^{\frac{1}{n-p}},
\]

ii) \( f(z) \) is convex of order \( \delta(0 \leq \delta < 1) \) in the disc

\[
|z| \leq r = \inf_{n \geq p+1} \left\{ \frac{\mu p(\beta - \alpha)(2 - n - \delta)}{n p(\beta - \alpha)(2 - n - \delta)} \right\}^{\frac{1}{n-p}},
\]

where \( n = 2, 3, \cdots, p \in N \). These results are sharp for the function

\[
f(z) = z^{n} - \frac{\mu p(\beta - \alpha)}{\left[ (n - p) + \mu(n\beta - \alpha p) \right] n(k)} z^{n} \quad (n = 2, 3, \cdots).
\]
Proof. i) \( f(z) \) is starlike of order \( \delta(0 \leq \delta < 1) \), we have \( \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \delta \). That is \( \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \delta \). Now

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \left| \frac{(p - 1)z^p - \sum_{n=p+1}^{\infty} (n - 1)a_n z^n}{z^p - \sum_{n=p+1}^{\infty} a_n z^n} \right| < 1 - \delta \leq \left| (p - 1)z^p - \sum_{n=p+1}^{\infty} (n - 1)a_n z^n \right| < (1 - \delta) \left| z^p - \sum_{n=p+1}^{\infty} a_n z^n \right|.
\]

Hence

\[
(2.7) \quad \sum_{n=p+1}^{\infty} \left( \frac{2 - n - \delta}{2 - p - \delta} \right) a_n |z|^{n-p} < 1.
\]

We note that \( f(z) \in S^{**}(\alpha, \beta) \) if and only if

\[
(2.8) \quad \sum_{n=p+1}^{\infty} \left[ \frac{(n - p) + \mu(n\beta - \alpha p)]n(k)a_n}{\mu p(\beta - \alpha)} \right] < 1
\]

Using (2.7) and (2.8) we get

\[
\left( \frac{2 - n - \delta}{2 - p - \delta} \right) |z|^{n-p} < \left[ \frac{(n - p) + \mu(n\beta - \alpha p)]n(k)}{\mu p(\beta - \alpha)} \right].
\]

Thus

\[
|z| \leq r = \inf_{n \geq p+1} \left\{ \frac{\left[ (n - p) + \mu(n\beta - \alpha p)](2 - p - \delta)n(k) \right]}{\mu p(\beta - \alpha)(2 - n - \delta)} \right\}^{1/(n-p)}
\]

for \( n = 2, 3, 4, \ldots, p \in N \), which proves starlikeness of family.
ii) \( f(z) \) is convex of order \( \delta (0 \leq \delta < 1) \), we have \( \text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \delta \), that is

\[
\left| \frac{zf''(z)}{f'(z)} \right| < 1 - \delta. \tag{2.9}
\]

Thus

\[
\left| \frac{p(p-1)z^{p-1} - \sum_{n=p+1}^{\infty} a_n n(n-1)z^{n-1}}{pz^{p-1} - \sum_{n=p+1}^{\infty} a_n nz^{n-1}} \right| < 1 - \delta,
\]

that is

\[
\sum_{n=p+1}^{\infty} \frac{n(2-n-\delta)}{p(2-p-\delta)} a_n |z|^{n-p} < 1. \tag{2.10}
\]

We note that \( f(z) \in S^{**}(\alpha, \beta) \) if and only if

\[
\sum_{n=p+1}^{\infty} \frac{[(n-p) + \mu(n\beta - \alpha p)]n(k)}{\mu p(\beta - \alpha)} a_n < 1. \tag{2.11}
\]

Using (2.10) and (2.11), we get

\[
\frac{n}{p} \left( \frac{2-n-\delta}{2-p-\delta} \right) |z|^{n-p} < \frac{[(n-p) + \mu(n\beta - \alpha p)]n(k)}{\mu p(\beta - \alpha)}.
\]

Thus

\[
|z| \leq r = \inf_{n \geq p+1} \left\{ \frac{p[(n-p) + \mu(n\beta - \alpha p)](2-p-\delta)n(k)}{n\mu p(\beta - \alpha)(2-n-\delta)} \right\} \frac{1}{n-p},
\]

for \( n = 2, 3, 4, \cdots, p \in N \) which proves convex property of the family. \( \square \)

### 3. Partial sums

In this we consider partial sums of functions in the class \( S^{**}(\alpha, \beta) \) and obtain sharp lower bounds for the ratios of real part of \( f(z) \), \( f_k(z) \) and \( f'(z) \) to \( f'_k(z) \). Silverman([8]) and Silvia([9]) have studied the partial sums of analytic functions.
Theorem 3.1. Let \( f(z) \in S^{**}(\alpha, \beta) \) be given by (1.1) and define the partial sums of \( f_1(z) \) to \( f_k(z) \) by \( f_1(z) = z^p \) and for \( k = 2, 3, \cdots \)

\[
(3.1) \quad f_k(z) = z^p + \sum_{n=p+1}^{k} a_n z^n.
\]

If \( \sum_{n=p+1}^{\infty} t_n |a_n| \leq 1 \) and

\[
(3.2) \quad t_n = \frac{[(n - p) + \mu(n\beta - \alpha p)]}{\mu p(\beta - \alpha)} n(k),
\]

then \( f(z) \in S^{**}(\alpha, \beta) \).

Furthermore,

\[
(3.3) \quad \text{Re} \left\{ \frac{f(z)}{f_k(z)} \right\} > 1 - \frac{1}{t_{k+1}}
\]

and

\[
(3.4) \quad \text{Re} \left\{ \frac{f_k(z)}{f(z)} \right\} > \frac{t_{k+1}}{1 + t_{k+1}},
\]

where \( z \in U, k \in N \).

Proof. For the coefficient \( t_n \) given by (3.2), it is not difficult to verify that,

\[
(3.5) \quad t_{n+1} > t_n > 1.
\]

Therefore we have

\[
(3.6) \quad \sum_{n=p+1}^{\infty} |a_n| + t_{k+1} \sum_{n=k+1}^{\infty} |a_n| \leq \sum_{n=p+1}^{\infty} t_n |a_n| \leq 1.
\]

By using (3.2) and by setting

\[
(3.7) \quad g_1(z) = t_{k+1} \left\{ \frac{f(z)}{f_k(z)} - \left( 1 - \frac{1}{t_{k+1}} \right) \right\}
\]

\[
= 1 + \frac{t_{k+1} \sum_{n=k+1}^{\infty} a_n z^n}{z^p + \sum_{n=p+1}^{k} a_n z^n}
\]
and using (3.6), we find that for \( z \in U \),

\[
g_1(z) - 1 \leq \frac{t_{k+1} \sum_{n=k+1}^{\infty} a_n z^n}{2z^p + 2 \sum_{n=p+1}^{k} a_n z^n + t_{k+1} \sum_{n=k+1}^{\infty} a_n z^n}
\]

\[
\leq \frac{t_{k+1} \sum_{n=k+1}^{\infty} |a_n|}{2 + 2 \sum_{n=p+1}^{k} |a_n| + t_{k+1} \sum_{n=k+1}^{\infty} |a_n|}
\]

\[
\leq 1
\]

which really gives the assertion (3.3) of Theorem 3.1. \( \square \)

In order to see that \( f(z) = z^p + \frac{z^{k+1}}{t_{k+1}} \) gives sharp result, we observe that for \( z = re^{i\pi/k} \), \( \frac{f(z)}{f_k(z)} = 1 + \frac{z^k}{t_{k+1}} \rightarrow 1 - \frac{1}{t_{k+1}} \) as \( z \rightarrow \). Similarly if we take

\[
g_2(z) = (1 + t_{k+1}) \left\{ \frac{f_k(z)}{f(z)} - \frac{t_{k+1}}{1 + t_{k+1}} \right\}
\]

\[
= 1 + \frac{(1 + t_{k+1}) \sum_{n=p+1}^{k} a_n z^n}{z^p + \sum_{n=p+1}^{\infty} a_n z^n}
\]

and making use of (3.6), we can deduce that

\[
\left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| \leq \frac{(1 + t_{k+1}) \sum_{n=p+1}^{k} a_n z^n}{2z^p + 2 \sum_{n=p+1}^{\infty} a_n z^n + (1 + t_{k+1}) \sum_{n=p+1}^{\infty} a_n z^n}
\]

\[
\leq \frac{(1 + t_{k+1}) \sum_{n=p+1}^{k} |a_n|}{2 + 2 \sum_{n=p+1}^{\infty} |a_n| + (1 + t_{k+1}) \sum_{n=p+1}^{k} |a_n|}
\]

which leads us to the assertion (3.4) of Theorem 3.1. The bound in (3.4) is sharp for each \( k \in N \) with the extremal function \( f(z) \) given.

**Theorem 3.2.** If \( f(z) \) of the form (1.1) satisfies the condition (2.1) then,

\[
\text{Re} \left\{ \frac{f'(z)}{f_k(z)} \right\} \geq 1 - \frac{k + 1}{t_{k+1}}.
\]
C. Consider

\[ g(z) = t_{k+1} \left\{ \frac{f'(z)}{f'_k(z)} - \left(1 - \frac{k+1}{t_{k+1}} \right) \right\} \]

\[ = 1 + \frac{t_{k+1}}{k+1} \sum_{n=k+1}^{\infty} a_n n^2 z^{n-1} \]

Now

\[ \frac{g(z) - 1}{g(z) + 1} \leq \frac{t_{k+1}}{k+1} \sum_{n=k+1}^{\infty} n |a_n| - 2p - 2 \sum_{n=p+1}^{k} n |a_n| - \frac{t_{k+1}}{k+1} \sum_{n=k+1}^{\infty} n |a_n| . \]

If

\[ \frac{t_{k+1}}{k+1} \sum_{n=k+1}^{\infty} n |a_n| - 2p - 2 \sum_{n=p+1}^{k} n |a_n| - \frac{t_{k+1}}{k+1} \sum_{n=k+1}^{\infty} n |a_n| \leq 1, \]

then

\[ \frac{t_{k+1}}{k+1} \sum_{n=k+1}^{\infty} n |a_n| \leq -2p - 2 \sum_{n=p+1}^{k} n |a_n| - \frac{t_{k+1}}{k+1} \sum_{n=p+1}^{\infty} n |a_n| , \]

that is

\[ -\frac{1}{p} \frac{t_{k+1}}{k+1} \sum_{n=k+1}^{\infty} n |a_n| - \frac{1}{p} \sum_{n=p+1}^{k} n |a_n| \leq 1. \]

Since the left hand size of (3.14) is bounded above by \( \sum_{n=p+1}^{k} t_n |a_n| \), if

\[ -\frac{1}{p} \frac{t_{k+1}}{k+1} \sum_{n=k+1}^{\infty} n |a_n| - \frac{1}{p} \sum_{n=p+1}^{k} n |a_n| \leq \sum_{n=p+1}^{k} t_n |a_n| \]

then

\[ \sum_{n=p+1}^{k} \left( t_n + \frac{n}{p} \right) |a_n| + \sum_{n=k+1}^{\infty} \left[ t_n + \frac{nt_k}{p(k+1)} \right] |a_n| \geq 0. \]

The result is sharp for the extremal function \( f(z) \). \( \square \)
Theorem 3.3. If $f(z)$ of the form (1.1) satisfies the condition (2.1) then,

$$\Re \left\{ \frac{f_k'(z)}{f'(z)} \right\} \geq \frac{t_{k+1}}{k+1 + t_{k+1}}.$$

Proof. By setting

$$g(z) = (k+1 + t_{k+1}) \left\{ \frac{f_k'(z)}{f'(z)} - \frac{t_{k+1}}{k+1 + t_{k+1}} \right\}$$

$$= 1 - \frac{\sum_{n=k+1}^{\infty} a_n n z^{n-1}}{p z^{p-1} + \sum_{n=p+1}^{\infty} a_n n z^{n-1}}.$$

Making use of (3.13), we deduce that

$$\frac{|g(z) - 1|}{g(z) + 1} \leq \frac{\sum_{n=k+1}^{\infty} n |a_n|}{2 p z^{p-1} + 2 \sum_{n=p+1}^{\infty} a_n n z^{n-1} - \frac{t_{k+1}}{k+1} \sum_{n=p+1}^{\infty} a_n n z^{n-1}} \leq 1.$$

The result is sharp for the function $f(z)$. \qed

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